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Analysis of a delayed predator-prey system with Holling type-IV functional response and impulsive diffusion between two patches

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Abstract. Due to the extensive existence of time delay for natural population, it is necessary to take the effect of time delay into account in forming a biologically meaningful mathematical model. In view of this, a delayed predator-prey system with Holling type-IV functional response and impulsive dispersal between two patches is formulated. By using comparison theorem of impulsive differential equation and some analysis techniques, we obtain a predator-extinction periodic solution, which is globally attractive. Furthermore, it is proved that the investigated system is permanent. Numerical simulations are carried out to illustrate the theoretical results.

Keywords: predator-prey; time-delay; impulsive dispersal; global attractivity; permanence.

1. Introduction

Dispersal is a ubiquitous phenomenon in natural world. Its importance in understanding the ecological and evolutionary dynamics of populations was mirrored by a large number of mathematical models devoted to it in the scientific literature. The persistence and extinction for ordinary differential equation and delayed differential equation models were investigated[1-3]. Global stability of equilibrium and periodic solution for diffusing models were studied[4-6].

However, in all of above population dispersing systems, it is always assumed that the dispersal occurs at every time. For example, in [3], Huang proposed the following periodic predator-prey system with Holling-IV functional response:

$$\begin{cases} \dot{x}_1 = x_1\left[b_1(t) - a_1(t)x_1 - \frac{c_1(t)y}{e(t)+\beta(t)x_1+x_1^2}\right] + D(t)(x_2 - x_1). \\ \dot{x}_2 = x_2[b_2(t) - a_2(t)x_2] + D(t)(x_1 - x_2). \\ \dot{y} = y\left[-d(t) + \frac{c_2(t)x_1}{e(t)+\beta(t)x_1+x_1^2} - q(t)y\right]. \end{cases} \tag{1.1}$$

The function $\frac{c(t)x_1(t)}{e(t)+\beta(t)x_1(t)+x_1^2(t)}$ represents the functional response of predator to the prey in patch 1. Let $\psi(t, x_1(t)) = \frac{c(t)x_1(t)}{e(t)+\beta(t)x_1(t)+x_1^2(t)}$, then we have

$$\frac{\partial}{\partial x_1}\psi(t, x_1(t)) \geq 0, 0 < x_1(t) \leq \sqrt{e(t)},$$

$$\frac{\partial}{\partial x_1}\psi(t, x_1(t)) < 0, x_1(t) > \sqrt{e(t)}.$$

In practice, it is often the case that diffusion occurs at certain moment. For example, when winter comes, birds will migrate between patches in search for a better environment, whereas they do not diffuse in other seasons, and the excursion of foliage seeds occurs at certain moment every year. Therefore, it is not reasonable to characterize the population movements in these cases with continuous dispersal models. This short-time scale dispersal is more appropriately assumed to be in the form of impulses in the modeling process. With the developments and applications of impulsive differential equations, theories of impulsive differential equations have been introduced into population dynamics, and many important studies have been performed [7-11]. Hui [8] proposed the following single model with impulsive diffusion:

$$\begin{cases} x_1'(t) = x_1(t)\big(a_1 - b_1x_1(t)\big), & \\ x_2'(t) = x_2(t)\big(a_2 - b_2x_2(t)\big), & t \neq n\tau, \\ \Delta x_1(t) = d_1\big(x_2(t) - x_1(t)\big), & t = n\tau, \\ \Delta x_2(t) = -d_2\big(x_2(t) - x_1(t)\big), & \end{cases} \tag{1.2}$$

where a_i, b_i (i = 1, 2) are the intrinsic growth and density-dependent parameters of the population in the *i* th patch, d_i is the net dispersal rate between the *i* th patch and *j* th patch ($i \neq j$, i, j = 1, 2). $\Delta x_i(t)$ =$x_i(n\tau^+)$−$x_i(n\tau)$, where $x_i(n\tau^+)$ represents the density of the population in the *i* th patch immediately after the *n* th diffusion pulse at time *t* = *n*τ, $x_i(n\tau)$ represents the density of the population in the *i*th patch before the *n*th diffusion pulse at time *t* = *n*τ,(*n* = 1, 2,···, *i* = 1, 2).

It is well known that the time delay is quite common for natural population. In order to reflect the dynamical behaviors of models that depend on the past history of system, it is necessary to take time delay into account in forming a biologically mathematical model. Delay differential equations have attracted a significant interest in recent years due to their frequent appearance in a wide range of applications, which serve as mathematical models describing various phenomena in physics, biology, physiology, and engineering, see [12-16] and references therein, their research topics include global asymptotic stability of the equilibria, existence of periodic solutions, complicated behaviors and chaos.

