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Extended rational (G'/G) expansion method for nonlinear partial differential equations

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Abstract. In this article, we use the extended rational (G'/G)- expansion function method to construct exact solutions for some nonlinear partial differential equations in mathematical physics via the (2+1)-dimensional Wu-Zhang equations, the KdV equation, and generalized Hirota–Satsuma coupled KdV equation in terms of the hyperbolic functions, trigonometric functions and rational function, when G satisfies a nonlinear second order ordinary differential equation. When the parameters are taken some special values, the solitary wave are derived from the traveling waves. This method is reliable, simple and gives many new exact solutions.

Keywords: The extended rational(G'/G)- expansion function method, Traveling wave solutions, The (2+1)-dimensional Wu-Zhang equations, The KdV equation, The generalized Hirota–Satsuma coupled KdV equation.

1. Introduction

Nonlinear partial differential equations play an important role in describing the various phenomena not only in physics, but also in biology and chemistry, and several other fields of science and engineering. It is one of the important jobs in the study of the nonlinear partial differential equations are searching for finding the traveling wave solutions. There are many methods for obtaining exact solutions to nonlinear partial differential equations such as the inverse scattering method [1], Hirota's bilinear method [2], Backlund transformation [3], the first integral method [4], Painlevé expansion [5], sine–cosine method [6], homogenous balance method [7], extended trial equation method [8,9], perturbation method [10,11], variation method [12], tanh - function method [13,14], Jacobi elliptic function expansion method [15,16], Exp-function method [17,18] and F-expansion method [19,20] . Wang etal [21] suggested a direct method called the (G'/G) expansion method to find the traveling wave solutions for nonlinear partial differential equations (NPDEs) . Zayed etal [22,23] have used the (G'/G) expansion method and modified (G'/G) expansion method to obtain more than traveling wave solutions for some nonlinear partial differential equations. Shehata [24] have successfully obtained more traveling wave solutions for some important NPDEs when G satisfies a linear differential equations $G''-\mu G=0$. In this paper we use the extended rational (G'/G)- expansion function method when G satisfies a nonlinear differential equations $AGG''(\xi)-BGG'-EG^2-CG'^2=0,$ where A,B,C,E are real arbitrary constants to find the traveling wave solutions for some nonlinear partial differential equations in mathematical physics namely the (2+1)-dimensional Wu-Zhang equations, the KdV equation and the generalized Hirota–Satsuma coupled KdV equation. We obtain some new kind of traveling wave solutions when the parameter are taking some special values.

2. Description of the extended rational (G'/G) expansion function method for NPDEs

In this part of the manuscript, the extended rational (G'/G) expansion method will be given. In order to apply this method to nonlinear partial differential equations we consider the following steps.

Step 1. We consider the nonlinear partial differential equation, say in two independent variables x and t is given by

$$P(u,u_t,u_x,u_{tt},u_{xx},u_{xt},..)=0, \tag{1}$$

where $u=u(x,t)$ is an unknown function, P is a polynomial in $u=u(x,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

Step 2. We use the following travelling wave transformation:

$$u=U(\xi), \qquad \xi = x-kt, \tag{2}$$

where k is a nonzero constant. We can rewrite Eq.(1) in the following form:

$$P(U,U',U'',...)=0 \tag{3}$$

Step 3. We assume that the solutions of Eq. (3) can be expressed in the following form:

$$U(\xi)=\sum_{i=-m}^{m}\frac{a_i\left(G'(\xi_n)/G(\xi_n)\right)^i}{\left[1+\alpha G'(\xi_n)/G(\xi_n)\right]^i}, \tag{4}$$

where $a_i(i=0,\pm1,...,\pm m)$ are arbitrary constants , α is nonzero constant to be determined later, m is a positive integer and $G(\xi)$ satisfies a nonlinear second order differential equation

$$AGG''(\xi)-BGG'-EG^2-CG'^2=0, \tag{5}$$

where A,B,C,E are real nonzero constants.

Step 4. Determine the positive integer m by balancing the highest order nonlinear term(s) and the highest order derivative in Eq (3).

Step 5. Substituting Eq. (4) into (3) along with (5), cleaning the denominator and then setting each coefficient of $(G'(\xi)/G(\xi))^i, i=0,\pm1,\pm2,..$ to be zero, yield a set of algebraic equations for $a_i(i=0,\pm1,...,\pm m)$, k and α .

Step 6. Solving these over-determined system of algebraic equations with the help of Maple software package to determine $a_i(i=0,\pm1,...,\pm m)$, k and α.

Step 7. The general solution of Eq. (5), takes the following cases :

(i) When $B\neq 0$, $B^2+4E(A-C)>0$, we obtain the hyperbolic exact solution of Eq.(5) takes the following form:

$$G(\xi)=\frac{e^{\frac{\xi B}{2(A-C)}}}{[B^2+4E(A-C)]^{\frac{A}{2(A-C)}}}\left[C_1\cosh\left(\frac{\sqrt{B^2+4E(A-C)}}{2A}\xi\right)+C_2\sinh\left(\frac{\sqrt{B^2+4E(A-C)}}{2A-C}\xi\right)\right]^{\frac{A}{(A-C)}} \tag{6}$$

where C_1 and C_2 are arbitrary constants. In this case the ratio between G' and G takes the form

$$\frac{G'}{G}=\frac{B}{2(A-C)}+\frac{\sqrt{B^2+4E(A-C)}}{2(A-C)}\left[\frac{C_1\sinh\left(\frac{\sqrt{B^2+4E(A-C)}}{2A}\xi\right)+C_2\cosh\left(\frac{\sqrt{B^2+4E(A-C)}}{2A}\xi\right)}{C_1\cosh\left(\frac{\sqrt{B^2+4E(A-C)}}{2A}\xi\right)+C_2\sinh\left(\frac{\sqrt{B^2+4E(A-C)}}{2A}\xi\right)}\right] \tag{7}$$

(ii) When $B\neq 0$, $B^2+4E(A-C)<0$, we obtain the trigonometric exact solution of Eq.(5) takes the form

$$\frac{G'}{G}=\frac{B}{2(A-C)}+\frac{\sqrt{-B^2-4E(A-C)]}}{2(A-C)}\left[\frac{-C_1\sin\left(\frac{\xi\sqrt{-B^2-4E(A-C)}}{2A}\right)+C_2\cos\left(\frac{\xi\sqrt{-B^2-4E(A-C)}}{2A}\right)}{C_1\cos\left(\frac{\xi\sqrt{-B^2-4E(A-C)}}{2A}\right)+C_2\sin\left(\frac{\xi\sqrt{-B^2-4E(A-C)}}{2A}\right)}\right] \tag{8}$$

(iii) When $B \neq 0$, $B^2 + 4E(A-C) = 0$, we obtain the rational exact solution of Eq.(5) takes the form

$$\frac{G'}{G} = \frac{B}{2(A-C)} + \frac{C_2}{C_1 + C_2\xi} \tag{9}$$

(iv) When $B = 0$, $4E(A-C) > 0$, we obtain the hyperbolic exact solution of Eq.(5) takes the following form:

$$\frac{G'}{G} = \frac{\sqrt{4E(A-C)}}{2(A-C)} \left[\frac{C_1 \sinh(\frac{\sqrt{4E(A-C)}}{2A}\xi) + C_2 \cosh(\frac{\sqrt{4E(A-C)}}{2A}\xi)}{C_1 \cosh(\frac{\sqrt{4E(A-C)}}{2A}\xi) + C_2 \sinh(\frac{\sqrt{4E(A-C)}}{2A}\xi)} \right] \tag{10}$$

(v) When $B = 0$, $4E(A-C) < 0$, we obtain the hyperbolic exact solution of Eq.(5) takes the following form:

$$\frac{G'}{G} = \frac{\sqrt{-4E(A-C)]}}{2(A-C)} \left[\frac{C_1 \sin(\frac{\xi\sqrt{-4E(A-C)}}{2A}) + C_2 \cos(\frac{\xi\sqrt{-4E(A-C)}}{2A})}{C_1 \cos(\frac{\xi\sqrt{-4E(A-C)}}{2A}) + C_2 \sin(\frac{\xi\sqrt{-4E(A-C)}}{2A})} \right] \tag{11}$$

Step 8. Substituting the constants $\alpha_i (i = 0, \pm 1, ..., \pm m)$, k and σ which obtained by solving the algebraic equations in Step 5, and the general solutions of Eq.(5) in step 6 into Eq.(4), we obtain more new exact solutions of Eq. (1) immediately.

3. Applications of extended rational (G'/G) expansion method for NPDEs

In this section, we use the extended rational (G'/G) expansion method to find the exact solutions for nonlinear evolution equations in mathematical physics namely the (2+1)-dimensional Wu-Zhang equations, the KdV equation and the generalized Hirota–Satsuma coupled KdV equation which are very important in the mathematical science and have been paid attention by many researchers in physics and engineering.

3.1. Example 1 . Extended rational (G'/G)- expansion method for KdV equation

In this section , we study the exact solution of the following KdV equation:-

$$u_t + \alpha u_x - \beta u u_x + \gamma u_{xxx} = 0 \tag{12}$$

where α and β are arbitrary constant. The Korteweg–de Vries equation is a nonlinear partial differential equation arising in the study of a number of different physical systems, e.g., water waves, plasma physics, an harmonic lattices, and elastic rods. It describes the long time evolution of small-but-finite amplitude dispersive waves [25]. Let us assume the traveling wave solutions of Eq. (12) in the following form:

$$u(x,t) = U(\xi), \quad \xi = x - kt \tag{13}$$

where k is an arbitrary constant. The transformation (13) permits us to convert PDE (12) to the following ODE :-

$$(\alpha - k)U' - \beta U U' + \gamma U''' = 0 \tag{14}$$

By balancing the highest order derivative term and nonlinear term in Eq. (14), we suppose the solution of Eq. (14) has the following form:

$$U = a_0 + \frac{a_1\left(\frac{G'(\xi)}{G(\xi)}\right)}{\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]} + \frac{a_2\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]}{\left(\frac{G'(\xi)}{G(\xi)}\right)} + \frac{a_3\left(\frac{G'(\xi)}{G(\xi)}\right)^2}{\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]^2} + \frac{a_4\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]^2}{\left(\frac{G'(\xi)}{G(\xi)}\right)^2}. \tag{15}$$

where a_0, a_1, a_2, a_3 and a_4 are constants to be determined later. Substituting Eq. (15) along with (5) into Eq. (14) and cleaning the denominator and collecting all terms with the same order of $(G'(\xi)/G(\xi))$ together, the left hand side of Eq. (14) are converted into polynomial in $(G'(\xi)/G(\xi))$. Setting each coefficient of these polynomials to be zero , we derive a set of algebraic equations for

$a_0, a_1, a_2, a_3, a_4, \alpha, \beta, \gamma, k$ and σ. Solving the set of algebraic equations by using Maple or Mathematica , software package to get the following results:

Case 1.

$$a_0 = \frac{-(-12\gamma\sigma^2 E^2 + 12\gamma\sigma BE - \gamma B^2 - 8\gamma CE + 8\gamma EA + A^2 k - A^2\alpha)}{\beta A^2},\ a_3 = \frac{12\gamma E(-2\sigma E + B)}{\beta A^2}, a_4 = \frac{12\gamma E^2}{\beta A^2},$$

$a_1 = a_2 = 0$,

where $C, B, E, A, \sigma, \alpha, \beta$ and γ are arbitrary constants. In this case the traveling wave solutions of the KdV equation take the following form:

Family 1. When $B \neq 0$, $\Omega = B^2 + 4E(A - C) > 0$, we obtain the hyperbolic exact solution of Eq.(15) takes the following:

$$U_1 = \frac{-(-12\gamma\sigma^2 E^2 + 12\gamma\sigma BE - \gamma B^2 - 8\gamma CE + 8\gamma EA + A^2 k - A^2\alpha)}{\beta A^2}$$
$$+ \frac{12\gamma E(-2\sigma E + B)\left[[BC_1 + C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega)}}{2A}\xi) + [BC_2 + C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega)}}{2A}\xi\right]^2}{\beta A^2\left[[(2(A-C)+\sigma B)C_1 + \sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi) + [(2(A-C)+\sigma B)C_2 + \sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]^2}$$
$$+ \frac{12\gamma E^2\left[[(2(A-C)+\sigma B)C_1 + \sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi) + [(2(A-C)+\sigma B)C_2 + \sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]^2}{\beta A^2\left[[BC_1 + C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega)}}{2A}\xi) + [BC_2 + C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega)}}{2A}\xi\right]^2}. \quad (17)$$

Family 2. When $B \neq 0$, $\Omega = B^2 + 4E(A - C) < 0$, we obtain the trigonometric exact solution of Eq.(15) takes the following form

$$U_2 = \frac{-(-12\gamma\sigma^2 E^2 + 12\gamma\sigma BE - \gamma B^2 - 8\gamma CE + 8\gamma EA + A^2 k - A^2\alpha)}{\beta A^2}$$
$$+ \frac{12\gamma E(-2\sigma E + B)\left[[BC_1 + C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi) + [BC_2 - C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]^2}{\beta A^2\left[[(2(A-C)+\sigma B)C_1 + \sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi) + [(2(A-C)+\sigma B)C_2 - \sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]^2}$$
$$+ \frac{12\gamma E^2\left[[(2(A-C)+\sigma B)C_1 + \sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi) + [(2(A-C)+\sigma B)C_2 - \sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]^2}{\beta A^2\left[[BC_1 + C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi) + [BC_2 - C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]^2} \quad (18)$$

Family 3. When $B \neq 0$, $B^2 + 4E(A - C) = 0$, we obtain the rational exact solution of Eq.(15) takes the following form:

$$U_3 = \frac{-(-12\gamma\sigma^2E^2 + 12\gamma\sigma BE - \gamma B^2 - 8\gamma CE + 8\gamma EA + A^2k - A^2\alpha)}{\beta A^2} + \frac{12\gamma E(-2\sigma E + B)[2C_2(A-C) + B(C_1 + C_2\xi)]^2}{\beta A^2[(C_1 + C_2\xi)(2(A-C) + \sigma B) + 2C_2\sigma(A-C)]^2} + \frac{12\gamma E^2[(C_1 + C_2\xi)(2(A-C) + \sigma B) + 2C_2\sigma(A-C)]^2}{\beta A^2[2C_2(A-C) + B(C_1 + C_2\xi)]^2}. \tag{19}$$

There are other families of exact solutions which omitted here for convenience .

