ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 12, No. 4, 2017, *pp*.291-303

Converting Z-number to Fuzzy Number using Fuzzy Expected Value

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(Received March 20, 2017, accepted May 11, 2017)

Abstract. In order to deal with uncertain information of real-world, in 2011 Zadeh suggested the concept of a Z-number, as an ordered pair of fuzzy numbers ($\tilde{A}$,$\tilde{B}$) that describes the restriction and the reliability of the evaluation. Due to the limitation of its basic properties, converting Z-number to classical fuzzy number is rather significant for application. In this paper, we will calculate fuzzy expected value of a Z-number with assuming uniform distribution and linear membership functions. This fuzzy expected value can be used instead of Z-number in applications. An Example is used to illustrate the procedure of the proposed approach.

Keywords: Z-number; Fuzzy Expected value; Fuzzy Set

1. Introduction

In the real world, decisions are based on information usually uncertain, imprecise and/or incomplete, such information is often characterized by fuzziness, but it is not sufficient and the other important property of information is that information must be reliable. Thus, fuzziness from the one side and reliability form the other side are strongly associated to each other. In order to take into account this fact, in 2011 Zadeh proposed a concept, namely Z-number, which is an order pair of fuzzy numbers ($\tilde{A}$, $\tilde{B}$). The first component, $\tilde{A}$, is a fuzzy restriction and the second component $\tilde{B}$ is a reliability of the first component. Typically, $\tilde{A}$ and $\tilde{B}$ are described in a natural language for example (about 25 minutes, very sure) [1, 2, 3].

In 2012, Yager used Z-number to provide a simple illustration of a Z-valuation (X, $\tilde{A}$, $\tilde{B}$). He showed that these Z-valuations essentially induce a possibility distribution G(p) over probability distributions associated with an uncertain variable X and used this representation to make decisions and answer questions. He suggested manipulation and combination of multiple Z-valuations. He showed the relationship between Z-numbers and linguistic summaries and provided for a representation of Z-valuations in terms of Dempster–Shafer belief structures, which made use of type-2 fuzzy sets [4,5].Yager found a fuzzy expected value of a Z-number whose probability density function is expressible in terms of an exponential distribution.

Kang et al. [6] proposed a simple method for converting Z-numbers with range [0, 1] to the classical fuzzy numbers in 3 step: first convert the second part (reliability) into a crisp number and then calculate the weighted Z-number by adding the weight of the second part (reliability) to the first part (restriction) and finally converting the weighted fuzzy number to normal fuzzy number. The benefit of the proposed method is represented by its low analytical and computational complexity, hence their theorem for converting Z-number to classical fuzzy number was used in some papers [7,8,9,10].

Gardashova [11] suggested an algorithm of decision making method using Z-numbers, in 5 steps: Construction of the fuzzy decision making matrix, transforming the linguistic value to numerical value, normalizing the fuzzy decision making matrix, Converting the Z-numbers to crisp number and Determining the priority weight of each alternative. He used a simple way of the canonical representation of multiplication operation on triangular fuzzy numbers [12] for the converting the Z-numbers to crisp number.

In this study we find a fuzzy expected value of a Z-number whose probability density function is expressible in terms of a uniform distribution. The result can be used in many applications to converting Z-number to fuzzy number.

The remainder of the paper is organized as follows: In Section 2 we discuss the concept of Z-number, Z-number with the probability based on uniform distribution and the probability of a restriction ($\tilde{A}$) on the values of uncertain variable (X) based on the parameters of uniform distribution. Section 3 explains the method of calculating the maximum probability of $\tilde{A}$. Later in section 4, the membership function of the possibility of the expected value of Z-number (μ) is proposed. Finally, the whole presented material is summed up in section 5.

2. The concept of Z-numbers

Any Z-number is an ordered pair of fuzzy numbers $(\tilde{A},\tilde{B})$ where $\tilde{A}$ is a restriction on the values of uncer tain variable, X, and $\tilde{B}$ is the level of confidence in $\tilde{A}$. Both $\tilde{A}$ and $\tilde{B}$ are fuzzy sets. Zadeh[1] defines Z-valuat ion as an ordered triple, (X,$\tilde{A}$,$\tilde{B}$), which is equivalent to " X is $(\tilde{A},\tilde{B})$". Moreover, Z-valuation can be defined as

$$Prob(X \ \ is \ A\tilde{}) \ is \ B\tilde{} \tag{1}$$

According to the Zadeh's equation, we express the probability that X is $\tilde{A}$ as:

$$Prob_p(X \ is \ \tilde{A}) = \int_R \ A(x)f(x)dx \tag{2}$$

where *A(x)* is the membership function of Ã and f(x) is the probability density function of X on R. Now, we can get *G(p)*, the degree to which p satisfies our Z-valuation, 〖Prob〗 _p (X is A˜) is B, as:

$$G(p) = B\left(Prob_p(X \ is \ A)\right) = B(\int_R \ A(x)f(x)dx) \tag{3}$$

Since the probability density function of X is unknown, according to Zadeh's simplifying assumption, a particular set of parametric distributions (e.g. normal, exponential and uniform distributions) can be applied based on available knowledge about the variable [1]. But when the knowledge available is not sufficient, the uniform distribution is proposed for a poorly known variable [13]. This distribution is sometimes referred to as the "no knowledge" distribution. Hence, we used uniform distribution in the current study.