Motivated by above analysis, in this paper, we will consider a delayed predator-prey system with Holling type-IV functional response and impulsive diffusion between two patches:

$$\begin{cases} x_1'(t) = x_1(t)[a_1 - b_1x_1(t) - \frac{c_1y(t)}{e+x_1(t)+x_1^2(t)}], & \\ x_2'(t) = x_2(t)\big(a_2 - b_2x_2(t)\big), & t \neq n\tau, \\ y'\ (t) = y(t)[-d + \frac{c_1x_1(t-\tau_1)}{e+x_1(t-\tau_1)+x_1^2(t-\tau_1)} - qy(t-\tau_2)], & \\ \Delta x_1(t) = d_1\big(x_2(t) - x_1(t)\big), & t = n\tau, \\ \Delta x_2(t) = -d_2\big(x_2(t) - x_1(t)\big), & \\ \Delta y(t) = 0, & \end{cases} \tag{1.3}$$

with initial conditions

$$x_1(s) = \phi_1(s), x_2(s) = \phi_2(s), y(s) = \phi_3(s),$$
$$\phi\ = (\phi_1, \phi_2, \phi_3)^T \in C([-\tilde{\tau}, 0], R_+^3), \phi_i(0) > 0, i = 1,2,3.$$

In this case, we suppose that the system is composed of two patches connected by diffusion and occupied by a single species. x_i (i = 1, 2) denotes the density of prey species in the *i*th patch, respectively, and *y* is the density of predator species. a_i and b_i denote the intrinsic growth rate and the density dependence rate of prey species in patch *i*(i = 1, 2), d is the death rate of the predator, and q represents the density dependence rate of predator species in patch 1, c_1 is the capturing rate of the predator, c_2/c_1 is the conversion rate of the nutrient into the production rate of the predator. $\tilde{\tau}$ = max{τ_1, τ_2}, $\tau_1 \geq 0$ is a constant delay due to the gestation of the predator. In addition, we have included the term $-qy(t-\tau_2)$ in the dynamics of predator y to incorporate the negative feedback of predator crowding. d_i is the net dispersal rate between the *i*th patch and *j*th patch (*i*≠*j*, *i*, *j* = 1, 2), 0 < d_i< 1 for *i* = 1, 2.

Other part of this paper is organized as follows. Some important Lemmas are presented in section. In section 3, the global attractively of the predator-extinction periodic solution and permanence of system (1.3) are investigated. In section 4, some numerical simulations are presented to illustrate the feasibility of our results. In the last section, we give a brief discussion of our results.

2. Preliminaries

In this section, we will give some definitions and lemmas.

Let $R_+ = [0, +\infty)$, $R_+^3 = \{x \in R^3, x \geq 0\}$, the map $f = (f_1, f_2, f_3)^T$ is defined by the right-hand sides of the first three equations of system (1.3), suppose $V: R_+ \times R_+^3 \to R_+$, then V is said to belong to V_0 if

(1) V is continuous in $[n\,\tau, (n\ +1)\ \tau\,] \times R_+^3$, and, for each $x \in R_+^3$, $n \in$ N, $\lim_{(t,y)\to(n\tau^+,x)} V(t,y) = V(n\tau^+, x)$ exists.

(2) *V* is locally Lipschitzian in *x*.

Definition 2.1. Let *V*∈ V_0, then for (*t*, *x*)∈[$n\tau$, (n + 1) τ] × R_+^3, the upper right derivative of *V* (*t*, *x*) with respect to the impulsive differential equation (1.3) is defined as

$$\mathrm{D}^{+}V(t,x)=\lim_{h\to 0^{+}} sup\frac{1}{h}(V(t+h,x+hf(t,x))-V(t,x)) \quad .$$

Definition 2.2. System (1.3) is said to be permanent if there exist positive constants m and M, such that each positive solution $(x_1(t),\, x_2(t),\, y(t))$ of system (1.3) satisfies $m \le x_1(t) \le M$, $m \le x_2(t) \le M$, $m \le y(t) \le M$ as $t \to \infty$.

The following lemmas are useful for the proof of the main results.

Lemma 2.1[17]. Suppose $V \in V_0$ and assume that

$$\begin{cases} D^{+}V(t,x) \le g(t,V(t,x)), t \ne n\tau, \\ V(t,x(t^{+})) \le \psi_n(V(t,x), t = n\tau, \end{cases}$$

where $g: R_{+} \times R_{+} \to R$ is continuous in $(n\tau,\ (n+1)\tau] \times R_{+}$, and for $u \in R_{+}$, $n \in N$, $\lim_{(t,y)\to(n\tau^{+},u)} g(t,y) = g(n\tau^{+},u)$ exists. $\psi_n: R_{+} \to R_{+}$ is non-decreasing. Let $r(t)$ be the maximal solution of the scalar impulsive differential equation

$$\begin{cases} u'\ (t) = g\big(t,u(t)\big), t \ne n\tau, \\ u(t^{+}) = \psi_n\big(u(t)\big),\ t = n\tau, \\ u(0^{+}) = u_0, \end{cases}$$

existing on $[0,\infty)$, then $V(0^{+}, x_0) \le u_0$ implies that $V(t, x(t)) \le r(t)(t \ge 0)$.