Case 2.

$$a_0 = \frac{-(-12\gamma\sigma^2E^2 + 12\gamma\sigma BE - \gamma B^2 - 8\gamma CE + 8\gamma EA + A^2k - A^2\alpha)}{\beta A^2}, a_2 = \frac{12(-E\sigma^2 + B\sigma - C + A)^2\gamma}{\beta A^2}$$

$$a_1 = \frac{-12(B^2\sigma + AB - CB - 3B\sigma^2E + 2E^2\sigma^3 + 2C\sigma E - 2A\sigma E)\gamma}{\beta A^2}, \tag{20}$$

$$a_3 = a_4 = 0$$

where $C, B, E, A, \alpha, \beta, \sigma$ and γ are arbitrary constants. In this case the following traveling wave solutions of nonlinear KdV equation takes the following form:

Family 4. When $B \neq 0$, $\Omega = B^2 + 4E(A-C) > 0$, we obtain the hyperbolic exact solution of Eq.(15) takes the following form:

$$U_4 = \frac{-(-12\gamma\sigma^2E^2 + 12\gamma\sigma BE - \gamma B^2 - 8\gamma CE + 8\gamma EA + A^2k - A^2\alpha)}{\beta A^2} - \frac{12(B^2\sigma + AB - CB - 3B\sigma^2E + 2E^2\sigma^3 + 2C\sigma E - 2A\sigma E)\gamma\left[[BC_1 + C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi) + [BC_2 + C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]}{\beta A^2\left[[(2(A-C) + \sigma B)C_1 + \sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi) + [(2(A-C) + \sigma B)C_2 + \sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]} + \frac{12(-E\sigma^2 + B\sigma - C + A)^2\gamma\left[[(2(A-C) + \sigma B)C_1 + \sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi) + [(2(A-C) + \sigma B)C_2 + \sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]}{\beta A^2\left[[BC_1 + C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi) + [BC_2 + C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]} \tag{21}$$

Family 5. When $B \neq 0$, $\Omega = B^2 + 4E(A-C) < 0$, we obtain the trigonometric traveling wave solution of Eq.(15) takes the following form:

$$U_5 = \frac{-(-12\gamma\sigma^2E^2 + 12\gamma\sigma BE - \gamma B^2 - 8\gamma CE + 8\gamma EA + A^2k - A^2\alpha)}{\beta A^2} - \frac{12(B^2\sigma + AB - CB - 3B\sigma^2E + 2E^2\sigma^3 + 2C\sigma E - 2A\sigma E)\gamma\left[[BC_1 + C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi) + [BC_2 - C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]}{\beta A^2\left[[(2(A-C) + \sigma B)C_1 + \sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi) + [(2(A-C) + \sigma B)C_2 - \sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]} + \frac{12(-E\sigma^2 + B\sigma - C + A)^2\gamma\left[[(2(A-C) + \sigma B)C_1 + \sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi) + [(2(A-C) + \sigma B)C_2 - \sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]}{\beta A^2\left[[BC_1 + C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega)}}{2A}\xi) + [BC_2 - C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega)}}{2A}\xi\right]}. \tag{22}$$

Family 6. When $B \neq 0$, $B^2 + 4E(A-C) = 0$, we obtain the rational exact solution of Eq.(15) takes the form:

$$U_6 = \frac{-(-12\gamma\sigma^2 E^2 + 12\gamma\sigma BE - \gamma B^2 - 8\gamma CE + 8\gamma EA + A^2 k - A^2\alpha)}{\beta A^2}$$
$$+ \frac{-12(B^2\sigma + AB - CB - 3B\sigma^2 E + 2E^2\sigma^3 + 2C\sigma E - 2A\sigma E)\gamma\left[2C_2(A-C) + B(C_1 + C_2\xi)\right]}{\beta A^2\left[(C_1 + C_2\xi)(2(A-C) + \sigma B) + 2C_2\sigma(A-C)\right]}$$
$$+ \frac{12(-E\sigma^2 + B\sigma - C + A)^2\gamma\left[(C_1 + C_2\xi)(2(A-C) + \sigma B) + 2C_2\sigma(A-C)\right]}{\beta A^2\left[2C_2(A-C) + B(C_1 + C_2\xi)\right]}. \quad (23)$$

There are other families of exact solutions which omitted here for convenience .

3.2. Numerical solutions for the exact solutions of the KdV equation

In this section we give some figures to illustrate some of our results which obtained in this section. To this end , we select some special values of the parameters to show the behavior of the extended rational (G'/G) expansion method for the KdV equation.

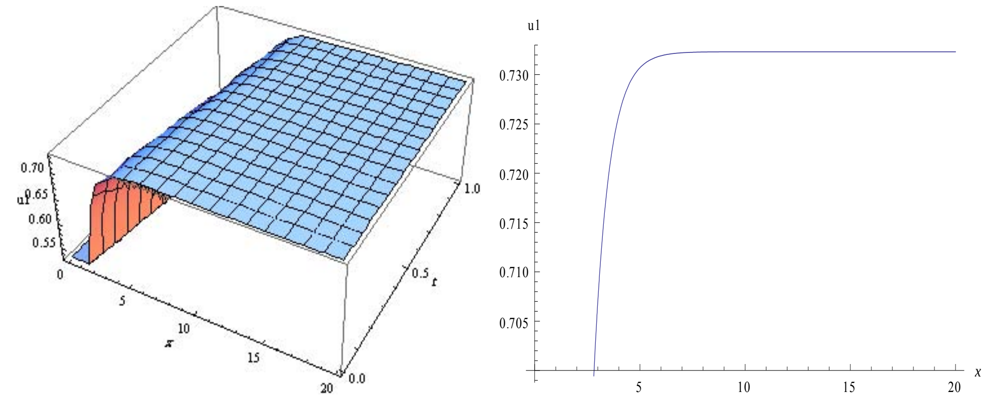

Figure 1. The exact extended (G'/G) expansion solution U_1 in Eq. (17) and its projection at $t = 0$ when the parameters take special values $E = 2,\ C1 = 0.5,\ A = 3,\ C = 1,\ k = 2.5,\ C2 = 0.75,\ \alpha = 1.75,$ $B = 1,\ \beta = 2.2,\ \gamma = 1.25$ and $\sigma = 0.25$.

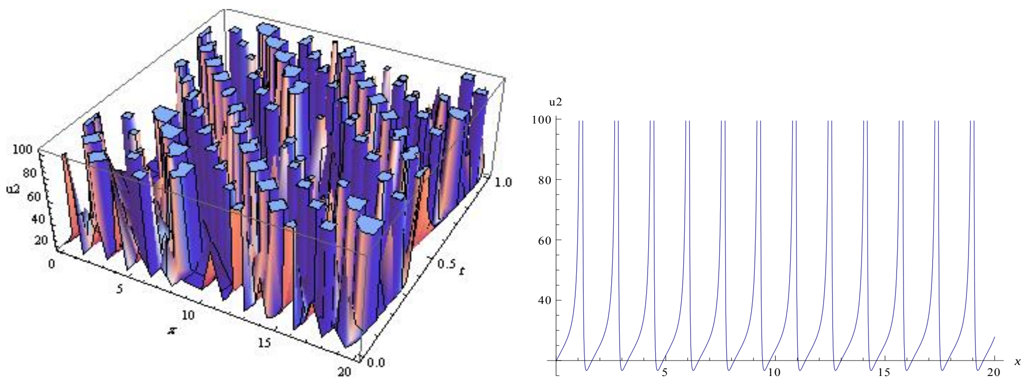

Figure 2. The exact extended (G'/G) expansion solution U_2 in Eq. (18) and its projection at $t = 0$ when the parameters take special values $E = 2,\ C1 = 0.5,\ A = 1,\ C = 3,\ k = 2.5,\ C2 = 0.75,\ \alpha = 1.75,$ $B = 1,\ \beta = 2.2,\ \gamma = 1.25$ and $\sigma = 0.25$.

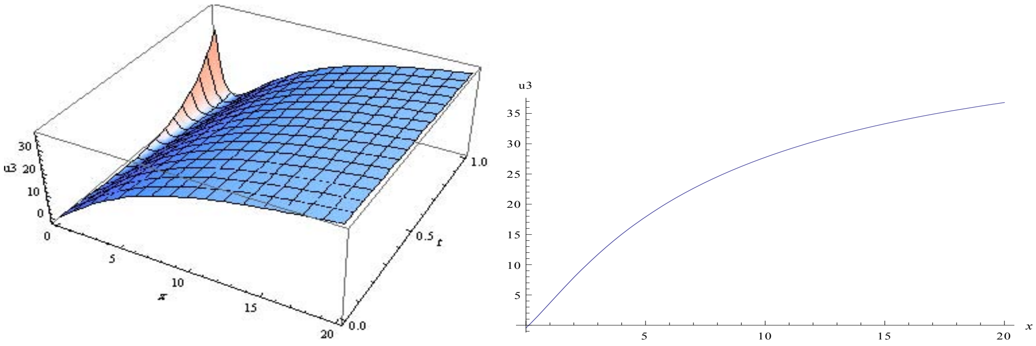

Figure 3.The exact extended (G'/G) expansion solution U_3 in Eq. (19) and its projection at $t=0$ when the parameters take special values $E=2,\ C1=0.5,\ A=3,\ C=1,\ k=2.5,\ C2=0.75,\ \alpha=1.75,\ B=1,\ \beta=2.2,\ \gamma=1.25$ and $\sigma=0.25$.

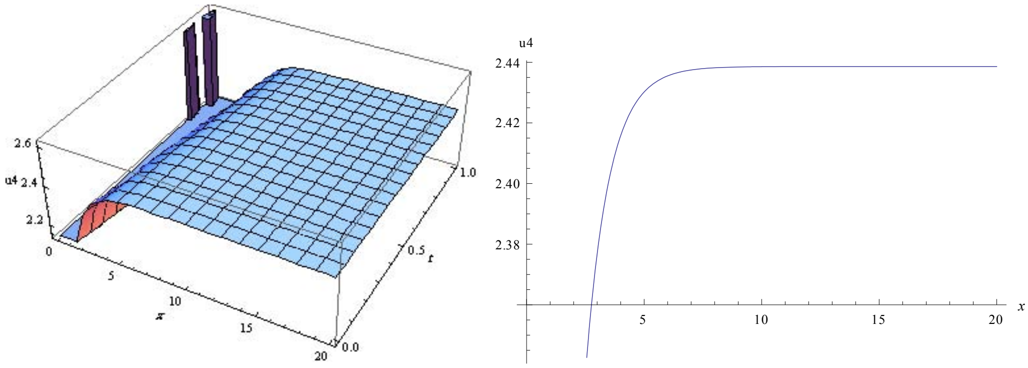

Figure 4. The exact extended (G'/G) expansion solution U_4 in Eq. (21) and its projection at $t=0$ when the parameters take special values $E=2,\ C1=0.5,\ A=3,\ C=1,\ k=2.5,\ C2=0.75,\ \alpha=1.75,\ B=1,\ \beta=2.2,\ \gamma=1.25$ and $\sigma=0.25$.

