ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 13, No. 4, 2018, pp.261-268

Hopf bifurcation analysis in a predator-prey model with square root response function with two time delays

Miao Peng [1] and Zhengdi Zhang [1+]

[1] Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu, P.R. China, 212013.

Email address: dyzhang@ujs.edu.cn, pengmiao199101@126.com.

(Received September 01 2018, accepted November 24 2018)

Abstract. In this paper, we investigate the local stability and Hopf bifurcation analysis in a predator-prey model with square root response function and two time delays. By choosing the two delays as the bifurcation parameter and by analyzing the corresponding characteristic equations, the conditions for the stability and existence of Hopf bifurcation for the system are obtained. Finally, the corresponding numerical simulations are carried out to support the theoretical analysis.

Keywords: Hopf bifurcation, predator-prey model, square root response function, time delay

1. Introduction

The interaction between the predator and the prey is the fundamental structure in population dynamics. Predator-prey models have been widely researched in both ecology and mathematical ecology[1-7]. There has been a great and continuing interest in predator–prey models with time delay, functional response, etc. The effect of these factors has explained more richer and complicated dynamics.

Time delay is a natural phenomenon which exists universally and unavoidably in nature. In biological systems, such as the gestation process of species, the migration process of species, the digestion and transformation process of species after capture of prey and its own mature time, etc, they all belong to time delay phenomena[8-12]. In [8], Song et al. presented a predator–prey system with stage structure and time delay for the prey. In [10], Peng and Zhang considered a delayed predator-prey system with Holling type III functional response incorporating a prey refuge and selective harvesting. In [12], Hao et al. studied a diffusive single species model with stage structure and strong Allee effect subject to homogeneous Neumann boundary condition. From the discussions of above references, we found that time delay destroyed the stability of the system at the equilibrium point and caused the periodic fluctuations of the species, which can lead to various forms of bifurcation behavior or chaotic movement in the related systems. Therefore, in order to maintain the ecological balance, it is meaningful to fully study the factor of time delay.

In the predator-prey model, one of the important factors affecting population dynamics is functional response, which reflects the predatory ability of predators. In [13], Salmanet al. researched a discrete predator-prey system with square root functional response, they derived the flip and Niemark-Sacker bifurcations. In [14], Braza analyzed a predator-prey model with square root functional responses in which a modified Lotka–Volterra interaction term is used as the functional response of the predator to the prey. Based on the above considerations, in this paper, we shall investigate the dynamic analysis in a predator-prey model with two time delays and square root functional response.

The remainder of this paper is organized as follows. In Section 2, we present a general description for the predator-prey model. In Section 3, the local stability and the existence of Hopf bifurcation at the positive equilibrium are discussed. In Section 4, we perform some numerical simulations which are revealed to illustrate the validity of the theoretical results. Finally, making a brief conclusion in Section 5.

2. The delayed predator-prey model with square root response function

In [14], Braza proposed a predator–prey model with square root functional responses:

$$\dot{x}(t) = x(t) - x^2(t) - \sqrt{x(t)}y(t),$$
$$\dot{y}(t) = -sy(t) + c\sqrt{x(t)}y(t),$$

(1)

where $x(t)$ and $y(t)$ can be described as the population densities of prey and predator at time t, s denotes the death rate of the predator, and c is the biomass conversion or consumption rate. He studied the dynamics of the square root system and compared with the dynamics of predator–prey systems that used a typical Lotka–Volterra interaction term.

However, the factor about time delay often appeared in actual situation. In this paper, motivated by Braza [14]and Zhu et al. [15], we introduce two time delays and square root functional responses into the following predator-prey model:

$$\dot{x}(t) = x(t) - x(t)x(t-\tau_1) - \sqrt{x(t)}y(t),$$
$$\dot{y}(t) = -sy(t) + c\sqrt{x(t-\tau_2)}y(t), \tag{2}$$

where the parameters $x(t), y(t), c, s$ are defined in system (1). τ_1 is the time delay due to the gestation of prey, and τ_2 is the feedback delay,that is to say, the predator takes the time to convert the food into its growth.

3. Local stability and Hopf bifurcation analysis

In this section, we shall discuss the stability of system (2) at the positive equilibrium and the existence of Hopf bifurcation by analyzing the corresponding linearized system.

It is obvious that system (2) has a unique positive equilibrium $E(x^*, y^*)$, where

$$x^* = \frac{s^2}{c^2},\ y^* = \frac{sc^2 - s^3}{c^3},$$

if the following condition (H1) $c > s$ satisfies.