Lemma 2.2[18]. Assume that for $y(t) > 0$, $t \ge 0$, it holds that

$$y'\ (t) \le y(t)[a - by(t-\tau)]$$

with initial conditions $y(s) = \phi(s)$ for $s \in [-\tau,\ 0]$, where a, b are positive constants. Then there exists a positive constant y^{*} such that

$$\lim_{t\to+\infty} sup\, y(t) \le y^{*} := \frac{ae^{\alpha\tau}}{b} . \tag{2.1}$$

Lemma 2.3[18]. Assume that for $y(t) > 0$, $t \ge 0$, it holds that

$$y'\ (t) \ge y(t)[a - by(t-\tau)]$$

with initial conditions $y(s) = \phi(s)$ for $s \in [-\tau,\ 0]$, where a, b are positive constants. If (2.1) holds, then there exists a positive constant y_{*} such that

$$\lim_{t\to+\infty} inf\, y(t) \ge y_{*} := \frac{ae^{\alpha(1-e^{\alpha\tau})\tau}}{b} .$$

Consider the system (1.2), integrating and solving the first two equations of system (1.2) between pulses, we have

$$x_i(t) = [\tfrac{b_i}{a_i} + \Big(\tfrac{1}{x_i(n\tau^{+})} - \tfrac{b_i}{a_i}\Big)\exp\big(-a_i(t-n\tau)\big)]^{-1},\ n\tau < t \le (n+1)\tau,\ i = 1, 2. \quad (2.2)$$

By considering the last two equations of system (1.2), we obtain the following stroboscopic map of system (1.2):

$$\begin{cases} x_{1,(n+1)\tau} = \dfrac{x_{1,n\tau}}{h_1 + c_1 x_{1,n\tau}} + d_1[\dfrac{x_{2,n\tau}}{h_2 + c_2 x_{2,n\tau}} - \dfrac{x_{1,n\tau}}{h_1 + c_1 x_{1,n\tau}}], \\ x_{2,(n+1)\tau} = \dfrac{x_{2,n\tau}}{h_2 + c_2 x_{2,n\tau}} - d_2[\dfrac{x_{2,n\tau}}{h_2 + c_2 x_{2,n\tau}} - \dfrac{x_{1,n\tau}}{h_1 + c_1 x_{1,n\tau}}], \end{cases} \tag{2.3}$$

where $x_{i,(n+1)\tau} = x_i[(n+1)\tau^{+}]$, $c_i = \frac{a_i}{b_i}(1 - \mathrm{e}^{-a_i\tau}) > 0$, and $h_i = \mathrm{e}^{-a_i\tau} < 1$. Equations (2.3) are differential equation, which describes the densities of population in the two patches at a pulse in terms of values at the previous pulse. In other words, we are stroboscopically sampling at its pulsing period. The dynamical behavior of system (2.3) coupled with (2.2) determine the dynamical behavior of system (1.2). To write system (1.2) as a map, we define the map $F: R_{+}^{2} \to R_{+}^{2}$:

$$\begin{cases} F_1(x) = \dfrac{x_1}{h_1 + c_1 x_1} + d_1[\dfrac{x_2}{h_2 + c_2 x_2} - \dfrac{x_1}{h_1 + c_1 x_1}], \\ F_2(x) = \dfrac{x_2}{h_2 + c_2 x_2} - d_2[\dfrac{x_2}{h_2 + c_2 x_2} - \dfrac{x_1}{h_1 + c_1 x_1}], \end{cases}$$

The set of all iterations of the map F is equivalent to the set of all density sequences generated by system (2.3) and $F(x)$ is the map evaluated at the point $x = (x_1, x_2) \in R_{+}^{2}$. Consequently, in system (2.3), F_n describes the population densities in the time $n\tau$.