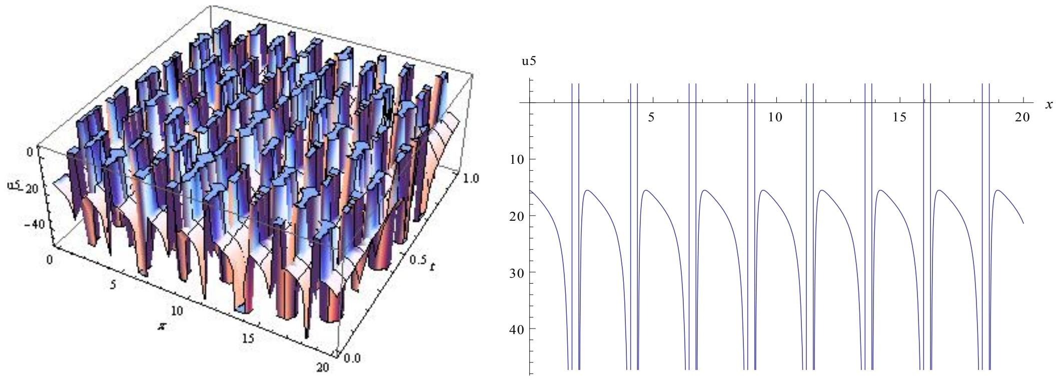

Figure 5. The exact extended (G'/G) expansion solution U_5 in Eq. (22) and its projection at $t=0$ when the parameters take special values $E=2,\ C1=0.5,\ A=1,\ C=3,\ k=2.5,\ C2=0.75,\ \alpha=1.75,\ B=1,\ \beta=2.2,\ \gamma=1.25$ and $\sigma=0.25$.

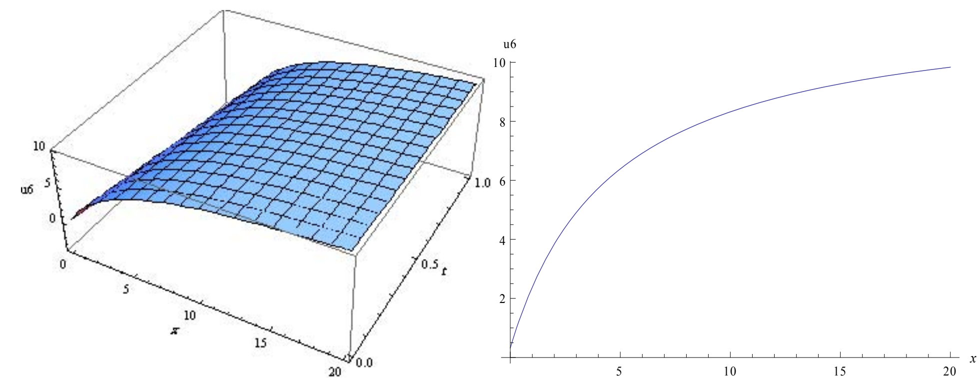

Figure 6. The exact extended (G'/G) expansion solution U_6 in Eq. (23) and its projection at $t=0$ when the parameters take special values $E=2,\ C1=0.5,\ A=3,\ C=1,\quad k=2.5,\quad C2=0.75,\quad \alpha=1.75,$ $B=1,\ \beta=2.2,\ \gamma=1.25$ and $\sigma=0.25$.

3.3. Example 2. Extended rational (G'/G) expansion function method for the (2+1)-dimensional Wu-Zhang equations

In this section, we study the (2+1)-dimensional Wu-Zhang equations [26,27].

$$
\begin{aligned}
&u_t + uu_x + vu_y + w_x = 0,\\
&v_t + uv_x + vv_y + w_y = 0,\\
&w_t + (uv)_x + (uw)_y + \frac{1}{3}(u_{xxx} + u_{xyy} + v_{xxy} + v_{yyy}) = 0.
\end{aligned} \tag{24}
$$

where w is the elevation of the water, u is the surface velocity of water along x-direction, and v is the surface velocity of water along y-direction. The explicit solutions of Eqs. (24) are very helpful for costal and civil engineers to apply the nonlinear water wave model in harbor and coastal design [26,27]. Let us assume the traveling wave solutions of Eqs. (24) in the following forms:

$$
u(x,y,t)=U(\xi),\quad v(x,y,t)=V(\xi),\quad w(x,y,t)=W(\xi),\quad \xi = x+y-kt, \tag{25}
$$

where k is an arbitrary constant. The transformations (25) permit us to convert NPDE's (24) to the following NODE's :

$$
\begin{aligned}
&-kU' + UU' + VU' + W' = 0,\\
&-kV' + UV' + VV' + W' = 0,\\
&-kW + UV + UW + \frac{2}{3}(U'' + V'') + L = 0,
\end{aligned} \tag{26}
$$

where L is the integration constant. By balancing the highest order derivative terms and nonlinear terms in Eqs. (26), we suppose that Eqs. (26) have the following solutions :

$$
\begin{aligned}
U &= a_0 + \frac{a_1\left(\frac{G'(\xi)}{G(\xi)}\right)}{\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]} + \frac{a_2\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]}{\left(\frac{G'(\xi)}{G(\xi)}\right)},\\
V &= b_0 + \frac{b_1\left(\frac{G'(\xi)}{G(\xi)}\right)}{\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]} + \frac{b_2\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]}{\left(\frac{G'(\xi)}{G(\xi)}\right)},\\
W &= c_0 + \frac{c_1\left(\frac{G'(\xi)}{G(\xi)}\right)}{\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]} + \frac{c_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2}{\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]^2} + \frac{c_3\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]}{\left(\frac{G'(\xi)}{G(\xi)}\right)} + \frac{c_4\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]^2}{\left(\frac{G'(\xi)}{G(\xi)}\right)^2},
\end{aligned} \tag{27}
$$

where $a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2, c_3$ and c_4 are arbitrary constants to be determined later. Substituting Eqs. (27) along with (5) into Eqs. (26) and cleaning the denominator and collecting all terms with the same order of $(G'(\xi)/G(\xi))$ together, the left hand side of Eqs. (26) are converted into polynomials in $(G'(\xi)/G(\xi))$. Setting each coefficient of these polynomials to be zero , we get a set of algebraic equations for $a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2, c_3, c_4, k, L$ and σ. Solving these set of algebraic equations by using the software backage such as Maple or Mathematica , we get the following results:

Case1.

$$a_1 = \pm\frac{-2\sqrt{6}(-\sigma^2 E - C + A + \sigma B)}{3A},\quad a_2 = \pm\frac{2\sqrt{6}E}{3A},\quad b_0 = \pm\frac{3}{\sqrt{6}A}\left[\pm\frac{\sqrt{6}}{3}\left(-2Ak + 2a_0 A + A\right) - 2B + 4E\sigma\right],$$

$$b_1 = \pm\frac{-2\sqrt{6}(-\sigma^2 E - C + A + \sigma B)}{3A},\qquad b_2 = \pm\frac{2\sqrt{6}E}{3A},\qquad c_2 = \frac{-8}{3A^2}(-\sigma^2 E - C + A + \sigma B)^2,$$

$$c_0 = -\frac{\pm 1}{\sqrt{6}A^2}[\pm\frac{\sqrt{6}}{3}\left(3A^2 - 6a_0A^2k + 4B^2 + 16E^2\sigma^2 - 6a_0A^2 - 16BE\sigma + 3a_0{}^2A^2 + 3k^2A^2 + 9kA^2\right)$$
$$+ 6kBA - 12E\sigma A + 6AB + 12a_0\sigma AE - 12k\sigma AE - 6a_0BA],$$

$$c_1 = \frac{-2}{A^2}(-\sigma^2 E - C + A + \sigma B)\left[\pm\frac{\sqrt{6}}{3}(-Ak + a_0A - A) - 2B + 4E\sigma\right],$$

$$c_3 = \frac{2E}{A^2}\left[\pm\frac{\sqrt{6}}{3}(-Ak + a_0A - A) - 2B + 4E\sigma\right],\qquad c_4 = \frac{8E^2}{3A^2},$$

$$L = \pm\frac{-1}{3\sqrt{6}A^3}[-64B^2\sigma E - 64EAB + 64CEB + 192E^2\sigma^2B - 18kA^2B + 36A^2k\sigma E - 128E^2\sigma C + 128\sigma E^2A$$
$$-128\sigma^3E^3 - 72A^2ka_0\sigma E + 36A^2k^2\sigma E + 36A^2a_0{}^2\sigma E + 36A^2ka_0B - 18A^2k^2B - 18A^2a_0{}^2B \pm\frac{\sqrt{6}}{3}(48AkEC$$
$$-48Aa_0EC + 9A^3a_0{}^3 - 9A^3k^3 - 27A^3k^2 - 9A^3k + 12Aa_0B^2 - 48A^2kE - 27A^3ka_0{}^2 + 27A^3a_0k^2 + 48A^2a_0E$$
$$+27A^3a_0k - 12AkB^2)],\tag{28}$$

where $k, a_0, A, B, C, E, \sigma$ are arbitrary constants. In this case the following traveling wave solutions of the (2+1)-dimensional Wu-Zhang equations take the following form:

Family 1. When $B \neq 0$, $\Omega = B^2 + 4E(A-C) > 0$, we obtain the hyperbolic exact solution of Eq.(27) takes the following form:

$$U_1 = a_0 \mp \frac{2\sqrt{6}(-\sigma^2E - C + A + \sigma B)\left[[BC_1 + C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi) + [BC_2 + C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi\right]}{3A\left[[(2(A-C)+\sigma B)C_1 + \sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi) + [(2(A-C)+\sigma B)C_2 + \sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]}$$
$$\pm\frac{2\sqrt{6}E\left[[(2(A-C)+\sigma B)C_1 + \sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi) + [(2(A-C)+\sigma B)C_2 + \sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]}{3A\left[[BC_1 + C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi) + [BC_2 + C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi\right]}.$$

$$V_1 = \pm\frac{3}{\sqrt{6}A}\left[\pm\frac{\sqrt{6}A}{3}\left(-2k + 2a_0 + 1\right) - 2B + 4E\sigma\right]$$
$$\mp\frac{2\sqrt{6}(-\sigma^2E - C + A + \sigma B)\left[[BC_1 + C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi) + [BC_2 + C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi\right]}{3A\left[[(2(A-C)+\sigma B)C_1 + \sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi) + [(2(A-C)+\sigma B)C_2 + \sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]}$$

$$\pm\frac{2\sqrt{6}E\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2+\sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]}{3A\left[[BC_1+C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[BC_2+C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi\right]}.$$

$$W_1=\mp\frac{1}{\sqrt{6}A^2}[\pm\frac{\sqrt{6}}{3}\left(3A^2-6a_0A^2k+4B^2+16E^2\sigma^2-6a_0A^2-16BE\sigma+3a_0{}^2A^2+3k^2A^2+9kA^2\right)$$
$$+6kBA-12E\sigma A+6AB+12a_0\sigma AE-12k\sigma AE-6a_0BA]$$
$$-\frac{2(-\sigma^2E-C+A+\sigma B)\left[\pm\sqrt{6}(-Ak+a_0A-A)-3(2B+4E\sigma)\right]\left[[BC_1+C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[BC_2+C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi\right]}{3A^2\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2+\sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]}$$
$$-\frac{8(-\sigma^2E-C+A+\sigma B)^2\left[[BC_1+C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[BC_2+C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi\right]^2}{3A^2\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2+\sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]^2}$$
$$+\frac{2E\left[\pm\sqrt{6}(-Ak+a_0A-A)-3(2B+4E\sigma)\right]\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2+\sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]}{3A^2\left[[BC_1+C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[BC_2+C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi\right]}$$
$$+\frac{8E^2\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2+\sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]^2}{3A^2\left[[BC_1+C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[BC_2+C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi\right]^2}. \quad (29)$$