Let *(A˜,B˜)* be a Z-number,where Ã is a triangular number, (*c-l* , *c* , *c+r*), with a membership function as follows:

$$A(x) = \begin{cases} 0 & if \quad x < c-l \\ (x-c+l)/l & if \quad c-l \le x < c \\ (c+r-x)/r & if \quad c \le x \le c+r \\ 0 & if \quad x > c+r \end{cases} \tag{4}$$

On the other hand, B˜, confidence, could be from the set {"Likely", "Usually", "Sure"} are modelled using the right hand fuzzy sets as follows (see Fig. 1)

$$B_L(P) = \begin{cases} 0 & P < 0.5 \\ \frac{P-0.5}{0.1} & 0.5 \le P \le 0.6 \\ 1 & 0.6 < P \end{cases} \tag{5}$$

$$B_U(P) = \begin{cases} 0 & P < 0.65 \\ \frac{P-0.65}{0.1} & 0.65 \le P \le 0.75 \\ 1 & 0.75 < P \end{cases} \tag{6}$$

$$B_S(P) = \begin{cases} 0 & P < 0.8 \\ \frac{P-0.8}{0.1} & 0.8 \le P \le 0.9 \\ 1 & 0.9 < P \end{cases} \tag{7}$$

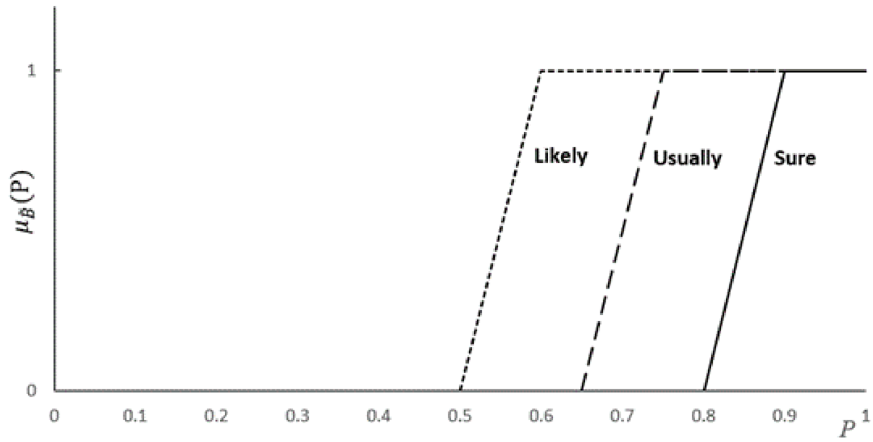

Fig 1. Fuzzy sets of linguistic reliability values.

Assuming that the probability density function, f(x), is uniform as:

$$f(x) = \frac{1}{b-a}, a \le x \le b. \tag{8}$$

Therefore, based on equation (2), the probability of Ã is:

$$P = \int_a^b f(x)A(x)dx = \frac{1}{b-a}\int_{max(a,c-l)}^{min(b,c+r)} A(x)dx \quad (9)$$

Now, having equation (9), P (the probability of Ã) can be determined based on the positions of a and b relative to the fuzzy number Ã (Table 1).

Table 1: The probability of Ã based on the positions of a and b relative to Ã

Category	**positions of a and b**	**Probability (Pi)**
1	$a \le c-l$, $b > c+r$	$P_1 = \frac{l+r}{2(b-a)}$
2	$a \le c-l$, $c \le b \le c+r$	$P_2 = \frac{l}{2(b-a)} + \frac{(b-c)(2r+c-b)}{2(b-a)r}$
3	$a < c-l$, $b < c$	$P_3 = \frac{(b-c+l)^2}{2(b-a)l}$
4	$c-l < a < c$, $c \le b < c+r$	$P_4 = \frac{(c-a)(2l+a-c)}{2(b-a)l} + \frac{(b-c)(2r+c-b)}{2(b-a)r}$
5	$c-l \le a \le c$, $b < c$	$P_5 = \frac{a+b-2c+2l}{2l}$
6	$c \le a \le c+r$, $c < b < c+r$	$P_6 = \frac{2r+2c-a-b}{2r}$
7	$c-l < a < c$, $c+r \le b$	$P_7 = \frac{(c-a)(2l+a-c)}{2l(b-a)} + \frac{r}{2(b-a)}$
8	$c \le a \le c+r$, $c+l \le b$	$P_8 = \frac{(r+c-a)^2}{2r(b-a)}$

3. Expected value of Z-number

We recall for the uniform distribution (4) that

$$E(x) = \mu = \frac{a+b}{2}. \quad (10)$$

As discussed earlier, the parameters of the density function (*a* and *b*) are unknown and μ, the expected value can be determined based on the given Z-number.

Assuming a particular μ will lead to various pairs of *(a,b)* with the mean value of μ which can then be used to calculate probability of $\tilde{A}$ according to *Table 1*.

Example 1. Let a Z-number be *((24,30,36), Sure)* which means fuzzy number ($\tilde{A}$) and confidence ($\tilde{B}$) are *(24,30,36)* and *"Sure"* respectively. By assuming uniform distribution and $\mu=29$, *Fig. 2* depicts change in the probability of $\tilde{A}$ for various pairs of *(a,b)*with the mean value of *29*.