Lemma 2.4[7]. There exists a unique positive fixed point $q = (q_1, q_2)$ of the map F, and for every $x = (x_1, x_2) > 0$, $F_n(x) \to q$ as $n \to \infty$. This implies the fixed point $q = (q_1, q_2)$ of F is globally stable. So all trajectories of system (1.2) approach the positive periodic solution $(x_1^{*}(\mathrm{t}), x_2^{*}(\mathrm{t}))$ with period τ , *i.e.,*

$$x_i^{*}(\mathrm{t}) = [\tfrac{b_i}{a_i} + \Big(\tfrac{1}{q_i} - \tfrac{b_i}{a_i}\Big)\exp\big(-a_i(t-n\tau)\big)]^{-1},\ n\tau < t \le (n+1)\tau,\ i = 1,2. \tag{2.4}$$

3. Main results

In this section, we will establish the sufficient conditions ensuring the global attractivity of predator-extinction periodic solution and the permanence of system (1.3).

Theorem 3.1. Assume that

$$\delta < \sqrt{e}, \tag{H1}$$

and

$$\frac{c_2[\frac{b_1}{a_1}+\left(\frac{1}{q_1}-\frac{b_1}{a_1}\right)exp(-a_1\tau)]^{-1}}{e+[\frac{b_1}{a_1}+\left(\frac{1}{q_1}-\frac{b_1}{a_1}\right)exp(-a_1\tau)]^{-1}+[\frac{b_1}{a_1}+\left(\frac{1}{q_1}-\frac{b_1}{a_1}\right)exp(-a_1\tau)]^{-2}}<d \tag{H2}$$

hold, and then system (1.3) admits a predator-extinction periodic solution, which is globally attractive, and δ is defined later.

Proof. Denote the predator-extinction periodic solution of system (1.3) by $(x_1^*(t), x_2^*(t), 0)$, then the existence of $(x_1^*(t), x_2^*(t), 0)$ can be derived easily from Lemma (2.4), we will prove that the predator-extinction periodic solution $(x_1^*(t), x_2^*(t), 0)$ of system (1.3) is globally attractive.

From the first equation of system (1.3), we have

$$x_1'(t) \le x_1(t)[a_1-b_1x_1(t)]. \tag{3.1}$$

Consider the following comparison system of the subsystem of (1.3)

$$\begin{cases} u_1'(t)=u_1(t)\big(a_1-b_1u_1(t)\big), & \\ u_2'(t)=u_2(t)\big(a_2-b_2u_2(t)\big), & t\neq n\tau, \\ \Delta u_1(t)=d_1\big(u_2(t)-u_1(t)\big), & t=n\tau, \\ \Delta u_2(t)=-d_2\big(u_2(t)-u_1(t)\big). & \end{cases} \tag{3.2}$$

From Lemma 2.4, we obtain the boundary periodic solution of system (3.2)

$$u_i^*(t)=[\frac{b_i}{a_i}+\left(\frac{1}{q_i}-\frac{b_i}{a_i}\right)\exp\big(-a_i(t-n\tau)\big)]^{-1}, n\tau<t\le(n+1)\tau\ (i=1, 2), \tag{3.3}$$

which is globally asymptotically stable. By Lemma 2.1 and Lemma 2.4, we have

$$x_i\ (t)\le u_i\ (t)\to x_i^*(t), t\to\infty. \tag{3.4}$$

For any $\varepsilon_1 > 0$, as long as ε_1 is small enough , there exists a nonnegative integer n_1 such that following inequalities hold.

$$x_1(t)<x_1^*(t)+\varepsilon_1\le[\frac{b_1}{a_1}+\left(\frac{1}{q_1}-\frac{b_1}{a_1}\right)exp(-a_1\tau)]^{-1}+\varepsilon_1:=\delta,$$

$$x_2(t)<x_2^*(t)+\varepsilon_1\le[\frac{b_2}{a_2}+\left(\frac{1}{q_2}-\frac{b_2}{a_2}\right)exp(-a_2\tau)]^{-1}+\varepsilon_1:=\vartheta, \tag{3.5}$$

where $n\tau < t \le (n+1)\tau$, $n\tau > n_1\tau$. From the third equation of system (1.3) and (3.5), we have

$$y'(t)\le y(t)\left[-d+\frac{c_2\delta}{e+\delta+\delta^2}-qy(t-\tau_2)\right], t\ >\ n_1\tau+\tilde{\tau}. \tag{3.6}$$