Let $\bar{x}(t) = x(t) - x^*, \bar{y}(t) = y(t) - y^*$ and represent $\bar{x}(t), \bar{y}(t)$ by $x(t), y(t)$, respectively. Using Taylor expansion to expand the system (2) at the posotive equilibrium $E(x^*, y^*)$, then we can get the linearized system of system (2) as follows:

$$\dot{x}(t) = a_1 x(t) + a_2 x(t-\tau_1) + a_3 y(t),$$
$$\dot{y}(t) = b_2 x(t-\tau_2) + b_3 y(t),$$

(3)

where

$$a_1 = 1 - x^* - \frac{y^*}{2\sqrt{x^*}},\ a_2 = -x^*,\ a_3 = -\sqrt{x^*},$$

$$b_2 = \frac{cy^*}{2\sqrt{x^*}},\ b_3 = -s + c\sqrt{x^*}.$$

The corresponding characteristic equation of system (3) is given by

$$\lambda^2 - (a_1 + b_3)\lambda + a_2(b_3 - \lambda)e^{-\lambda\tau_1} - a_3 b_2 e^{-\lambda\tau_2} + a_1 b_3 = 0.$$

(4)

In order to investigate the distribution of roots of the transcendental equation (4), the result of Ruan and Wei [16] is introduced here.

Lemma 1 For the transcendental equation

$$p(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m}) = \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)}$$
$$+ [p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}]e^{-\lambda\tau_1} + \cdots$$
$$+ [p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}]e^{-\lambda\tau_m} = 0.$$

As $(\tau_1, \tau_2, \tau_3, \ldots, \tau_m)$ vary, the sum of orders of the zeros of $p(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m})$ in the open right half plane can change, and only a zero appears on or crosses the imaginary axis.

Due to the time delay τ_1 and τ_2 are considered as the bifurcation parameters, next, we will discuss the following three cases.

Case 1: $\tau_1 = \tau_2 = 0$. The characteristic equation (4) reduces to

$$\lambda^2 - (a_1 + a_2 + b_3)\lambda + a_1 b_3 + a_2 b_3 - a_3 b_2 = 0 . \tag{5}$$

According to the Routh-Hurwitz criteria, a set of necessary and sufficient conditions for all roots of Eq. (5) to have a negative real part are obtained, we have

(H2) $a_1 + a_2 + b_3 < 0$,

then the equilibrium point $E(x^*, y^*)$ is locally asymptotically stable when the condition (H2) holds.

Case 2: $\tau_1 > 0$, $\tau_2 = 0$. Eq. (4) becomes

$$\lambda^2 - m_1\lambda + m_2 + (m_3 - m_4\lambda)e^{-\lambda\tau_1} = 0 , \tag{6}$$

where $m_1 = a_1 + b_3, m_2 = a_1 b_3 - a_3 b_2, m_3 = a_2 b_3, m_4 = a_2$.

Now we substitute $i\omega_1$ ($\omega_1 > 0$) into Eq. (6), separating the real and imaginary parts gives

$$\begin{aligned} -\omega_1{}^2 + m_2 + m_3 \cos(\omega_1\tau) - m_4\omega_1 \sin(\omega_1\tau) = 0, \\ -m_1\omega_1 - m_3 \sin(\omega_1\tau) - m_4\omega_1 \cos(\omega_1\tau) = 0, \end{aligned} \tag{7}$$

which leads to

$$\omega_1^4 + (m_1^2 - 2m_2 - m_4^2)\omega_1^2 + m_2^2 - m_3^2 = 0 . \tag{8}$$

It is easy to see that if the condition

(H3) $m_1^2 - 2m_2 - m_4^2 > 0$, $m_2^2 - m_3^2 > 0$,

satisfy, then Eq. (8) has no positive roots. Hence, all roots of Eq. (6) have negative real parts when $\tau_1 \in [0, \infty)$ under the conditions (H2) and (H3). Further, if (H2) and

(H4) $m_1^2 - 2m_2 - m_4^2 > 0$, $m_2^2 - m_3^2 < 0$,

hold, then Eq. (6) has positive roots. Without loss of generality, we assume that it has two positive roots, which are denoted by ω_{1k}, $k = 1,2$.

By Eq. (7), we have $\left\{\tau_{1k}^j \middle| k = 1,2; j = 0,1,2,\cdots\right\}$. Thus, $\pm\omega_{1k}$ is a pair of purely imaginary roots of Eq. (6) with $\tau_1 = \tau_{1k}^{(j)}$, and let $\tau_{10} = \min\left\{\tau_{1k}^j \middle| k = 1,2; j = 0,1,2,\cdots\right\}$, $\omega_{10} = \omega_{1k_0}$.