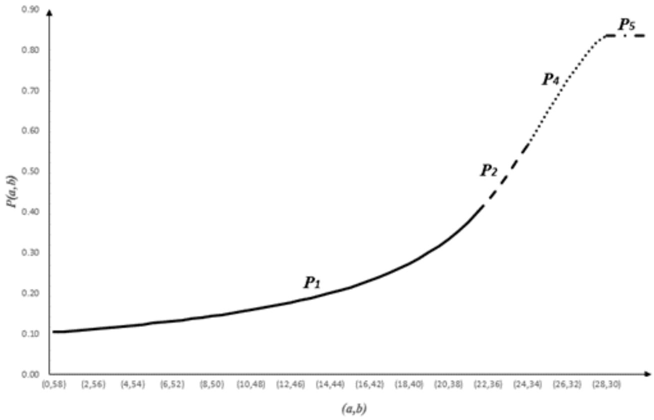

Fig 2. Changes in probability for μ = 29

Fuzzy membership degrees of each *(a,b)* can be computed based on the membership function of confidence, $\tilde{B}$, in the Z-number. Since the level of confidence was set at *"Sure"*, membership values of ordered pairs *(a,b)*, with the mean value of 29, was determined based on (7). Then an appropriate *S*-norm, here maximum, can be used to obtain the membership function *G* over the space $\mu \in R$:

$$G(\mu) = \max_{(a,b)}\{B\big(P(a,b)\big)\big|a+b=\mu\}, \mu \in R, \tag{11}$$

Since higher probability of *(a,b)* increases their membership degree in the confidence function, the (11) can be rewritten as:

$$G(\mu) = B(\max_{(a,b)}\{P(a,b)|a+b=\mu\}), \mu \in R \tag{12}$$

Then we can first find the maximum probability of Ã with the particular μ and then compute its possibility by using equation (12). For example, In Example 1, the maximum probability of $\tilde{A}$ with μ = 29 is 0.83. According to equation (7), the membership value of this event in the *"Sure"* is 0.3.

In order to find the maximum probability of Ã for different values of μ, we evaluated it in four intervals (*Table 2*).

Table 2- Four intervals of μ to find the maximum PROBABILITY of Ã

State	Interval
1	$\mu \le c - \frac{l}{2} \to a+b \le 2c-l$
2	$c - l/2 < \mu \le c \to 2c-l < a+b \le 2c$
3	$c < \mu \le c + r/2 \to 2c < a+b \le 2c+r$
4	$c + \frac{r}{2} < \mu \to 2c+r < a+b$

If *a* and *b* fall in any of the eight categories summarized in Table 1, for particular μs, the maximum probability of Ã in the above-mentioned Intervals will be as follows:

3.1 First Interval ($\mu \le c - \frac{l}{2}$)

If *a* and *b* falling in the first category of *Table 1*, the maximum probability of $\tilde{A}$ for $\mu \le c - \frac{l}{2}$ is obtained as follows (Fig. 3):

$$\begin{cases} a \le c-l \\ b > c+r \end{cases} \to P_1 = \frac{l+r}{2(b-a)} = \frac{l+r}{4(b-\mu)} \tag{13}$$

$$\frac{dP_1}{db} = \frac{-(l+r)}{4(b-\mu)^2} \to \frac{dP_1}{db} < 0 \to maxP_1 = \frac{l+r}{4(c+r-\mu)} \tag{14}$$

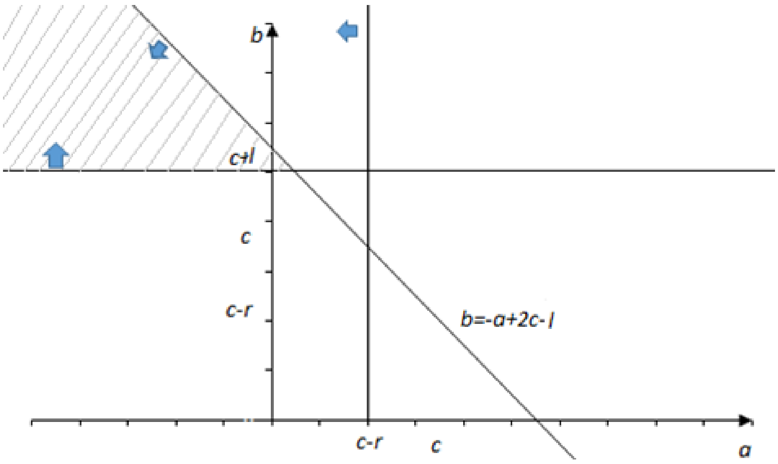

Fig 3. The area involved in calculation of the maximum probability (P1) when $\mu \le c - l/2$

When *a* and *b* belong to the second category of *Table 1*, the maximum probability of $\tilde{A}$ with $\mu \le c - \frac{l}{2}$ is obtained as:

$$\begin{cases} a \le c - l \\ c \le b \le c + r \end{cases} \rightarrow P_2 = \frac{lr + (b-c)(2r+c-b)}{2(b-a)r} \quad (15)$$

$$\frac{dP_2}{db} = 0 \rightarrow b^* = \sqrt{(c+r-\mu)^2 - lr - r^2} + \mu \quad (16)$$

In other words, if no restriction is imposed, the maximum of P_2 based on *b* will occur at a point like b^*.

Theorem 1. If $\mu \le c - \frac{l}{2}$, then $b^* \ge c$

Proof by contradiction: If $b^* < c$, then

$$b^* < c \rightarrow \sqrt{(c+r-\mu)^2 - lr - r^2} + \mu < c \rightarrow (c+r-\mu)^2 - lr - r^2 < (c-\mu)^2 \rightarrow c - \frac{l}{2} < \mu$$

Which is a contradiction since we had an initial assumption of $\mu \le c - \frac{l}{2}$. Therefore, $b^* \ge c$ is correct.