On the other hand, from (H1), (H2) and (3.5), we can obtain

$$\frac{c_2\delta}{e+\delta+\delta^2}<d. \tag{3.7}$$

From (3.6), (3.7) and Lemma 2.2, we have $y(t) \le 0$ as $t \to \infty$. Incorporating the positivity of $y(t)$, we have $\lim_{t\to\infty} y(t) = 0$. Hence, for any $\varepsilon_2 > 0$, if ε_2 is small enough, there exists an integer n_2 ($n_2\tau > n_1\tau + \tilde{\tau}$) such that the following inequality holds:

$$0<y(t)<\varepsilon_2, t\ >n_2\tau. \tag{3.8}$$

From the first equation of (1.3) and (3.8), we have

$$\begin{aligned} x_1'(t) &\ge x_1(t)[a_1-b_1x_1(t)-\frac{c_1\varepsilon_2}{e+x_1(t)+x_1^2(t)}] \\ &\ge x_1(t)[a_1-b_1x_1(t)-\frac{c_1\varepsilon_2}{e}] \end{aligned} \tag{3.9}$$

for all $t\ >n_2\tau$. Consider the comparison system of the subsystem of (1.3):

$$\begin{cases} w_1'(t) = w_1(t)\left(a_1 - \frac{c_1\varepsilon_2}{e} - b_1 w_1(t)\right), & \\ w_2'(t) = w_2(t)\big(a_2 - b_2 w_2(t)\big), & t \neq n\tau, \\ \Delta w_1(t) = d_1\big(w_2(t) - w_1(t)\big), & t = n\tau, \\ \Delta w_2(t) = -d_2\big(w_2(t) - w_1(t)\big), & \end{cases} \tag{3.10}$$

from Lemma 2.4, we can obtain the boundary periodic solutions of (3.10)

$$w_1^*(t) = [\frac{b_1}{a_1 - \frac{c_1\varepsilon_2}{e}} + \left(\frac{1}{q_1^*} - \frac{b_1}{a_1 - \frac{c_1\varepsilon_2}{e}}\right) exp\left(-(a_1 - \frac{c_1\varepsilon_2}{e})(t - n\tau)\right)]^{-1},$$

$$w_2^*(t) = [\frac{b_2}{a_2} + \left(\frac{1}{q_2^*} - \frac{b_2}{a_2}\right) exp(-a_2(t - n\tau))]^{-1}, \tag{3.11}$$

which is globally asymptotically stable, where $n\tau < t \leq (n+1)\tau$, $n > n_2$, q_1^* and q_2^* are confirmed homoplastically as q_1 and q_2. By Lemma 2.1 and Lemma 2.4, it can be gotten that

$$x_1(t) \geq w_1(t) \rightarrow w_1^*(t), x_2(t) \geq w_2(t) \rightarrow w_2^*(t), n\tau < t \leq (n+1)\tau, n\tau > n_2\tau.$$

On the other hand, let $\varepsilon_2 \rightarrow 0$, and we have

$$w_1^*(t) \rightarrow x_1^*(t), w_2^*(t) \rightarrow x_2^*(t) \tag{3.12}$$

for large enough t. Therefore, for any small enough $\varepsilon_3 > 0$, there exists an integer n_3 ($n_3 > n_2$) such that the following inequalities hold:

$$x_1(t) > x_1^*(t) - \varepsilon_1, x_2(t) > x_2^*(t) - \varepsilon_3, t > n_3\tau. \tag{3.13}$$

Let $\varepsilon_1 \rightarrow 0$, $\varepsilon_3 \rightarrow 0$, combining (3.5) and (3.13), we have

$$x_1(t) \rightarrow x_1^*(t), x_2(t) \rightarrow x_2^*(t), y(t) = 0, t \rightarrow \infty.$$

Therefore, the predator-extinction periodic solution $(x_1^*(t), x_2(t), 0)$ of system (1.3) is globally attractive.

Remark 3.1. If assumptions (*H*1) and (*H*2) hold, then the prey species must be permanent. If not, then it may be extinct, as a result, the predator species y will also be extinct because its survival is absolutely dependent on x_1. However, when y is extinct, x_1 will not turn to extinction because Lemma 2.4 shows that x_1 has a globally asymptotically stable positive τ -periodic solution $x_1^*(t)$.

Theorem 3.2. Assume that (*H*1) and

$$\frac{c_2\tilde{q}_1}{e + \tilde{q}_1 + \tilde{q}_1^{\,2}} > d \tag{H3}$$

hold, where $\tilde{q}_1$ is to be defined, then system (1.3) is permanent.