Let $\lambda(\tau) = \zeta(\tau) + i\omega(\tau)$ be the root of Eq. (6) near $\tau = \tau_{10}$ satisfying $\zeta(\tau_0) = 0$, $\omega(\tau_{10}) = \omega_{10}$. According to the Hopf bifurcation theorem [17], we will verify the transversality condition in the following.

Lemma 2 Suppose that

(H5): $$\frac{\sin(\omega_{10}\tau_{10})(-2\omega_{10}^2 m_4 - m_1 m_3) + \omega_{10}\cos(\omega_{10}\tau_{10})(2m_3 - m_1 m_4)}{m_3\omega_{10} + m_4^2\omega_{10}^3} + \frac{1}{\omega_{10}^2} \neq 0$$

holds, then the following transversality condition is satisfied:

$$\left\{\frac{d(\mathrm{Re}\,\lambda)}{d\tau_1}\right\}_{\lambda = i\omega_{10}} \neq 0 .$$

Proof Taking the derivative of λ with respect to τ_1 in Eq. (6), we obtain

$$(\frac{d\lambda}{d\tau_1})^{-1} = \frac{2\lambda - m_1}{(m_3 - m_4\lambda)\lambda e^{-\lambda\tau_1}} - \frac{m_4}{\lambda(m_3 - m_4\lambda)} - \frac{\tau_1}{\lambda},$$

(9)

which implies

$$\mathrm{Re}(\frac{d\lambda}{d\tau_1})^{-1} = \mathrm{Re}(\frac{2\lambda - m_1}{(m_3 - m_4\lambda)\lambda e^{-\lambda\tau_1}}) - \mathrm{Re}(\frac{m_4}{\lambda(m_3 - m_4\lambda)})$$

$$= \frac{\sin(\omega_{10}\tau_{10})(-2\omega_{10}^2 m_4 - m_1 m_3) + \omega_{10}\cos(\omega_{10}\tau_{10})(2m_3 - m_1 m_4)}{m_3\omega_{10} + m_4^2\omega_{10}^3} + \frac{1}{\omega_{10}^2} \neq 0.$$

Noting that $\left\{\frac{d(\mathrm{Re}\lambda)}{d\tau_1}\right\}_{\lambda=i\omega_{10}}$ and $\left\{\mathrm{Re}(\frac{d\lambda}{d\tau_1})^{-1}\right\}_{\lambda=i\omega_{10}}$ have the same sign, then

$$sign\left\{\frac{d(\mathrm{Re}\lambda)}{d\tau_1}\right\}_{\lambda=i\omega_{10}} = sign\left\{\mathrm{Re}(\frac{d\lambda}{d\tau_1})^{-1}\right\}_{\lambda=i\omega_{10}} =$$

$$sign(\frac{\sin(\omega_{10}\tau_{10})(-2\omega_{10}^2 m_4 - m_1 m_3) + \omega_{10}\cos(\omega_{10}\tau_{10})(2m_3 - m_1 m_4)}{m_3\omega_{10} + m_4^2\omega_{10}^3} + \frac{1}{\omega_{10}^2}) \neq 0.$$

(10)

Therefore, $\left\{\frac{d(\mathrm{Re}\lambda)}{d\tau_1}\right\}_{\lambda=i\omega_{10}} \neq 0$ if (H5) holds. The proof is complete.

According to the analysis above, we have the following results.

Theorem 1 For $\tau_2 = 0$, assume that (H1) and (H2) are satisfied, then the following conclusions hold:

(i) If (H3) holds, then the positive equilibrium $E^*(x^*, y^*)$ of the system (2) is asymptotically stable for all $\tau_1 \geq 0$.

(ii) If (H4) and (H5) hold, then the positive equilibrium $E^*(x^*, y^*)$ of the system (2) is asymptotically stable for all $\tau_1 \in [0, \tau_{10})$ and unstable for $\tau_1 > \tau_{10}$. Furthermore, the system (2) undergoes a Hopf bifurcation at the positive equilibrium $E^*(x^*, y^*)$ when $\tau_1 = \tau_{10}$.

Case 3: $\tau_1 \in [0, \tau_{10})$, $\tau_2 > 0$ and $\tau_1 \neq \tau_2$.