Now, the maximum of P_2 based on *b* and the position of b^* can be determined as (Fig. 4):

$$maxP_2 = \begin{cases} \frac{lr + (b^*-c)(2r+c-b^*)}{4(b^*-\mu)r} & c < b^* < c + r \\ \frac{l+r}{2(c+r-\mu)} & c + r \le b^* \end{cases} \quad (17)$$

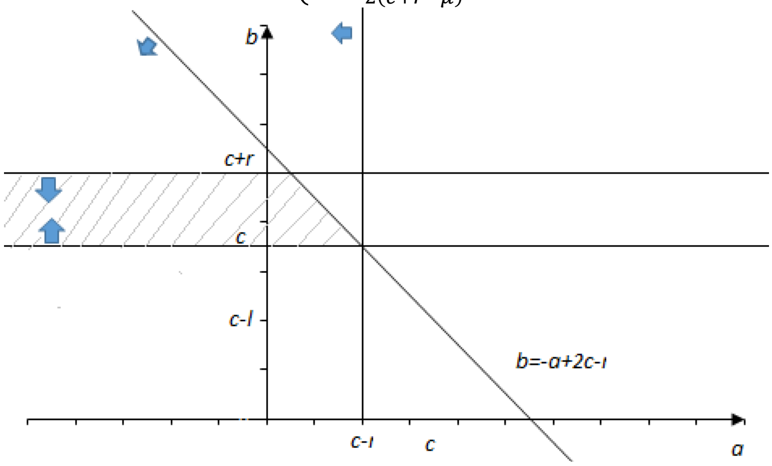

Fig 4.The area involved in calculation of the maximum probability (P2) with $\mu \le c - l/2$

If *a* and *b* fit in the third category of *Table 1*, the maximum probability of $\tilde{A}$ with $\mu \le c - \frac{l}{2}$ can be calculated as (Fig. 5):

$$\begin{cases} a < c - l \\ b < c \end{cases} \rightarrow P_3 = \frac{(b-c+l)^2}{4(b-\mu)l}$$

$$\frac{dP_3}{db} = \frac{(-a+c-l)(b-c+l)}{4l(b-\mu)^2} \xrightarrow[\substack{a<c-l\to(-a+c-l)>0 \\ b>c-l\to(b-c+l)>0}]{} \frac{dP_3}{db} > 0 \to$$

$$maxP_3 = \frac{(c-c+l)^2}{4(c-\mu)l} = \frac{l}{4(c-\mu)} \quad (18)$$

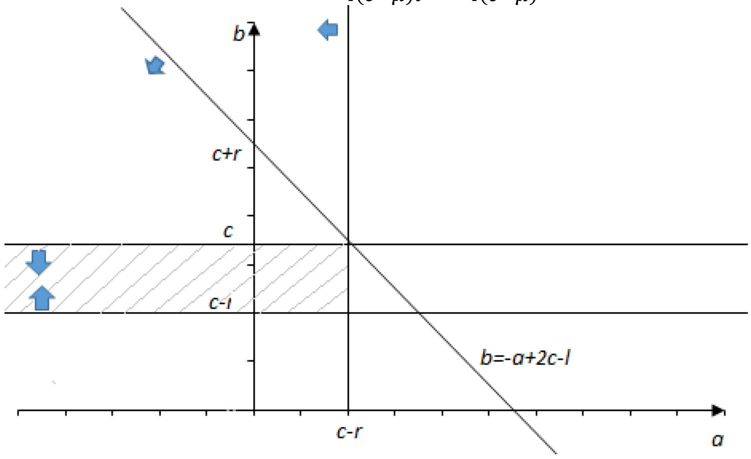

Fig 5. The area involved in calculation of the maximum probability (P3) when μ ≤ c - $l/2$

With a and b lying in the fourth category of Table 1, the maximum probability of $\tilde{A}$ with $\mu \le c$ - $\frac{l}{2}$ can be computed as:

$$\begin{cases} c-l<a<c \\ c \le b < c+r \end{cases} \to 2c-l<a+b<2c+r \to c-l/2<\mu \to P_4 = 0 \quad (19)$$

The maximum probability of $\tilde{A}$ with $\mu \le c$ - $\frac{l}{2}$ when a and b belong to the fifth category of *Table 1* is determined as (Fig. 6):

$$\begin{cases} c-l \le a \le c \\ b<c \end{cases} \to P_5 = \frac{a+b-2c+2l}{2l} = \frac{\mu-c+l}{l} \xrightarrow[P_5=cte]{} maxP_5 = \frac{\mu-c+l}{l} \quad (20)$$

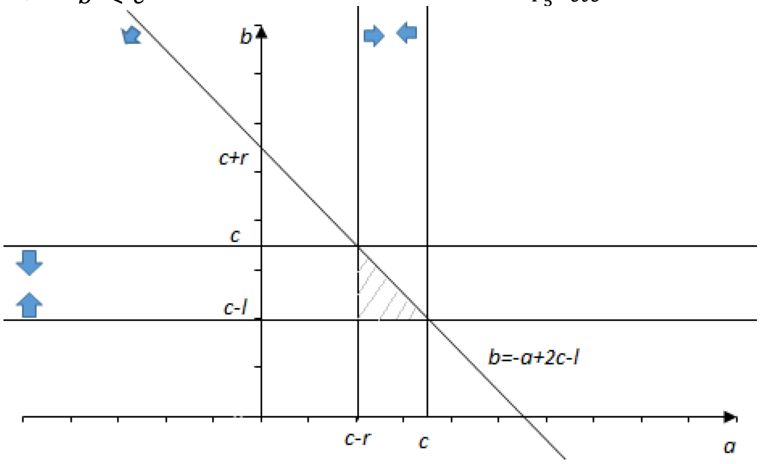

Fig 6. The area involved in calculation of the maximum probability (P5) when μ ≤ c - $l/2$