Proof. Firstly, we suppose that $(x_1(t), x_2(t), y(t))$ are ultimately bounded, e.g., there exists a constant $M > 0$ such that $x_1(t) < M$, $x_2(t) < M$, $y(t) < M$ as $t \rightarrow \infty$. From system (1.3), it can be obtained that

$$\begin{cases} x_1'(t) \leq x_1(t)\big(a_1 - b_1 x_1(t)\big), & \\ x_2'(t) = x_2(t)\big(a_2 - b_2 x_2(t)\big), & t \neq n\tau, \\ \Delta x_1(t) = d_1\big(x_2(t) - x_1(t)\big), & t = n\tau. \\ \Delta x_2(t) = -d_2\big(x_2(t) - x_1(t)\big), & \end{cases} \tag{3.14}$$

Via similar argument as in the proof of Theorem 3.1, we have

$$x_1(t) < x_1^*(t) + \varepsilon_1 \leq [\frac{b_1}{a_1} + \left(\frac{1}{q_1} - \frac{b_1}{a_1}\right) exp(-a_1\tau)]^{-1} + \varepsilon_1 \coloneqq \delta,$$

$$x_2(t) < x_2^*(t) + \varepsilon_1 \leq [\frac{b_2}{a_2} + \left(\frac{1}{q_2} - \frac{b_2}{a_2}\right) exp(-a_2\tau)]^{-1} + \varepsilon_1 \coloneqq \vartheta, \tag{3.15}$$

where $n\tau < t \leq (n+1)\tau$, $n\tau > n_1\tau$. From the third equation of system (1.3), we have

$$y'(t) \leq y(t)[-d + \frac{c_2\delta}{e+\delta+\delta^2} - qy(t - \tau_2)], t > n_1\tau + \tilde{\tau}. \tag{3.16}$$

From (*H*1), (*H*3) and (3.15), we have

$$\frac{c_2\delta}{e+\delta+\delta^2} > d. \tag{3.17}$$

By Lemma 2.2, we can obtain

$$\lim_{t\rightarrow+\infty} sup\, y(t) \leq \frac{\frac{c_2\delta}{e+\delta+\delta^2} - d}{q} \exp\left(\left(\frac{c_2\delta}{e+\delta+\delta^2} - d\right)\tau_2\right) \coloneqq \eta. \tag{3.18}$$

Take $M = \max\{\delta, \vartheta, \eta\}$, then $x_1(t)$, $x_2(t)$, $y(t)$ are ultimately bounded.

Secondly, we prove that there exists a positive constant $m > 0$ ($m < M$) such $x_1(t) > m$, $x_2(t) > m$, $y(t) > m$ as $t \rightarrow \infty$. From system (1.3) and (3.18), we have

$$\begin{cases} x_1'(t) \geq x_1(t)\left(a_1 - \frac{c_1\eta}{e} - b_1x_1(t)\right), & \\ & t \neq n\tau, \\ x_2'(t) = x_2(t)\big(a_2 - b_2x_2(t)\big), & \\ \Delta x_1(t) = d_1\big(x_2(t) - x_1(t)\big), & t = n\tau. \\ \Delta x_2(t) = -d_2\big(x_2(t) - x_1(t)\big), & \end{cases} \tag{3.19}$$

Consider the comparison system of (3.19), we have

$$\begin{cases} v_1'(t) = v_1(t)\left(a_1 - \frac{c_1\eta}{e} - b_1v_1(t)\right), & \\ & t \neq n\tau, \\ v_2'(t) = v_2(t)\big(a_2 - b_2v_2(t)\big), & \\ \Delta x_1(t) = d_1\big(v_2(t) - v_1(t)\big), & t = n\tau. \\ \Delta x_2(t) = -d_2\big(v_2(t) - v_1(t)\big), & \end{cases} \tag{3.20}$$

From Lemma 2.4, we obtain the boundary periodic solution of (3.20) as follows

$$v_1^*(t) = [\frac{b_1}{a_1 - \frac{c_1\varepsilon_2}{e}} + \left(\frac{1}{\tilde{q}_1} - \frac{b_1}{a_1 - \frac{c_1\varepsilon_2}{e}}\right) exp\left(-(a_1 - \frac{c_1\varepsilon_2}{e})(t - n\tau)\right)]^{-1},$$

$$v_2^*(t) = [\frac{b_2}{a_2} + \left(\frac{1}{\tilde{q}_2} - \frac{b_2}{a_2}\right) exp(-a_2(t - n\tau))]^{-1}, \tag{3.21}$$

which is globally asymptotically stable, where $n\tau < t \leq (n+1)\tau$, $\tilde{q}_1$ and $\tilde{q}_2$ are confirmed homoplastically as q_1 and q_2. By Lemma 2.1 and Lemma 2.4, we have

$$x_1(t) \geq v_1(t) \to v_1^*(t), v_2(t) \geq v_2(t) \to v_2^*(t), n\tau < t \leq (n+1)\tau . \tag{3.22}$$

Let $m_1 = \inf\{v_1^*(t) | \ t \in [0, \tau]\}$, $m2 = \inf\{v_2^*(t) | \ t \in [0, \tau]\}$, for any $\varepsilon_4 > 0$, if it is small enough, there exists an integer n_4 such that the following inequalities hold:

$$x_1(t) \geq v_1^*(t) - \varepsilon_4 \geq m_1 - \varepsilon_4, x_2(t) \geq v_2^*(t) - \varepsilon_4 \geq m_2 - \varepsilon_4, t > n_4\tau. \tag{3.23}$$