Family 2. When $B\neq 0$, $\Omega=B^2+4E(A-C)<0$, we obtain the trigonometric exact solution of Eq.(27) takes the following form:

$$U_2=a_0\mp\frac{2\sqrt{6}(-\sigma^2E-C+A+\sigma B)\left[[BC_1+C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[BC_2-C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]}{3A\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2-\sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]}$$
$$\pm\frac{2\sqrt{6}E\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2-\sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]}{3A\left[[BC_1+C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[BC_2-C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]}.$$

$$V_2=\pm\frac{3}{\sqrt{6}A}\left[\pm\frac{\sqrt{6}}{3}(-2Ak+2a_0A+A)-2B+4E\sigma\right]$$
$$\mp\frac{2\sqrt{6}(-\sigma^2E-C+A+\sigma B)\left[[BC_1+C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[BC_2-C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]}{3A\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2-\sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]}$$
$$\pm\frac{2\sqrt{6}E\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2-\sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]}{3A\left[[BC_1+C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[BC_2-C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]}.$$

$$W_2=-\frac{\pm1}{\sqrt{6}A^2}[\pm\frac{\sqrt{6}}{3}\left(3A^2-6a_0A^2k+4B^2+16E^2\sigma^2-6a_0A^2-16BE\sigma+3a_0{}^2A^2+3k^2A^2+9kA^2\right)$$
$$+6kBA-12E\sigma A+6AB+12a_0\sigma AE-12k\sigma AE-6a_0BA]$$
$$-\frac{2(-\sigma^2E-C+A+\sigma B)\left[\pm\sqrt{6}(-Ak+a_0A-A)-3(2B+4E\sigma)\right]\left[[BC_1+C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[BC_2-C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]}{3A^2\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2-\sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]}$$
$$-\frac{8(-\sigma^2E-C+A+\sigma B)^2\left[[BC_1+C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[BC_2-C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]^2}{3A^2\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2-\sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]^2}$$
$$+\frac{2E\left[\pm\sqrt{6}(-Ak+a_0A-A)-3(2B+4E\sigma)\right]\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2-\sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]}{3A^2\left[[BC_1+C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[BC_2-C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]}$$
$$+\frac{8E^2\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2-\sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]^2}{3A^2\left[[BC_1+C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[BC_2-C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]^2}.$$

(30)

Family 3. When $B\neq0$, $B^2+4E(A-C)=0$, we obtain the rational exact solution of Eq.(27) takes the following form:

$$U_3=a_0\mp\frac{2\sqrt{6}(-\sigma^2E-C+A+\sigma B)[2C_2(A-C)+B(C_1+C_2\xi)]}{3A[(C_1+C_2\xi)(2(A-C)+\sigma B)+2C_2\sigma(A-C)]}$$
$$\pm\frac{2\sqrt{6}E[(C_1+C_2\xi)(2(A-C)+\sigma B)+2C_2\sigma(A-C)]}{3A[2C_2(A-C)+B(C_1+C_2\xi)]}.$$
$$V_3=\pm\frac{3}{\sqrt{6}A}\left[\pm\frac{\sqrt{6}}{3}(-2Ak+2a_0A+A)-2B+4E\sigma\right]$$
$$\mp\frac{2\sqrt{6}(-\sigma^2E-C+A+\sigma B)[2C_2(A-C)+B(C_1+C_2\xi)]}{3A[(C_1+C_2\xi)(2(A-C)+\sigma B)+2C_2\sigma(A-C)]}$$
$$\pm\frac{2\sqrt{6}E[(C_1+C_2\xi)(2(A-C)+\sigma B)+2C_2\sigma(A-C)]}{3A[2C_2(A-C)+B(C_1+C_2\xi)]}.$$

$$W_3 = -\frac{\pm 1}{\sqrt{6}A^2}[\pm\frac{\sqrt{6}}{3}\left(3A^2 - 6a_0A^2k + 4B^2 + 16E^2\sigma^2 - 6a_0A^2 - 16BE\sigma + 3a_0{}^2A^2 + 3k^2A^2 + 9kA^2\right)$$
$$+ 6kBA - 12E\sigma A + 6AB + 12a_0\sigma AE - 12k\sigma AE - 6a_0BA]$$
$$- \frac{2(-\sigma^2E - C + A + \sigma B)\left[\pm\sqrt{6}(-Ak + a_0A - A) - 3(2B + 4E\sigma)\right][2C_2(A-C) + B(C_1 + C_2\xi)]}{3A^2[(C_1 + C_2\xi)(2(A-C) + \sigma B) + 2C_2\sigma(A-C)]} \quad (31)$$
$$- \frac{8(-\sigma^2E - C + A + \sigma B)^2[2C_2(A-C) + B(C_1 + C_2\xi)]^2}{3A^2[(C_1 + C_2\xi)(2(A-C) + \sigma B) + 2C_2\sigma(A-C)]^2}$$
$$+ \frac{2E\left[\pm\sqrt{6}(-Ak + a_0A - A) - 3(2B + 4E\sigma)\right][(C_1 + C_2\xi)(2(A-C) + \sigma B) + 2C_2\sigma(A-C)]}{3A^2[2C_2(A-C) + B(C_1 + C_2\xi)]}$$
$$+ \frac{8E^2[(C_1 + C_2\xi)(2(A-C) + \sigma B) + 2C_2\sigma(A-C)]^2}{3A^2[2C_2(A-C) + B(C_1 + C_2\xi)]^2}.$$

There are other families of exact are omitted here for convenience .

Case 2.

$$a_0 = \mp\frac{3}{\sqrt{6}A}\left[\pm\frac{\sqrt{6}}{3}(A + Ak) + 4E\sigma - 2B\right], \qquad a_1 = \pm\frac{2\sqrt{6}}{3A}(-\sigma^2E - C + A + \sigma B),$$

$$b_0 = \mp\frac{3}{\sqrt{6}A}\left[\pm\frac{\sqrt{6}}{3}A - 4E\sigma + 2B\right], \qquad b_1 = \pm\frac{2\sqrt{6}}{3A}\left[-\sigma^2E - C + A + \sigma B\right],$$

$$c_0 = \frac{1}{3A^2}(-24E^2\sigma^2 - 3kA^2 + 24B\sigma E + 8EA - 4B^2 - 8CE),$$
$$c_2 = \frac{-8}{3A^2}(\sigma^4E^2 - 2BE\sigma^3 + 2EC\sigma^2 - 2EA\sigma^2 + B^2\sigma^2 - 2BC\sigma + 2\sigma BA + A^2 + C^2 - 2CA),$$

$$L = \frac{-1}{A^3}[3A^3 + 6kA^3 \pm \frac{\sqrt{6}}{3}(36A^2Ek\sigma - 18A^2Bk - 18A^2B + 36A^2E\sigma - 32EA\sigma + 16BEA - 192BE^2\sigma^2 + 88B^2E\sigma +$$
$$32CE^2\sigma - 16BCE - 12B^3 + 128E^3\sigma^3) - 8A^2E + 22B^2A - 96\sigma EAB + 96E^2\sigma^2A + 8CAE].$$
$$a_2 = b_2 = c_1 = c_3 = c_4 = 0. \quad (32)$$

where A, B, C, E, k, σ are arbitrary constants .

In this case the following traveling wave solutions of the (2+1)-dimensional Wu-Zhang equations take the following form:

Family 4. When $B \neq 0$, $\Omega = B^2 + 4E(A-C) > 0$, we obtain the hyperbolic exact solution of Eq.(15) takes the following form:

$$U_4 = \mp\frac{3}{\sqrt{6}A}\left[\pm\frac{\sqrt{6}}{3}(A + Ak) + 4E\sigma - 2B\right]$$
$$\pm\frac{2\sqrt{6}(-\sigma^2E - C + A + \sigma B)\left[[BC_1 + C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi) + [BC_2 + C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi\right]}{3A\left[[(2(A-C) + \sigma B)C_1 + \sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi) + [(2(A-C) + \sigma B)C_2 + \sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]}.$$

$$V_4 = \mp\frac{3}{\sqrt{6}A}\left[\pm\frac{\sqrt{6}}{3}A - 4E\sigma + 2B\right]$$
$$\pm\frac{2\sqrt{6}(-\sigma^2E - C + A + \sigma B)\left[[BC_1 + C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi) + [BC_2 + C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi\right]}{3A\left[[(2(A-C) + \sigma B)C_1 + \sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi) + [(2(A-C) + \sigma B)C_2 + \sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]}.$$

$$W_4 = \frac{1}{3A^2}(-24E^2\sigma^2 - 3kA^2 + 24B\sigma E + 8EA - 4B^2 - 8CE)$$

$$-\frac{8(\sigma^4E^2 - 2BE\sigma^3 + 2EC\sigma^2 - 2EA\sigma^2 + B^2\sigma^2 - 2BC\sigma + 2\sigma BA + A^2 + C^2 - 2CA)\left[[BC_1 + C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi) + [BC_2 + C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi\right]^2}{3A^2\left[[(2(A-C)+\sigma B)C_1 + \sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi) + [(2(A-C)+\sigma B)C_2 + \sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]^2}$$

(33)

Family 5. When $B \neq 0$, $\Omega = B^2 + 4E(A-C) < 0$, we obtain the trigonometric exact solution of Eq.(27) takes the following form

$$U_5 = \mp\frac{3}{\sqrt{6}A}\left[\pm\frac{\sqrt{6}}{3}(A+Ak) + 4E\sigma - 2B\right]$$

$$\pm\frac{2\sqrt{6}(-\sigma^2E - C + A + \sigma B)\left[[BC_1 + C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi) + [BC_2 - C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]}{3A\left[[(2(A-C)+\sigma B)C_1 + \sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi) + [(2(A-C)+\sigma B)C_2 - \sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]}.$$

$$V_5 = \mp\frac{3}{\sqrt{6}A}\left[\pm\frac{\sqrt{6}}{3}A - 4E\sigma + 2B\right]$$

$$\pm\frac{2\sqrt{6}(-\sigma^2E - C + A + \sigma B)\left[[BC_1 + C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi) + [BC_2 - C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]}{3A\left[[(2(A-C)+\sigma B)C_1 + \sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi) + [(2(A-C)+\sigma B)C_2 - \sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]}.$$

$$W_5 = \frac{1}{3A^2}(-24E^2\sigma^2 - 3kA^2 + 24B\sigma E + 8EA - 4B^2 - 8CE)$$

$$-\frac{8(\sigma^4E^2 - 2BE\sigma^3 + 2EC\sigma^2 - 2EA\sigma^2 + B^2\sigma^2 - 2BC\sigma + 2\sigma BA + A^2 + C^2 - 2CA)\left[[BC_1 + C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi) + [BC_2 - C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]^2}{3A^2\left[[(2(A-C)+\sigma B)C_1 + \sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi) + [(2(A-C)+\sigma B)C_2 - \sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]^2}.$$

(34)

Family 6. When $B \neq 0$, $B^2 + 4E(A-C) = 0$, we obtain the rational exact solution of Eq.(27) takes the following form:

$$U_6 = \mp\frac{3}{\sqrt{6}A}\left[\pm\frac{\sqrt{6}}{3}(A+Ak) + 4E\sigma - 2B\right] \pm \frac{2\sqrt{6}(-\sigma^2E - C + A + \sigma B)[2C_2(A-C) + B(C_1 + C_2\xi)]}{3A[(C_1 + C_2\xi)(2(A-C)+\sigma B) + 2C_2\sigma(A-C)]}.$$

$$V_6 = \mp\frac{3}{\sqrt{6}A}\left[\pm\frac{\sqrt{6}}{3}A - 4E\sigma + 2B\right] \pm \frac{2\sqrt{6}(-\sigma^2E - C + A + \sigma B)[2C_2(A-C) + B(C_1 + C_2\xi)]}{3A[(C_1 + C_2\xi)(2(A-C)+\sigma B) + 2C_2\sigma(A-C)]}.$$

$$W_6 = \frac{1}{3A^2}(-24E^2\sigma^2 - 3kA^2 + 24B\sigma E + 8EA - 4B^2 - 8CE)$$
$$- \frac{8(\sigma^4 E^2 - 2BE\sigma^3 + 2EC\sigma^2 - 2EA\sigma^2 + B^2\sigma^2 - 2BC\sigma + 2\sigma BA + A^2 + C^2 - 2CA)\left[2C_2(A-C) + B(C_1 + C_2\xi)\right]^2}{3A^2\left[(C_1 + C_2\xi)(2(A-C) + \sigma B) + 2C_2\sigma(A-C)\right]^2}. \quad (35)$$

There are other cases of exact solutions are omitted here for convenience .