The maximum probability of $\tilde{A}$ with $\mu \le c$ - $\frac{l}{2}$ when a and b fall into the sixth-eighth categories of *Table 1* can be defined as:

$$\begin{cases} c-r<a<c \\ c+r \le b \end{cases} \to 2c < a+b \xrightarrow[a+b\le 2c-r]{} P_7 = 0 \quad (21)$$

$$\begin{cases} c \le a \le c+r \\ c+r \le b \end{cases} \to 2c+r < a+b \xrightarrow[a+b\le 2c-r]{} P_8 = 0 \quad (22)$$

Therefore, according to equations (14),0, 错误!未找到引用源。, and (20), when $\mu \le c$ - $\frac{l}{2}$, the maximum probability of $\tilde{A}$ can be stated as:

$$maxP(\mu) = max\{maxP_1, maxP_2, maxP_3, maxP_5\} \quad (23)$$

And based on 0:

$$maxP(\mu) = \begin{cases} max\left\{\frac{l+r}{4(c+r-\mu)}, \frac{l}{4(c-\mu)}, \frac{\mu-c+l}{l}\right\} & c+r \le b^* \\ max\left\{\frac{lr+(b^*-c)(2r+c-b^*)}{4(b^*-\mu)r}, \frac{\mu-c+l}{l}\right\} & c < b^* < c+r \end{cases} \quad (24)$$

If $c + r \le b^*$, then:

$$\frac{l}{4(c-\mu)} - \frac{\mu-c+l}{l} = \frac{l^2-4(c-\mu)(\mu-c+l)}{4l(c-\mu)} = \frac{\left(l-2(c-\mu)\right)^2}{4l(c-\mu)} > 0 \to \frac{l}{4(c-\mu)} > \frac{\mu-c+l}{l} \quad (25)$$

and

$$\frac{l}{4(c-\mu)} - \frac{l+r}{4(c+r-\mu)} = \frac{r}{4} \times \frac{(l-c+\mu)}{(c+r-\mu)(c-\mu)}$$

Now, two cases may occur:

$$① \ \mu \le c-l \to \begin{cases} -c+r+\mu \le 0 \\ c+r-\mu > 0 \\ c-\mu > 0 \end{cases} \to \frac{l}{4(c-\mu)} < \frac{l+r}{4(c+r-\mu)}$$

$$② \ c-l < \mu \le c-l/2 \to \begin{cases} -c+l+\mu > 0 \\ c+r-\mu > 0 \\ c-\mu < 0 \end{cases} \to \frac{l}{4(c-\mu)} < \frac{l+r}{4(c+r-\mu)}$$

Hence, in both cases, the following is true:

$$\frac{l}{4(c-\mu)} < \frac{l+r}{4(c+r-\mu)} \quad (26)$$

Based on (26)-(27), it can be concluded that:

$$max\left\{\frac{l+r}{4(c+r-\mu)}, \frac{l}{4(c-\mu)}, \frac{\mu-c+l}{l}\right\} = \frac{l+r}{4(c+r-\mu)} \quad (27)$$

Or:

$$\begin{cases} \mu \le c-l/2 \\ c+r \le b^* \end{cases} \to maxP(\mu) = maxP_2 = maxP_1 = \frac{l+r}{4(c+r-\mu)} \quad (28)$$

If $c < b^* < c + r$, then equation (17) entails that:

$$\frac{lr+(b^*-c)(2r+c-b^*)}{4(b^*-\mu)r} > \frac{\mu-c+l}{l}$$

So

$$\begin{cases} \mu \le c-\frac{l}{2} \\ c < b^* < c+r \end{cases} \to maxP(\mu) = maxP_2 = \frac{lr+(b^*-c)(2r+c-b^*)}{4(b^*-\mu)r} \quad (29)$$

Finally, under the first condition, $\mu \le c - \frac{l}{2}$, equations (27) and (29) indicate that:

$$\mu \le c-\frac{l}{2} \to maxP(\mu) = maxP_2 \quad (30)$$

3.2 Second Interval ($c - \frac{l}{2} < \mu \le c$)

According to equation (14), the maximum probability of $\tilde{A}$ *with* $c - \frac{l}{2} < \mu \le c$ when a and b belong to the first category of *Table 1* is (Fig. 7):

$$\begin{cases} a \le c-l \\ b > c+r \end{cases} \to maxP_1 = \frac{l+r}{4(c+r-\mu)} \quad (31)$$

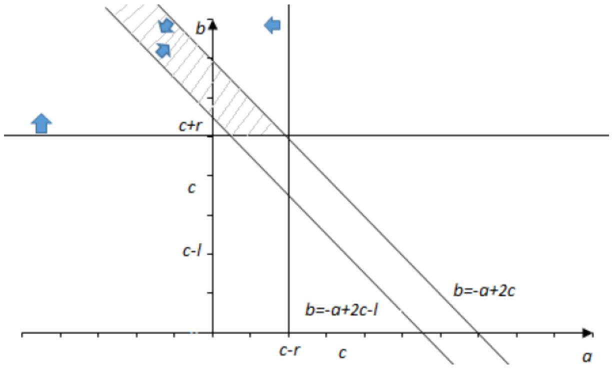

Fig 7. The area involved in calculation of the maximum probability (P1) when c - $l/2 < \mu \le$ c

With a and b fitting into the second category of Table 1, the maximum probability of $\tilde{A}$ *with* $c - \frac{l}{2} < \mu \le c$ can be obtained based on equations (15) and (17):

$$\begin{cases} a \le c - l \\ c \le b \le c + r \end{cases} \rightarrow P_2 = \frac{lr + 2br + 2bc - b^2 - 2cr - c^2}{4(b-\mu)r} \tag{32}$$

$$\frac{dP_2}{db} = 0 \rightarrow b^* = \sqrt{(c + r - \mu)^2 - lr - r^2} + \mu \tag{33}$$