Let $\varepsilon_4 \to 0$, we have

$$\lim_{t\to+\infty} inf\, x_1(t) \geq m_1, \lim_{t\to+\infty} inf\, x_2(t) \geq m_2. \tag{3.24}$$

According to *H*3, for above small enough ε_4, we have

$$\frac{c_2(\tilde{q}_1 - \varepsilon_4)}{e + (\tilde{q}_1 - \varepsilon_4) + (\tilde{q}_1 - \varepsilon_4)^2} > d. \tag{3.25}$$

In view of (3.21) and (3.23), for above small enough$\varepsilon_4 > 0$, when $t \to \infty$, we have

$$m_1 \geq \tilde{q}_1 - \varepsilon_4. \tag{3.26}$$

Combining (3.25) and (3.26), we have

$$\frac{c_2 m_1}{e + m_1 + m_1^2} > d. \tag{3.27}$$

From the third equation of system (1.3) and (3.24), we have

$$y'\ (t) \geq y(t)[-d + \frac{c_2 m_1}{e + m_1 + m_1^2} - qy(t - \tau_2)]. \tag{3.28}$$

According to (3.27), (3.28) and Lemma 2.3, we have

$$\lim_{t\to+\infty} inf\, y(t) \geq \frac{\frac{c_2 m_1}{e + m_1 + m_1^2} - d}{q} \exp\{(\frac{c_2 m_1}{e + m_1 + m_1^2} - d)$$

$$[1 - \exp((\frac{c_2 m_1}{e + m_1 + m_1^2} - d)\,\tau_2)]\tau_2\} := m_3.$$

Then, take $m = \min\{m_1, m_2, m_3\}$, and we have $x_1(t) > m$, $x_2(t) > m$, $y(t) > m$ as $t \to \infty$.

Remark 3.2. From Theorem 3.2, it is shown that if we can guarantee that the growth of the predator by foraging minus its death rate is positive, system (1.3) is permanent.

4. Numerical simulation and discussion

In this paper, a delayed predator-prey model with Holling-IV functional response and impulsive diffusion is studied. By using comparison theorem of impulsive differential equation and other analysis methods, the global attractivity of predator-extinction periodic solution and permanence of system (1.3) are established.

Numerical verification of the results is necessary for completeness of the analytical study, therefore, we present some numerical simulations to substantiate and argument our analytical findings of system (1.3). In system (1.3), we choose $\tau = 2$.

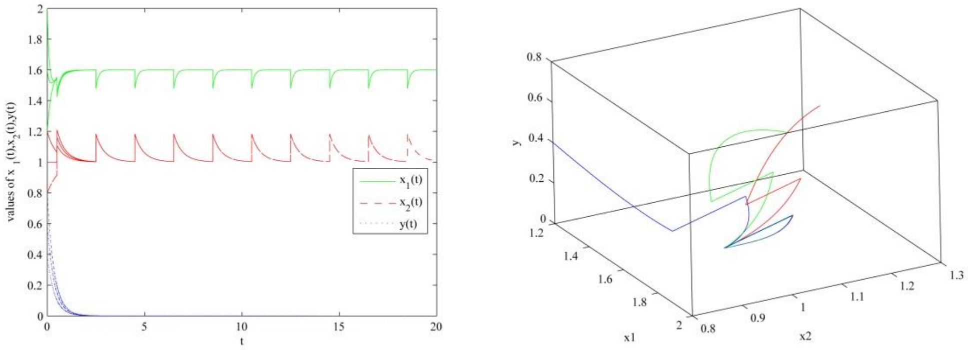
In Table 1, we give some parameter values and emphasize the parameters c_2, d, τ_1 and τ_2. Furthermore, if, in Table 1, we keep some parameter values unchanged, just adjust the parameters c_2, d, τ_1 and τ_2, and give different cases. The details are given in Table 2.

Table 1. Parameter values used in the simulations of model (1.3).