3.4. Numerical solutions for the exact solutions for (2+1)-dimensional Wu-Zhang equations

In this section we give some figures to illustrate the behavior of the exact solutions which obtained in above section To this end , we select some special values of the parameters to show the behavior of extended rational (G'/G) - expansion method for (2+1)-dimensional Wu-Zhang equations

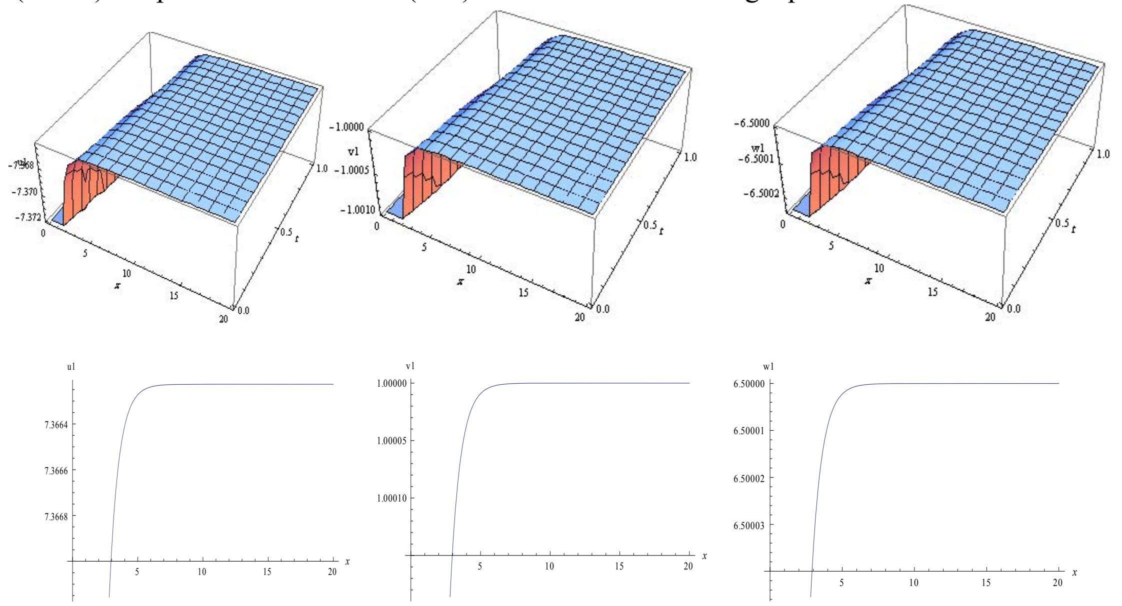

Figure 7. The exact extended (G'/G) expansion solutions U_1, V_1 and W_1 in E q. (29) and it s projection at $t = 0$ when the parameters take special values $A = 3,\ C = 1,\ E = 2,\ B = 1,\quad C_1 = 0.5,\quad y = 3,\quad a_0 = 1.5,$ $C_2 = 0.75,\ k = 2.5$ and $\sigma = 0.25.$

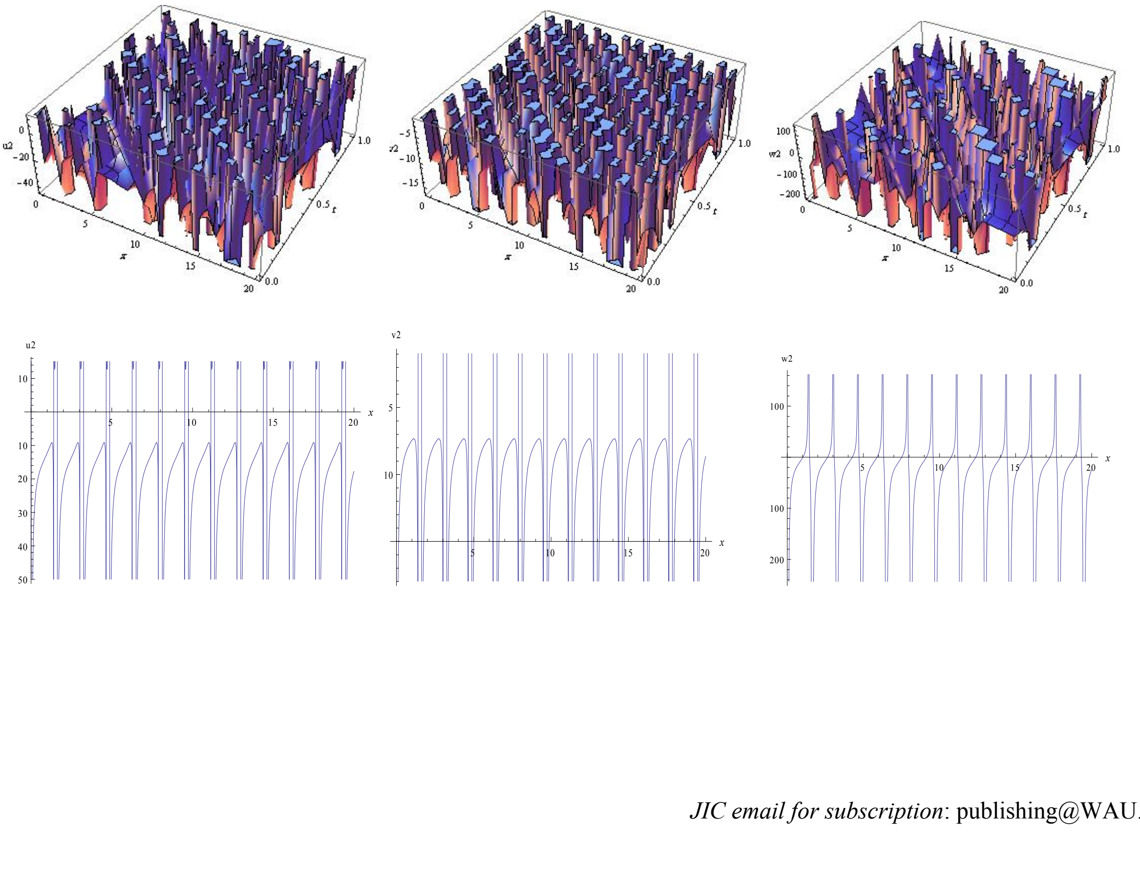

Figure 8. The exact extended (G'/G) expansion solutions U_2, V_2 and W_2 in Eq. (30) and its projection at $t=0$ when the parameters take special values $B=1,\ A=1,\ C=3,\ E=2,\quad C1=0.5,\quad y=3,\quad a_0=1.5,$ $C2=0.75,\ k=2.5$ and $\sigma=0.25.$

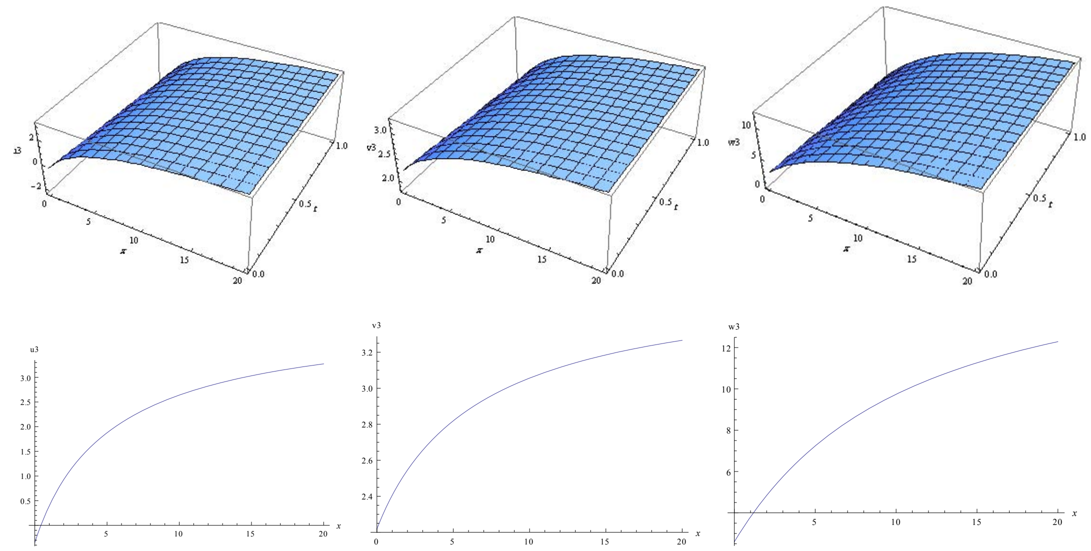

Figure 9. The exact extended (G'/G) expansion solutions U_3, V_3 and W_3 in Eq. (31) and its projection at $t=0$ when the parameters take special values $A=3,\ C=1,\ E=2,\ B=1,\quad C1=0.5,\quad y=3,\quad a_0=1.5,$ $C2=0.75,\ k=2.5$ and $\sigma=0.25.$

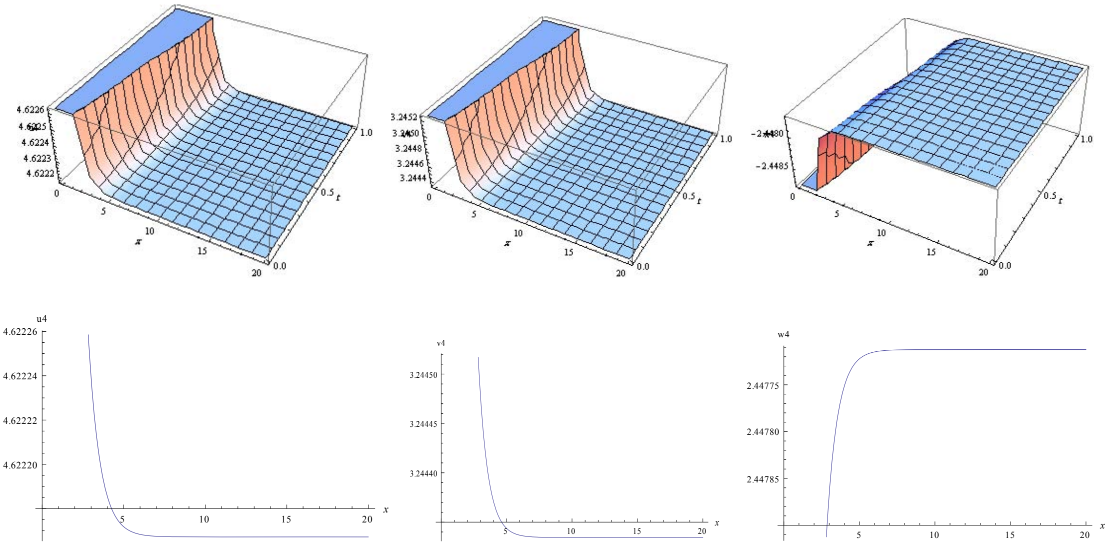

Figure 10. The exact extended (G'/G) expansion solutions U_4, V_4 and W_4 in Eq. (33) and its projection at $t=0$ when the parameters take special values $A=3,\ C=1,\ E=2,\ B=1,\quad C1=0.5,\quad y=3,\quad a_0=1.5,$ $C2=0.75,\ k=2.5$ and $\sigma=0.25.$

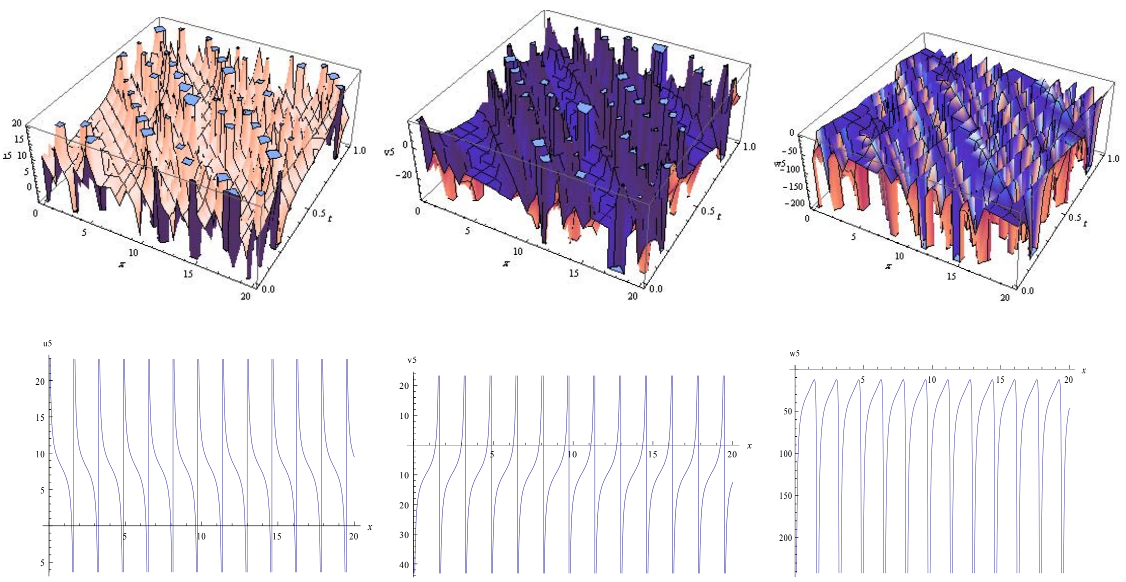

Figure 11. The exact extended (G'/G) expansion solutions U_5, V_5 and W_5 in Eq. (34) and its projection at $t=0$ when the parameters take special values $A=1,$ $C=3,$ $E=2,$ $B=1,$ $C1=0.5,$ $y=3,$ $a_0=1.5,$ $C2=0.75,$ $k=2.5$ and $\sigma=0.25.$