Which leads to three distinct conditions:

$$\begin{cases} b^* \le c \rightarrow \frac{dP_2}{db} < 0 \rightarrow maxP_2 = \frac{l}{4(c-\mu)} \\ c < b^* < c + r \rightarrow maxP_2 = \frac{lr + (b^* - c)(2r + c - b^*)}{4(b^* - \mu)r} \\ c + r \le b^* \rightarrow \frac{dP_2}{db} > 0 \rightarrow maxP_2 = \frac{l + r}{4(c + r - \mu)} \end{cases} \tag{34}$$

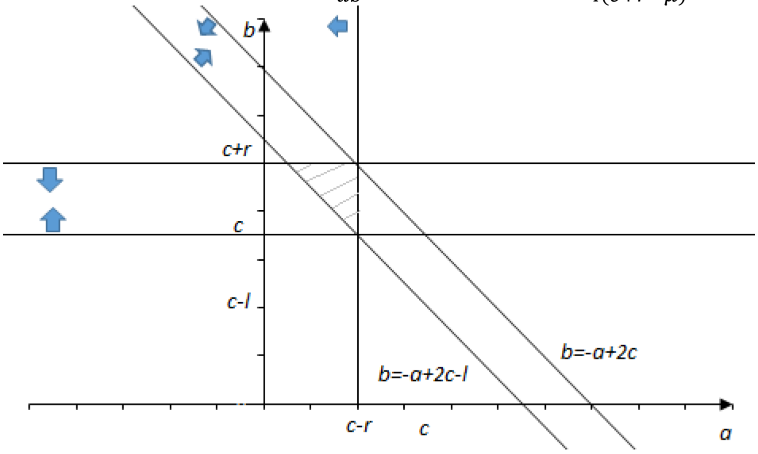

Fig 8. The area involved in calculation of the maximum probability (P2) when c - $l/2 < \mu \le$ c

When *a* and *b* fall into the third category of *Table 1*, the maximum probability of $\tilde{A}$ with $c - \frac{l}{2} < \mu \le c$ can be determined as:

$$\begin{cases} a < c - l \\ b < c \end{cases} \rightarrow a + b < 2c - l \rightarrow \mu < c - l/2 \rightarrow maxP_3 = 0 \tag{35}$$

The maximum probability of $\tilde{A}$ with $c - \frac{l}{2} < \mu \le c$ when *a* and *b* belong to the fourth category of Table 1 will be:

$$\begin{cases} c - l < a < c \\ c \le b < c + r \end{cases} \rightarrow P_4 = \frac{(c-a)(2l + a - c)}{4(b-\mu)l} + \frac{(b-c)(2r + c - b)}{4(b-\mu)r} \tag{36}$$

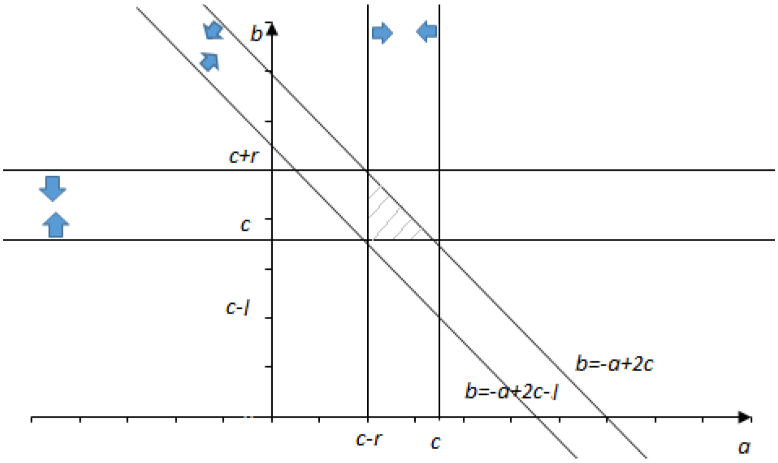

Fig 9. The area involved in calculation of the maximum probability (P4) when c - $l/2 < \mu \le$ c

With *a* and *b* lying in the fifth category of *Table 1*, the maximum probability of $\tilde{A}$ *with* $c - \frac{l}{2} < \mu \le c$ can be calculated as:

$$\begin{cases} c - l \le a \le c \\ b < c \end{cases} \rightarrow maxP_5 = \frac{\mu - c + l}{l} \tag{37}$$

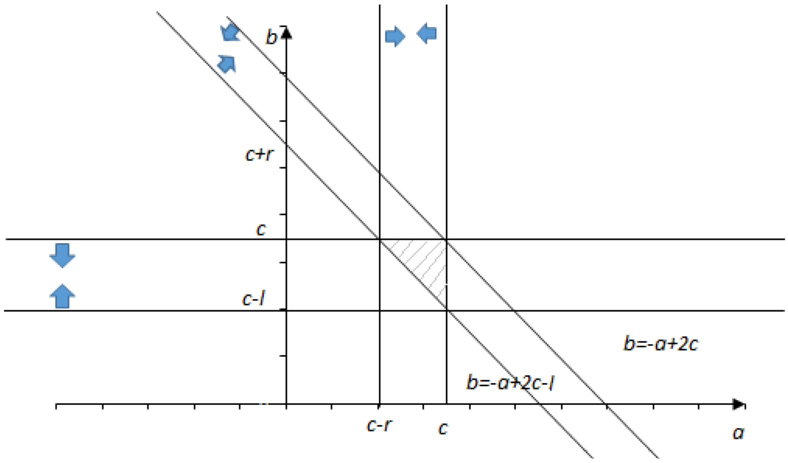

Fig 10. The area involved in calculation of the maximum probability (P5) when c - $l/2$< μ ≤ c

Theorem 2. P5> P4 is always true.