	Parameter Interpretation	**Value**
a_1	the intrinsic growth rate of prey in Patch 1	8
b_1	the self-inhibition coefficient of prey in Patch 1	5
c_1	the capture rate of predator on prey	10
a_2	the intrinsic growth rate of prey in Patch 2	2
b_2	the self-inhibition coefficient of prey in Patch 2	2
c_2	the conversion rate of nutrition	–
e	the half-saturation constant	4
d	the death rate of the predator in Patch 1	–
q	the self-inhibition coefficient of predator in Patch 1	0.4
τ_1	the gestation period of the predator	–
τ_2	the feedback time delay of predator	–
d_1	the net dispersal rate of prey between Patch 1 and Patch 2	0.2
d_2	the net dispersal rate of prey between Patch 1 and Patch 2	0.3

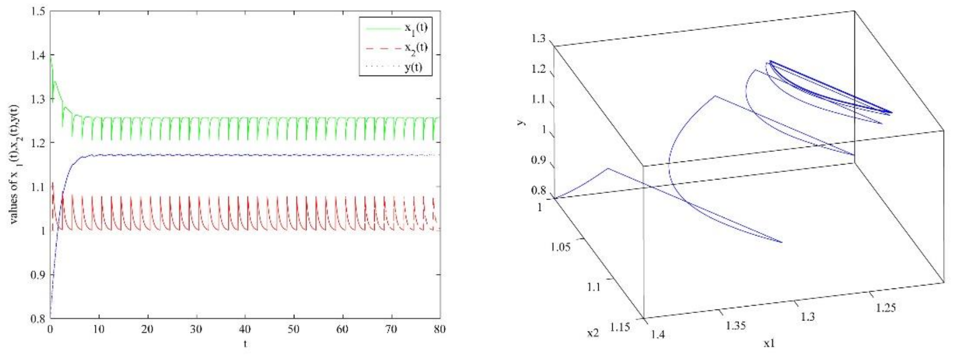
Table 2. Simulations of model (1.3).

Case	**d**	c_2	τ_1	τ_2	x_1	x_2	**y**	**Fig.**
1	3	2	0.1	0.1	Permanent	Permanent	Extinct	Fig. 1
2	1	8	0.1	0.1	Permanent	Permanent	Permanent	Fig. 2
3	1	8	2	2	Permanent	Permanent	Permanent	Fig. 3



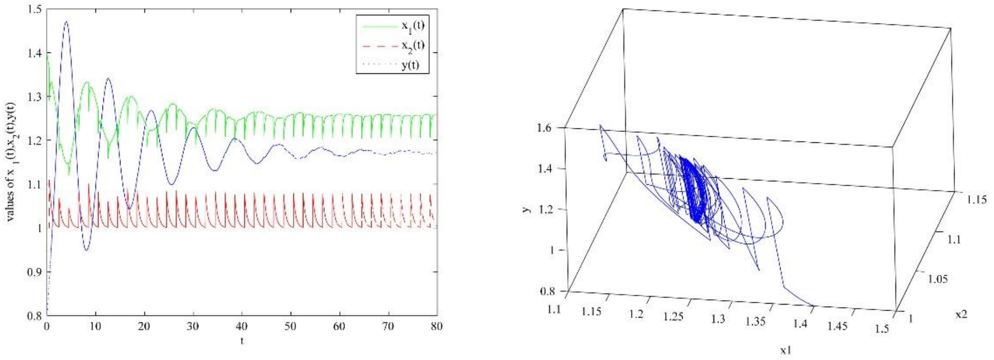
(a) (b)

Fig. 1. Dynamical behavior of system (1.3) with $d = 3$, $c_2 = 2$, $\tau_1 = 0.1$, $\tau_2 = 0.1$, initial values are taken as $(x_1(0), x_2(0), y(0)) = (2,1,0.8), (1.6,1.2,0.6), (1.2,0.8,0.4)$, respectively.
(a) time-series of system (1.3); (b) the phase portrait of system (1.3).



(a) (b)

Fig. 2. Dynamical behavior of system (1.3) with d = 1, c_2= 8, τ_1= 0.1, τ_2 = 0.1, and initial values ($x_1(0)$, $x_2(0)$, y(0)) = (1.4,1,0.8). (a) time-series of system (1.3); (b) the phase portrait of system (1.3).



(a) (b)

Fig. 3. Dynamical behavior of system (1.3) with d = 1, c_2= 8, τ_1= 2, τ_2 = 2, and initial values ($x_1(0)$,$x_2(0)$, y(0))= (1.4,1,0.8). (a) time-series of system (1.3), (b) the phase portrait of system (1.3).

According to Fig. 1 and Fig. 2, we can observe that: Fig. 1 shows that the prey species will be globally attractive and the predator species will be extinct when conditions (*H*1) and (*H*2) hold. Fig. 2 shows that the prey species and the predator species will be permanent when condition (*H*3) holds.

From the case 1 and the case 2 in Table 1, it is found that the parameters *d* and c_2 play an important role in the permanence of system (1.3).

From Fig. 2 and Fig. 3, we can obtain that time delays (τ_1 and τ_2) are harmless for permanence of populations even though time delays (τ_1 and τ_2) have a tendency to produce oscillations.

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