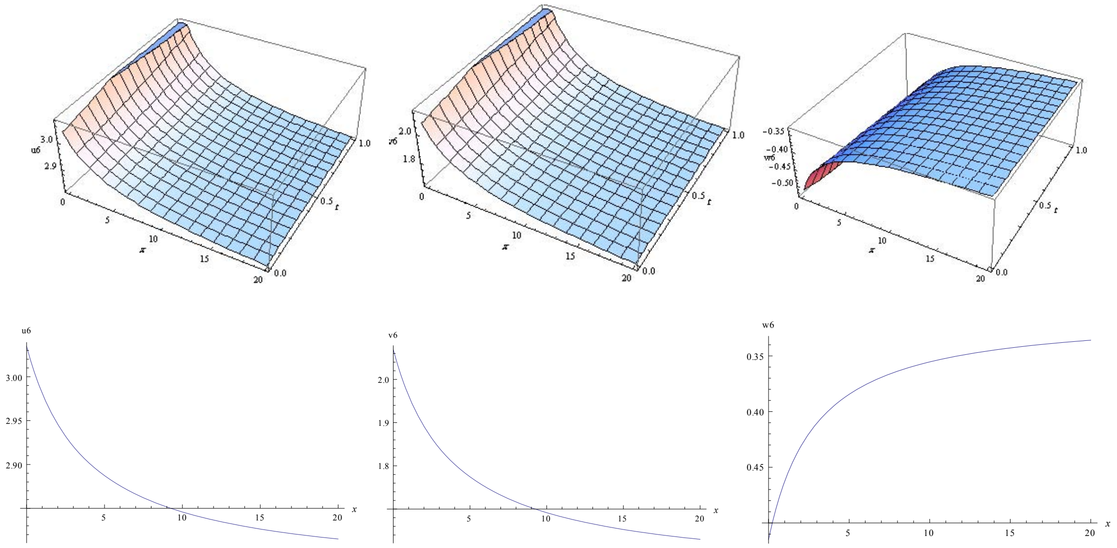

Figure 12. The exact extended (G'/G) expansion solutions U_6, V_6 and W_6 in Eq. (35) and its projection at $t=0$ when the parameters take special values $A=3,$ $C=1,$ $E=2,$ $B=1,$ $C1=0.5,$ $y=3,$ $a_0=1.5,$ $C2=0.75,$ $k=2.5$ and $\sigma=0.25.$

3.5. Example 3. Extended (G'/G) expansion method for generalized Hirota–Satsuma coupled KdV equations

In this section we study the following generalized Hirota–Satsuma coupled KdV equations by use the extended rational (G'/G) expansion method [28].

$$\begin{aligned} &u_t - \frac{1}{2}u_{xxx} + 3uu_x - 3(vw)_x = 0 \\ &v_t + v_{xxx} - 3uv_x = 0 \\ &w_t + w_{xxx} - 3uw_x = 0 \end{aligned} \tag{36}$$

The Hirota-Satsuma equations are widely used as models to describe complex physical phenomena in various fields of science, especially in fluid mechanics, solid stat physics, plasma physics. Various methods have been used to explore different kinds of solutions of physical models described by nonlinear PDEs [29]. Let us assume the traveling wave solutions of Eqs (36) in the following forms:

$$u(x,t)=U(\xi),\quad v(x,t)=V(\xi),\quad w(x,t)=W(\xi),\quad \xi=x-kt \tag{37}$$

where k is an arbitrary constant. Substituting (37) into Eqs. (36), we have:

$$\begin{aligned}
&-kU'-\frac{1}{2}U'''+3UU'-3(V'W+VW')=0\\
&-kV'+V'''-3UV'=0\\
&-kW'+W'''-3UW'=0
\end{aligned} \tag{38}$$

By balancing the highest order derivative terms and nonlinear terms in Eqs. (38), we suppose that Eqs. (38) own the solutions in the following:

$$U=a_0+\frac{a_1\left(\frac{G'(\xi)}{G(\xi)}\right)}{\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]}+\frac{a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2}{\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]^2}+\frac{a_3\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]}{\left(\frac{G'(\xi)}{G(\xi)}\right)}+\frac{a_4\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]^2}{\left(\frac{G'(\xi)}{G(\xi)}\right)^2},$$

$$V=b_0+\frac{b_1\left(\frac{G'(\xi)}{G(\xi)}\right)}{\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]}+\frac{b_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2}{\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]^2}+\frac{b_3\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]}{\left(\frac{G'(\xi)}{G(\xi)}\right)}+\frac{b_4\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]^2}{\left(\frac{G'(\xi)}{G(\xi)}\right)^2},$$

$$W=c_0+\frac{c_1\left(\frac{G'(\xi)}{G(\xi)}\right)}{\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]}+\frac{c_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2}{\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]^2}+\frac{c_3\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]}{\left(\frac{G'(\xi)}{G(\xi)}\right)}+\frac{c_4\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]^2}{\left(\frac{G'(\xi)}{G(\xi)}\right)^2}, \tag{39}$$

where $a_0,a_1,a_2,a_3,a_4,b_0,b_1,b_2,b_3,b_4,c_0,c_1,c_2,c_3$ and c_4 are constants to be determined later. Substituting Eqs. (39) along with (5) into Eqs. (38) and cleaning the denominator and collecting all terms with the same order of ($G'(\xi)/G(\xi)$) together, the left hand side of Eqs. (38) are converted into polynomials in ($G'(\xi)/G(\xi)$). Setting each coefficient of these polynomials to be zero , we derive a set of algebraic equations for $a_0,a_1,a_2,a_3,a_4,b_0,b_1,b_2,b_3,b_4,c_0,c_1,c_2,c_3,c_4,k$ and σ . Solving the set of algebraic equations by using Maple or Mathematica , software backage to get the following results:

Case 1:

$$a_1=\frac{-2}{A^2}(2E^2\sigma^3-3B\sigma^2E+2C\sigma E-2\sigma AE+B^2\sigma-CB+AB),\quad a_0=\frac{-A^2k+12\sigma^2E^2-12EB\sigma-8EA+8EC+B^2}{3A^2},$$

$$a_2=\frac{2}{A^2}(\sigma^2B^2-2B\sigma^3E-2BC\sigma+2B\sigma A-2AC+2\sigma^2CE-2\sigma^2EA+C^2+A^2+E^2\sigma^4),$$

$$b_1=\frac{-b_3}{E}(\sigma B+A-\sigma^2E-C),\qquad a_3=\frac{2E(-2E\sigma+B)}{A^2},\qquad a_4=\frac{2E^2}{A^2}, \tag{40}$$

$$c_0=\frac{-E}{3{b_3}^2A^4}(-8kb_3E\sigma A^2+20b_3BEA-40b_3E^2\sigma A+4kb_3BA^2+12E^3\sigma^2b_0+8b_0CE^2-12E^2\sigma Bb_0+$$

$$b_0EB^2-8Ab_0E^2-b_3B^3+40b_3CE^2\sigma-20b_3CBE-72b_3E^2\sigma^2B+26b_3B^2E\sigma+48b_3E^3\sigma^3-4b_0EA^2k),$$

$$c_1=\frac{E}{3A^4b_3}(12EB\sigma+4A^2k+8EA-8EC-12\sigma^2E^2-B^2)(\sigma B+A-\sigma^2E-C),$$

$$c_3 = -\frac{E^2}{3A^4 b_3}(12E\sigma B + 4A^2 k + 8EA - 8EC - 12\sigma^2 E^2 - B^2).$$

$$b_2 = b_4 = c_2 = c_4 = 0$$

where $C, B, E, A, \sigma, b_0, b_3$ and k are arbitrary constants. In this case the following traveling wave solutions of the generalized Hirota–Satsuma coupled KdV equations take the following forms:

Family 1. When $B \neq 0$, $\Omega = B^2 + 4E(A - C) > 0$, we obtain the hyperbolic exact solutions of Eqs.(39) take the following forms:

$$
\begin{aligned}
U_1 &= \frac{-A^2k+12\sigma^2E^2-12EB\sigma-8EA+8EC+B^2}{3A^2} \\
&- \frac{2(2E^2\sigma^3-3B\sigma^2E+2C\sigma E-2\sigma AE+B^2\sigma-CB+AB)\left[[BC_1+C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[BC_2+C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi\right]}{A^2\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2+\sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]} \\
&+ \frac{2(\sigma^2B^2-2B\sigma^3E-2BC\sigma+2B\sigma A-2AC+2\sigma^2CE-2\sigma^2EA+C^2+A^2+E^2\sigma^4)\left[[BC_1+C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[BC_2+C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi\right]^2}{A^2\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2+\sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]^2} \\
&+ \frac{2E(-2E\sigma+B)\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2+\sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]}{A^2\left[[BC_1+C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[BC_2+C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi\right]} \\
&+ \frac{2E^2\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2+\sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]^2}{A^2\left[[BC_1+C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[BC_2+C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi\right]^2}.
\end{aligned}
$$

$$
\begin{aligned}
V_1 &= b_0 - \frac{b_3(\sigma B+A-\sigma^2E-C)\left[[BC_1+C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[BC_2+C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi\right]}{E\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2+\sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]} \\
&+ \frac{b_3\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2+\sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]}{\left[[BC_1+C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[BC_2+C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi\right]}.
\end{aligned}
$$

$$
\begin{aligned}
W_1 &= \frac{-E}{3{b_3}^2A^4}(-8kb_3E\sigma A^2+20b_3BEA-40b_3E^2\sigma A+4kb_3BA^2+12E^3\sigma^2b_0+8b_0CE^2-12E^2\sigma Bb_0+ \\
&b_0EB^2-8Ab_0E^2-b_3B^3+40b_3CE^2\sigma-20b_3CBE-72b_3E^2\sigma^2B+26b_3B^2E\sigma+48b_3E^3\sigma^3-4b_0EA^2k) \\
&+ \frac{E(12EB\sigma+4A^2k+8EA-8EC-12\sigma^2E^2-B^2)(\sigma B+A-\sigma^2E-C)\left[[BC_1+C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega)}}{2A}\xi)+[BC_2+C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega)}}{2A}\xi\right]}{3A^4b_3\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2+\sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]} \\
&- \frac{E^2(12E\sigma B+4A^2k+8EA-8EC-12\sigma^2E^2-B^2)\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2+\sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]}{3A^4b_3\left[[BC_1+C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega)}}{2A}\xi)+[BC_2+C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega)}}{2A}\xi\right]}.
\end{aligned}
$$

(41)

Family 2. When $B \neq 0$, $\Omega = B^2 + 4E(A-C) < 0$, we obtain the trigonometric exact solutions of Eqs.(39) take the following forms:

$$U_2=\frac{-A^2k+12\sigma^2E^2-12EB\sigma-8EA+8EC+B^2}{3A^2}$$

$$-\frac{2(2E^2\sigma^3-3B\sigma^2E+2C\sigma E-2\sigma AE+B^2\sigma-CB+AB)\left[[BC_1+C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[BC_2-C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]}{A^2\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2-\sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]}$$

$$+\frac{2(\sigma^2B^2-2B\sigma^3E-2BC\sigma+2B\sigma A-2AC+2\sigma^2CE-2\sigma^2EA+C^2+A^2+E^2\sigma^4)\left[[BC_1+C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[BC_2-C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]^2}{A^2\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2-\sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]^2}$$

$$+\frac{2E(-2E\sigma+B)\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2-\sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]}{A^2\left[[BC_1+C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[BC_2-C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]}$$

$$+\frac{2E^2\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2-\sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]^2}{A^2\left[[BC_1+C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[BC_2-C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]^2}.$$

$$V_2=b_0-\frac{b_3(\sigma B+A-\sigma^2E-C)\left[[BC_1+C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[BC_2-C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]}{E\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2-\sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]}$$

$$+\frac{b_3\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2-\sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]}{\left[[BC_1+C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[BC_2-C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]}.$$

$$W_2=\frac{-E}{3b_3^{\,2}A^4}(-8kb_3E\sigma A^2+20b_3BEA-40b_3E^2\sigma A+4kb_3BA^2+12E^3\sigma^2b_0+8b_0CE^2-12E^2\sigma Bb_0+$$

$$b_0EB^2-8Ab_0E^2-b_3B^3+40b_3CE^2\sigma-20b_3CBE-72b_3E^2\sigma^2B+26b_3B^2E\sigma+48b_3E^3\sigma^3-4b_0EA^2k)$$

$$+\frac{E(12EB\sigma+4A^2k+8EA-8EC-12\sigma^2E^2-B^2)(\sigma B+A-\sigma^2E-C)\left[[BC_1+C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[BC_2-C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]}{3A^4b_3\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2-\sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]}$$

$$-\frac{E^2(12E\sigma B+4A^2k+8EA-8EC-12\sigma^2E^2-B^2)\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2-\sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]}{3A^4b_3\left[[BC_1+C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[BC_2-C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]}.$$

(42)