Proof:

$$P_5 - P_4 = \frac{r(b-c)^2 + l(b-c)^2}{4(b-\mu)lr} \geq 0 \rightarrow P_5 \geq P_4$$

Finally, the maximum probability of $\tilde{A}$ with $c - \frac{l}{2} < \mu \leq c$ when a and b belong to the sixth, seventh and eighth categories of *Table 1* can be stated as:

$$\begin{cases} c \leq a \leq c+r \\ c < b < c+r \end{cases} \rightarrow 2c < a+b \xRightarrow[a+b \leq 2c]{} P_6 = 0 \quad (38)$$

$$\begin{cases} c-r < a < c \\ c+r \leq b \end{cases} \rightarrow 2c < a+b \xRightarrow[a+b \leq 2c]{} P_7 = 0 \quad (39)$$

$$\begin{cases} c \leq a \leq c+r \\ c+r \leq b \end{cases} \rightarrow 2c+r < a+b \xRightarrow[a+b \leq 2c]{} P_8 = 0 \quad (40)$$

According to the equation (34):

$$maxP_2 \geq maxP_1 \quad (41)$$

And based on *Theorem 2*:

$$maxP_5 \geq maxP_4 \quad (42)$$

Therefore, when $c - \frac{l}{2} < \mu \leq c$, equations (41) and (42) lead to the following equation:

$$maxP(\mu) = max\{maxP_2, maxP_5\} \quad (43)$$

Theorem 3. If c - $\frac{l}{2}$<μ ≤ c, then P5>P2 is always true.

Proof:

$$P_2 - P_5 = \frac{-4r(\mu - a - l/2)(\mu - c + l/2) - l(b-c)^2}{2(b-a)rl} \xRightarrow[\begin{cases} c-l/2<\mu\leq c \rightarrow c-l<\mu-l/2 \\ a<c-l \end{cases} \rightarrow \begin{cases} 0<\mu-c+l/2 \\ 0<\mu-\frac{l}{2}-a \end{cases}]{} P_2 - P_5 < 0$$

$$\rightarrow P_5 > P_2$$

Based on Theorem 3 if $c - \frac{l}{2} < \mu \leq c$, then $maxP_5 > maxP_2$ is always true.

Hence, when c - $\frac{l}{2}$< μ ≤ c, it can be concluded that:

$$c - \frac{l}{2} < \mu \leq c \rightarrow maxP(\mu) = maxP_5 \quad (44)$$

3.3 Third Interval ($c < \mu \leq c + r/2$)

With calculations Similar to second interval, we have:

$$c < \mu \leq c + \frac{r}{2} \rightarrow maxP(\mu) = maxP_6 = \frac{r+c-\mu}{r} \quad (45)$$

3.4 Forth Interval ($c + r/2 < \mu$)

With calculations Similar to first interval, we have:

$$c + \frac{r}{2} < \mu \rightarrow maxP(\mu) = maxP_7 \quad (46)$$

Maximum of P_7 can be computed as:

$$maxP_7 = \begin{cases} \frac{lr + (c - a^*)(2l + a^* - c)}{-4l(\mu - a^*)} & c - l < a^* < c \\ \frac{l + r}{4(c - l - \mu)} & a^* < c - l \end{cases}$$

Where

$$a^* = \sqrt{(\mu - c + l)^2 - lr - l^2} + \mu$$

Now, according to the equations (30) and (44)-(46), it can be inferred that:

$$maxP(\mu) = \begin{cases} maxP_2 & \mu \leq c - l/2 \\ maxP_5 & c - l/2 < \mu \leq c \\ maxP_6 & c < \mu \leq c + r/2 \\ maxP_7 & c + r/2 < \mu \end{cases} \quad (47)$$

Theorem 4. Minimum of $maxP_5$ equals 0.5.

Proof:

$$maxP_5 = \frac{\mu-c+l}{l} \rightarrow \frac{dmaxP_5}{d\mu} > 0 \xrightarrow[\mu=c-\frac{l}{2}]{} \frac{c-\frac{l}{2}-c+l}{l} = 0.5 \quad (48)$$

Theorem 5. Minimum of $maxP_6$ equals 0.5.

Proof: Similar to Theorem 4.

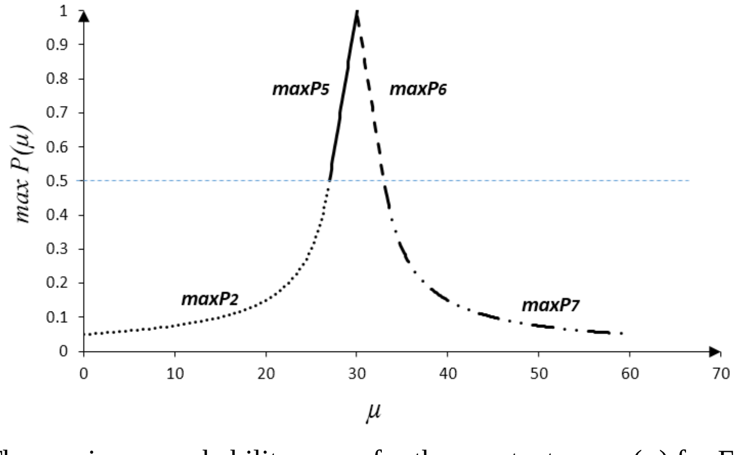

Fig 11. The maximum probability curve for the constant mean (μ) for Example 1

After calculating the maximum probability of $\tilde{A}$, the membership function of the μ at three levels of the possibility (L, U, and S) can be computed using equations (5)-(7).

4. Calculating the membership function of the expected value of Z-number (μ)

By replacing probability values (P), obtained from equation (47), in relevant equation of (5)-(7), the membership function of the μ can be determined.

$$G_{\tilde{\mu}_L}(p) = \begin{cases} 0 & p < c - 0.5l \\ \frac{p-c+0.5l}{0.1l} & c - 0.5l \le p < c - 0.4l \\ 1 & c - 0.4l \le p < c + 0.4r \\ \frac{c+0.5r-p}{0.1r} & c + 0.4r \le p < c + 0.5r \\ 0 & c + 0.5r \le p \end{cases} \tag{49}$$

$$G_{\tilde{\mu}_U}(p) = \begin{cases} 0 & p < c - 0.35l \\ \frac{p-c+0.35l}{0.1l} & c - 0.35l \le p < c - 0.25l \\ 1 & c - 0.25l \le p < c + 0.25r \\ \frac{c+0.35r-p}{0.1r} & c + 0.25r \le p < c + 0.35r \\ 0 & c + 0.35r \le p \end{cases} \tag{50}$$

$$G_{\tilde{\mu}_S}(p) = \begin{cases} 0 & p < c - 0.2l \\ \frac{p-c+0.2l}{0.1l} & c - 0.2l \le p < c - 0.1l \\ 1 & c - 0.1l \le p < c + 0.1r \\ \frac{c+0.2r-p}{0.1r} & c + 0.1r \le p < c + 0.2r \\ 0 & c + 0.2r \le p \end{cases} \tag{51}$$

Thus, the μ in L, U, and S levels of the confidence are trapezoidal fuzzy numbers shown as:

$$\tilde{\mu}_L = (c - 0.5l, c - 0.4l, c + 0.4r, c + 0.5r) \tag{52}$$

$$\tilde{\mu}_U = (c - 0.35l, c - 0.25l, c + 0.25r, c + 0.35r) \tag{53}$$

$$\tilde{\mu}_S = (c - 0.2l, c - 0.1l, c + 0.1r, c + 0.2r) \tag{54}$$

This result for *Example1* is

$$G_{\tilde{\mu}}(p) = \begin{cases} 0 & p < 28.8 \\ \frac{p-28.8}{0.6} & 28.8 \le p < 29.4 \\ 1 & 29.4 \le p < 30.6 \\ \frac{31.2-p}{0.6} & 30.6 \le p < 31.2 \\ 0 & 31.2 \le p \end{cases} \tag{55}$$

Or the fuzzy expected value is: $\tilde{\mu} = (28.8\,, 29.4\,, 30.6\,, 31.2)$.

We can compare this result with fuzzy expectation proposed by Kang et al [6]. In their method first, the reliability of Z-number ($\tilde{B}$) is converted to crisp number with the center of gravity method.

$$\alpha = \frac{\int x\mu_S(x)dx}{\int \mu_S(x)dx} = 0.922$$

Then the weighted Z-number is obtained by adding the α to the first part (Ã). (Ã , α) =((24,30,36);0.922).

Finally, weighted Z-number is converted to regular fuzzy number:

$$\tilde{\mu} = \left(\sqrt{0.922} * 24, \sqrt{0.922} * 30, \sqrt{0.922} * 36\right) = (28.8, 5.76, 5.76)$$

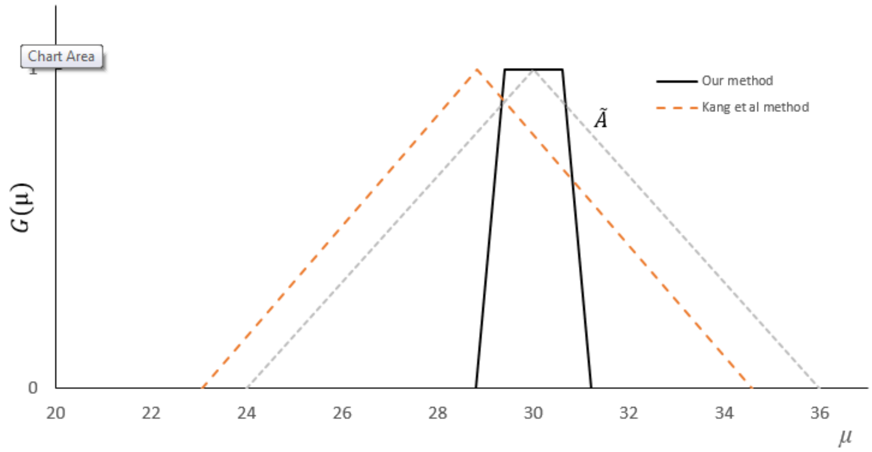

Fig 12.The membership function of the possibility ofμ

As you can see in equations (52)-(54), in this approach the more reliability results smaller support set of the obtained fuzzy expected value. Also the center of the first part of Z-number ($\tilde{A}$) doesn't change in this approach, however using the Kang et al [6] method, if the first part of Z-number ($\tilde{A}$) is positive, the more reliability results the smaller fuzzy expectation, without any reason.

5. Conclusion

The main contribution of the present paper is converting a Z-number to a fuzzy number which is very significant for application. Hence the probability density function of uncertain variable, X, in Z-valuation (X, $\tilde{A}$, $\tilde{B}$) is unknown, according to Zadeh's simplifying assumption, a particular set of parametric distributions can be applied based on available knowledge about the variable. But when the knowledge available is not sufficient, the uniform distribution is proposed for a poorly known variable. Therefore, in practice, most applications use linear membership functions to describe fuzzy sets, in this study linear right-hand membership functions are used for reliability values. With above assumptions, that is proved that the fuzzy expected value of a Z-number is a trapezoidal fuzzy number.

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