Family 3. When $B\neq 0$, $B^2+4E(A-C)=0$, we obtain the rational exact solutions of Eqs.(39) take the forms:

$$U_3 = \frac{-A^2k + 12\sigma^2E^2 - 12EB\sigma - 8EA + 8EC + B^2}{3A^2}$$
$$-\frac{2(2E^2\sigma^3 - 3B\sigma^2E + 2C\sigma E - 2\sigma AE + B^2\sigma - CB + AB)[2C_2(A-C) + B(C_1 + C_2\xi)]}{A^2[(C_1 + C_2\xi)(2(A-C) + \sigma B) + 2C_2\sigma(A-C)]}$$
$$+\frac{2(\sigma^2B^2 - 2B\sigma^3E - 2BC\sigma + 2B\sigma A - 2AC + 2\sigma^2CE - 2\sigma^2EA + C^2 + A^2 + E^2\sigma^4)[2C_2(A-C) + B(C_1 + C_2\xi)]^2}{A^2[(C_1 + C_2\xi)(2(A-C) + \sigma B) + 2C_2\sigma(A-C)]^2}$$
$$+\frac{2E(-2E\sigma + B)[(C_1 + C_2\xi)(2(A-C) + \sigma B) + 2C_2\sigma(A-C)]}{A^2[2C_2(A-C) + B(C_1 + C_2\xi)]}$$
$$+\frac{2E^2[(C_1 + C_2\xi)(2(A-C) + \sigma B) + 2C_2\sigma(A-C)]^2}{A^2[2C_2(A-C) + B(C_1 + C_2\xi)]^2}.$$
$$V_3 = b_0 - \frac{b_3(\sigma B + A - \sigma^2E - C)[2C_2(A-C) + B(C_1 + C_2\xi)]}{E[(C_1 + C_2\xi)(2(A-C) + \sigma B) + 2C_2\sigma(A-C)]} + \frac{b_3[(C_1 + C_2\xi)(2(A-C) + \sigma B) + 2C_2\sigma(A-C)]}{[2C_2(A-C) + B(C_1 + C_2\xi)]}.$$
$$W_3 = \frac{-E}{3b_3^{\,2}A^4}(-8kb_3E\sigma A^2 + 20b_3BEA - 40b_3E^2\sigma A + 4kb_3BA^2 + 12E^3\sigma^2b_0 + 8b_0CE^2 - 12E^2\sigma Bb_0 +$$
$$b_0EB^2 - 8Ab_0E^2 - b_3B^3 + 40b_3CE^2\sigma - 20b_3CBE - 72b_3E^2\sigma^2B + 26b_3B^2E\sigma + 48b_3E^3\sigma^3 - 4b_0EA^2k)$$
$$+\frac{E(12EB\sigma + 4A^2k + 8EA - 8EC - 12\sigma^2E^2 - B^2)(\sigma B + A - \sigma^2E - C)[2C_2(A-C) + B(C_1 + C_2\xi)]}{3A^4b_3[(C_1 + C_2\xi)(2(A-C) + \sigma B) + 2C_2\sigma(A-C)]}$$
$$-\frac{E^2(12E\sigma B + 4A^2k + 8EA - 8EC - 12\sigma^2E^2 - B^2)[(C_1 + C_2\xi)(2(A-C) + \sigma B) + 2C_2\sigma(A-C)]}{3A^4b_3[2C_2(A-C) + B(C_1 + C_2\xi)]}.$$

(43)

There are other cases of exact solutions are omitted here for convenience .

Case 2:

$$a_0 = \frac{-A^2k + 12\sigma^2E^2 - 12EB\sigma - 8EA + 8EC + B^2}{3A^2}, \quad a_3 = \frac{4E(-2E\sigma + B)}{A^2}, \quad a_4 = \frac{4E^2}{A^2},$$
$$b_3 = \frac{b_4(-2E\sigma + B)}{E}, \quad c_3 = \frac{4E^3(-2E\sigma + B)}{A^4b_4}, \quad c_4 = \frac{4E^4}{A^4b_4},$$
$$c_0 = \frac{-2E}{3b_4^{\,2}A^4}(8b_4AE - 8b_4CE + 12b_4B\sigma E - 12b_4E^2\sigma^2 - b_4B^2 + 4kA^2b_4 + 6b_0E^2),$$
$$a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 0. \tag{44}$$

where $C, B, E, A, \sigma, b_0, b_4$ and k are arbitrary constants. In this case the following traveling wave solutions of the generalized Hirota–Satsuma coupled KdV equations take the following forms:

Family 4. When $B \neq 0$, $\Omega = B^2 + 4E(A-C) > 0$, we obtain the hyperbolic exact solutions of Eqs.(39) take the following forms:

$$U_4=\frac{-A^2k+12\sigma^2E^2-12EB\sigma-8EA+8EC+B^2}{3A^2}$$
$$+\frac{4E(-2E\sigma+B)\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2+\sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]}{A^2\left[[BC_1+C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[BC_2+C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi\right]}$$
$$+\frac{4E^2\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2+\sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]^2}{A^2\left[[BC_1+C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[BC_2+C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi\right]^2}.$$
$$V_4=b_0+\frac{b_4(-2\sigma E+B)\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2+\sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]}{E\left[[BC_1+C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[BC_2+C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi\right]}$$
$$+\frac{b_4\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2+\sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]^2}{\left[[BC_1+C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[BC_2+C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi\right]^2}.$$
$$W_4=\frac{-2E^2}{3{b_4}^2A^4}(8b_4AE-8b_4CE+12b_4B\sigma E-12b_4E^2\sigma^2-b_4B^2+4kA^2b_4+6b_0E^2)$$
$$+\frac{4E^3(-2\sigma E+B)\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2+\sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]}{A^4b_4\left[[BC_1+C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega)}}{2A}\xi)+[BC_2+C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega)}}{2A}\xi\right]}$$
$$+\frac{4E^4\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2+\sigma C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega}}{2A}\xi)\right]^2}{A^4b_4\left[[BC_1+C_2\sqrt{\Omega}]\cosh(\frac{\sqrt{\Omega)}}{2A}\xi)+[BC_2+C_1\sqrt{\Omega}]\sinh(\frac{\sqrt{\Omega)}}{2A}\xi\right]^2}.$$

(45)

Family 5. When $B\neq 0$, $\Omega=B^2+4E(A-C)<0$, we obtain the trigonometric exact solutions of Eqs.(39) take the following forms:

$$U_5=\frac{-A^2k+12\sigma^2E^2-12EB\sigma-8EA+8EC+B^2}{3A^2}$$
$$+\frac{4E(-2E\sigma+B)\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2-\sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]}{A^2\left[[BC_1+C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[BC_2-C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]}$$
$$+\frac{4E^2\left[[(2(A-C)+\sigma B)C_1+\sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[(2(A-C)+\sigma B)C_2-\sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]^2}{A^2\left[[BC_1+C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi)+[BC_2-C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]^2}.$$

$$V_5 = b_0 + \frac{b_4(-2\sigma E + B)\left[[(2(A-C)+\sigma B)C_1 + \sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi) + [(2(A-C)+\sigma B)C_2 - \sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]}{E\left[[BC_1 + C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi) + [BC_2 - C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]}$$

$$+\frac{b_4\left[[(2(A-C)+\sigma B)C_1 + \sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi) + [(2(A-C)+\sigma B)C_2 - \sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]^2}{\left[[BC_1 + C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi) + [BC_2 - C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]^2}.$$

$$W_5 = \frac{-2E^2}{3b_4^{\,2}A^4}(8b_4AE - 8b_4CE + 12b_4B\sigma E - 12b_4E^2\sigma^2 - b_4B^2 + 4kA^2b_4 + 6b_0E^2)$$

$$+\frac{4E^3(-2\sigma E + B)\left[[(2(A-C)+\sigma B)C_1 + \sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi) + [(2(A-C)+\sigma B)C_2 - \sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]}{A^4b_4\left[[BC_1 + C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi) + [BC_2 - C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]}$$

$$+\frac{4E^4\left[[(2(A-C)+\sigma B)C_1 + \sigma C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi) + [(2(A-C)+\sigma B)C_2 - \sigma C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi)\right]^2}{A^4b_4\left[[BC_1 + C_2\sqrt{-\Omega}]\cos(\frac{\sqrt{-\Omega}}{2A}\xi) + [BC_2 - C_1\sqrt{-\Omega}]\sin(\frac{\sqrt{-\Omega}}{2A}\xi\right]^2}.$$

(46)

Family 6. When $B \neq 0$, $B^2 + 4E(A-C) = 0$, we obtain the rational exact solutions of Eqs.(39) take the forms:

$$U_6 = \frac{-A^2k + 12\sigma^2E^2 - 12EB\sigma - 8EA + 8EC + B^2}{3A^2}$$

$$+\frac{4E(-2E\sigma + B)[(C_1 + C_2\xi)(2(A-C)+\sigma B) + 2C_2\sigma(A-C)]}{A^2[2C_2(A-C) + B(C_1 + C_2\xi)]}$$

$$+\frac{4E^2[(C_1 + C_2\xi)(2(A-C)+\sigma B) + 2C_2\sigma(A-C)]^2}{A^2[2C_2(A-C) + B(C_1 + C_2\xi)]^2}.$$

$$V_6 = b_0 + \frac{b_4(-2\sigma E + B)[(C_1 + C_2\xi)(2(A-C)+\sigma B) + 2C_2\sigma(A-C)]}{E[2C_2(A-C) + B(C_1 + C_2\xi)]}$$

$$+\frac{b_4[(C_1 + C_2\xi)(2(A-C)+\sigma B) + 2C_2\sigma(A-C)]^2}{[2C_2(A-C) + B(C_1 + C_2\xi)]^2}.$$

$$W_6 = \frac{-2E^2}{3b_4^{\,2}A^4}(8b_4AE - 8b_4CE + 12b_4B\sigma E - 12b_4E^2\sigma^2 - b_4B^2 + 4kA^2b_4 + 6b_0E^2)$$

$$+\frac{4E^3(-2\sigma E + B)[(C_1 + C_2\xi)(2(A-C)+\sigma B) + 2C_2\sigma(A-C)]}{A^4b_4[2C_2(A-C) + B(C_1 + C_2\xi)]}$$

$$+.\frac{4E^4[(C_1 + C_2\xi)(2(A-C)+\sigma B) + 2C_2\sigma(A-C)]^2}{A^4b_4[2C_2(A-C) + B(C_1 + C_2\xi)]^2}.$$

(47)

There are other families of exact are omitted here for convenience .

3.6. Numerical solutions of the generalized Hirota–Satsuma coupled KdV equations

In this section we give some figures to illustrate some of our results which obtained in this section. To this end , we select some special values of the parameters to show the behavior of extended (G'/G) expansion method for the generalized Hirota–Satsuma coupled KdV equations.

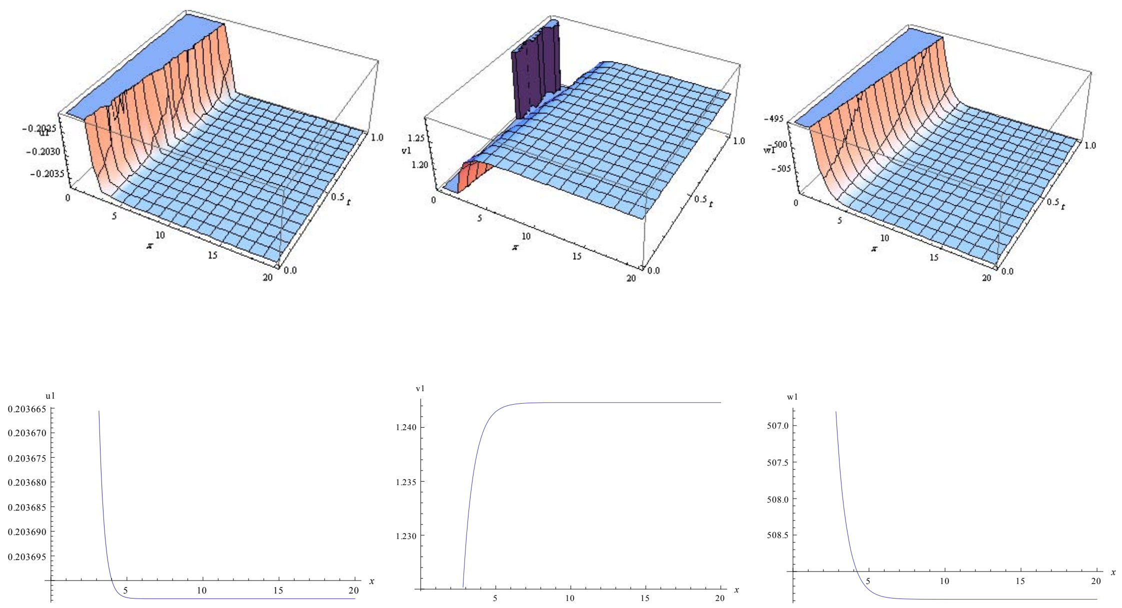

Figure 13.The exact extended (G'/G) expansion solutions U_1, V_1 and W_1 in Eqs.(41) and its projection at $t=0$ when the parameters take special values $A=3,\ C=1,\ E=2,\ C1=0.5,\ b_0=1.5,\ C2=0.75,\ B=1,\ k=2.5,\ b_3=1.75$ and $\sigma=0.25$.

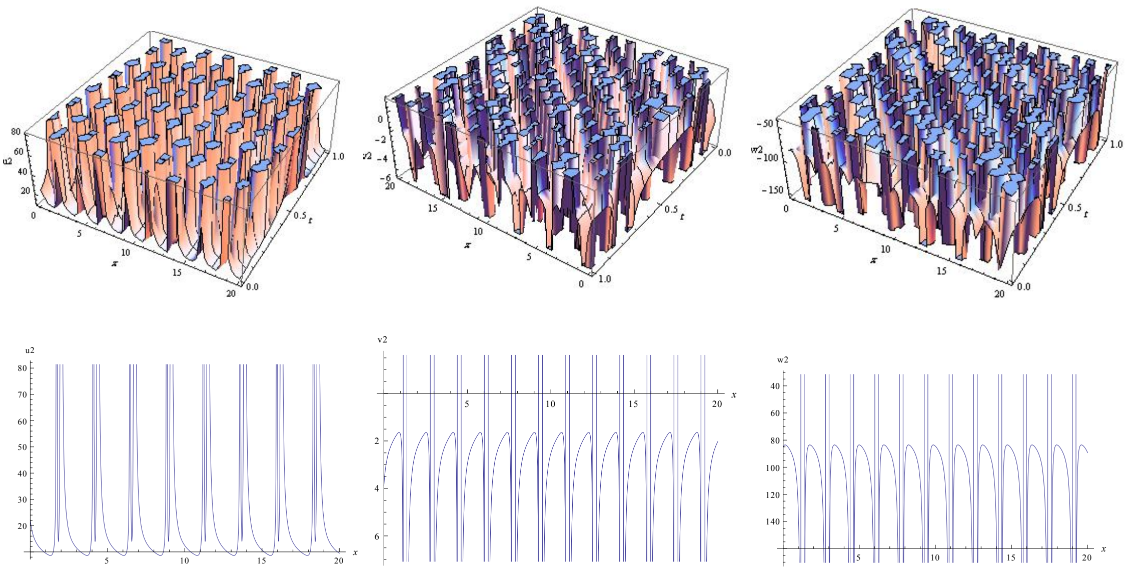

Figure 14.The exact extended (G'/G) expansion solutions U_2, V_2 and W_2 in Eqs. (42) and its projection at $t=0$ when the parameters take special values $A=1,\ C=3,\ E=2,\ C1=0.5,\ B=1,\ b_0=1.5,\ C2=0.75,\ k=2.5,\ b_3=1.75$ and $\sigma=0.25$.

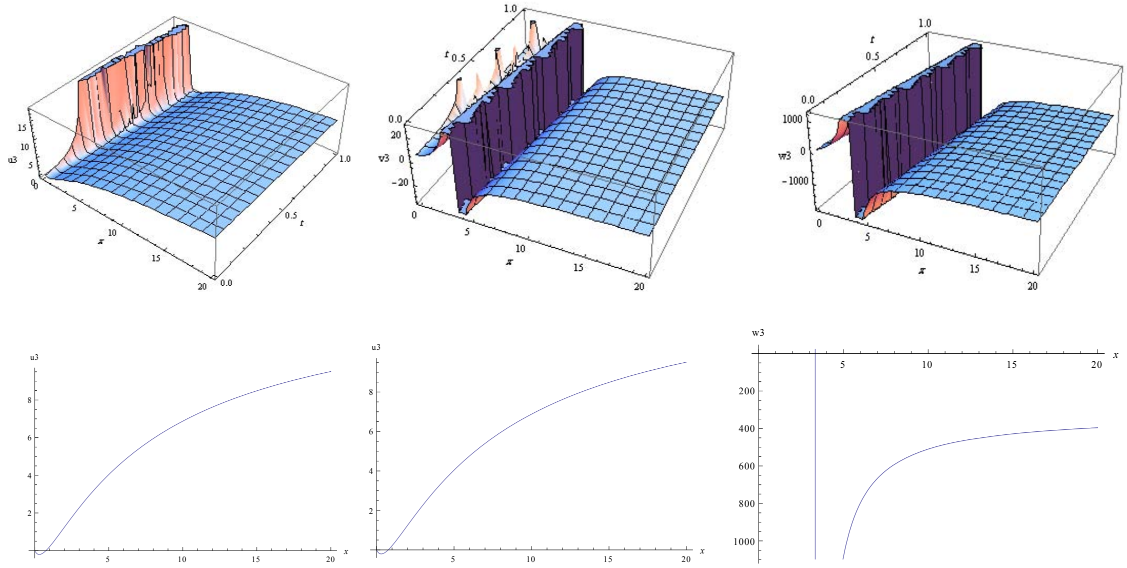

Figure 15.The exact extended (G'/G) expansion solutions U_3, V_3 and W_3 in Eqs. (43) and its projection at $t=0$ when the parameters take special values $A=3,$ $C=1,$ $E=2,$ $C1=0.5,$ $b_0=1.5,$ $C2=0.75,$ $k=2.5,$ $B=1,$ $b_3=1.75$ and $\sigma=0.25.$

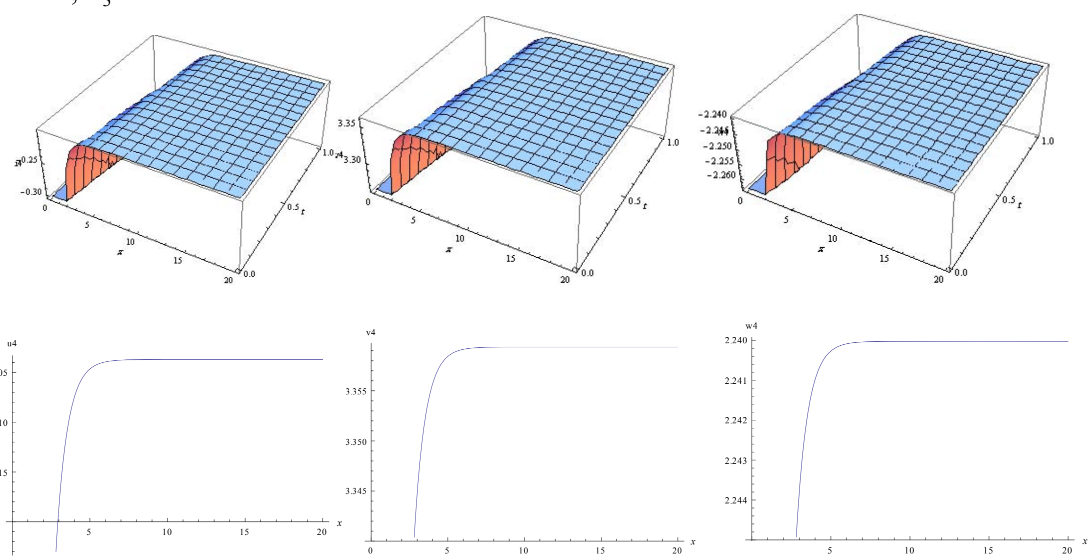

Figure 16.The exact extended (G'/G) expansion solutions U_4, V_4 and W_4 in Eqs. (45) and its projection at $t=0$ when th e parameters take special values $A=3,$ $C=1,$ $E=2,$ $B=1,$ $C1=0.5,$ $b_0=1.5,$ $C2=0.75,$ $k=2.5,$ $b_4=1.75$ and $\sigma=0.25.$

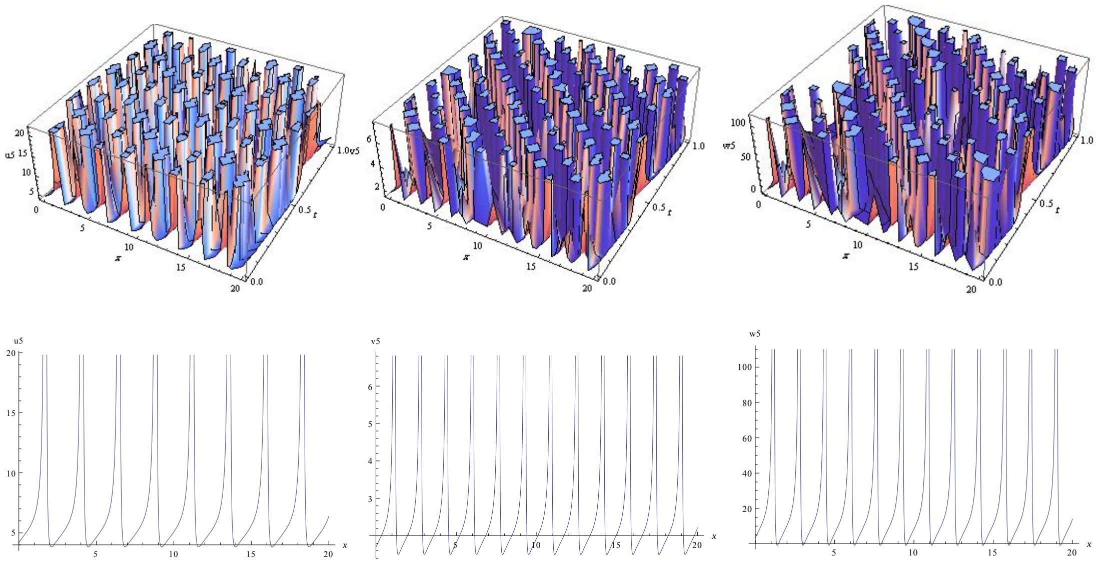

Figure 17.The exact extended (G'/G) expansion solutions U_5, V_5 and W_5 in Eqs. (46) and its projection at $t=0$ when the parameters take special values $A=1$, $C=3$, $E=2$, $B=1$, $C1=0.5$, $b_0=1.5$, $C2=0.75$, $k=2.5$, $b_4=1.75$ and $\sigma=0.25$.

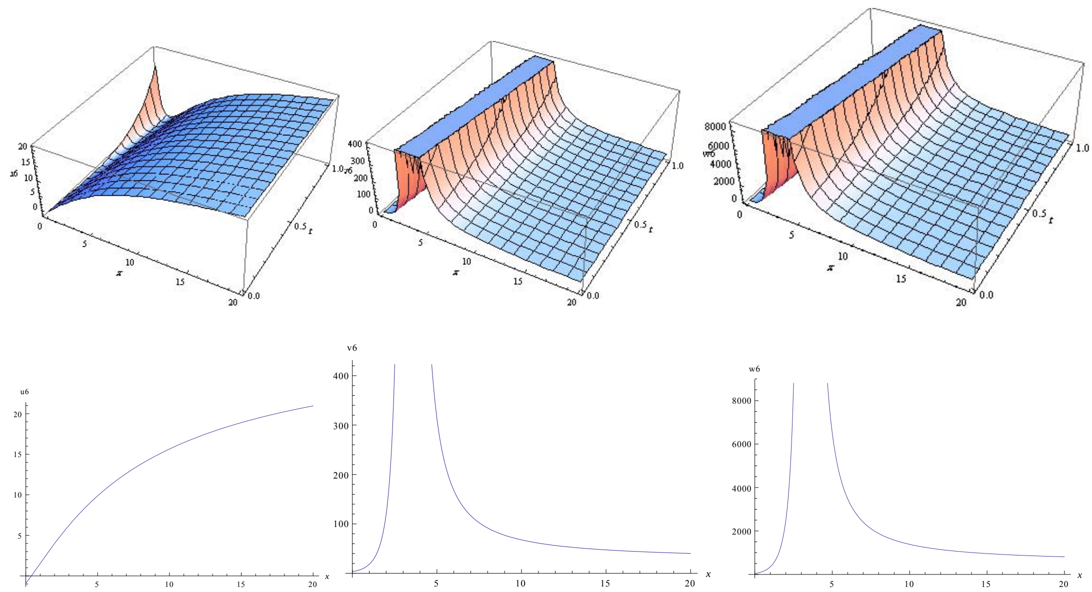

Figure 18. The exact extended (G'/G) expansion solutions U_6, V_6 and W_6 in Eqs. (47) and its projection at $t=0$ when the parameters take special values $A=1$, $C=3$, $E=2$, $B=1$, $C1=0.5$, $b_0=1.5$, $C2=0.75$, $k=2.5$, $b_4=1.75$ and $\sigma=0.25$.

4. Conclusion

In this paper we use the extended (G'/G) expansion method to construct a series of some new traveling wave solutions for some nonlinear partial differential equations in the mathematical physics. We constructed the rational exact solutions in many different functions such as hyperbolic function solutions, trigonometric function solutions and rational exact solution. The performance of this method reliable, effective and powerful for solving the nonlinear partial differential equations.

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