We consider Eq. (4) with τ_1 in its stable interval, and τ_2 is regarded as the parameter. The corresponding characteristic equation is written as

$$\lambda^2 - n_1\lambda + (n_2 - n_3\lambda)e^{-\lambda\tau_1} - n_4 e^{-\lambda\tau_2} + n_5 = 0,$$

(11)

where $n_1 = a_1 + b_3, n_2 = a_2 b_3, n_3 = a_2, n_4 = a_3 b_2, n_5 = a_1 b_3.$

Let $i\omega_2(\omega_2 > 0)$ be the root of Eq. (11), it follows that

$$-\omega_2{}^2 + n_2\cos(\omega_2\tau_1) - n_3\omega_2\sin(\omega_2\tau_1) + n_5 = n_4\cos(\omega_2\tau_2),$$
$$-n_1\omega_2 - n_2\sin(\omega_2\tau_1) - n_3\omega_2\cos(\omega_2\tau_1) = -n_4\sin(\omega_2\tau_2),$$

(12)

and ω_2 should satisfies

$$\omega_2^4 + 2n_3\sin(\omega_2\tau_1)\omega_2{}^3 + [(-2n_2 + 2n_1 n_3)\cos(\omega_2\tau_1) + n_1^2 + n_3^2 - 2n_5]\omega_2{}^2 +$$
$$(-2n_1 n_2 - 2n_3 n_5)\sin(\omega_2\tau_1)\omega_2 + n_2^2 - n_4^2 + n_5^2 + 2n_2 n_5\cos(\omega_2\tau_1) = 0.$$

(13)

In order to give the main results, we provide the following assumption.

(H6) Eq. (13) has at least finite positive root.

We denote the positive roots of Eq. (13) as $\omega_2^{(1)}, \omega_2^{(2)}, \omega_2^{(3)}$ and $\omega_2^{(4)}$. For every $\omega_2^{(i)} (i = 1,2,3,4)$, the corresponding critical value of time delay $\tau_{2i}^{(j)}$, $j = 1,2,3\ldots$ is

$$\tau_{2i}^{(j)} = \frac{1}{\omega_2} \arccos \left\{ \frac{-\omega_2^2 + n_2 \cos(\omega_2 \tau_1) - n_3 \omega_2 \sin(\omega_2 \tau_1) + n_5}{n_4} + 2\pi j \right\}_{\omega_2 = \omega_2^i}$$

$$i = 1,2,3,4;\ j = 0,1,2\ldots. \quad (14)$$

Let $\tau_{20} = \min\left\{\tau_{2i}^{(j)} \middle| \, i = 1,2,3,4;\ j = 0,1,2\ldots\right\}$, ω_{20} is the corresponding root of Eq. (11) with τ_{20}. In the following, we assume that

(H7) $\left\{ \frac{d(\mathrm{Re}\,\lambda)}{d\tau_2} \right\}_{\lambda = i\omega_{20}} \neq 0$.

We have the following theorem.

Theorem 2 For system (2), $\tau_2 > 0, \tau_1 \in [0, \tau_{10})$ and $\tau_1 \neq \tau_2$. Suppose that the conditions (H6) and (H7) hold, then the positive equilibrium $E^*(x^*, y^*)$ is asymptotically stable for all $\tau_2 \in [0, \tau_{20})$ and unstable for $\tau_2 > \tau_{20}$. Furthermore, the system (1.2) undergoes a Hopf bifurcation at the positive equilibrium $E^*(x^*, y^*)$ when $\tau_2 = \tau_{20}$.

4. Numerical simulation

In this section, we give some numerical simulations by using matlab to explain the analytical results in the above previous section.

Let $s = 0.35, c = 0.55$, then we obtain the following particular example of system (2):

$$\dot{x}(t) = x(t) - x(t)x(t - \tau_1) - \sqrt{x(t)}\,y(t),$$
$$\dot{y}(t) = -0.35 y(t) + 0.55\sqrt{x(t - \tau_2)}\,y(t),$$

(15)

It is not difficult to verify that the condition (H1) holds, we obtain the positive equilibrium $E^*(0.4050, 0.3787)$.

For $\tau_1 > 0, \tau_2 = 0$, we obtain $\omega_{10} \approx 0.4881, \tau_{10} \approx 1.5277$. From Theorem 1, we know that the positive equilibrium E^* is asymptotically stable when $\tau_1 \in [0, \tau_{10})$, when the time delay τ_1 passes through the critical value τ_{10}, the positive equilibrium E^* will lose its stability and a Hopf bifurcation occurs, and a family of periodic solutions bifurcate from the positive equilibrium E^*. The corresponding waveform and the phase plots are shown in Fig. 1 and Fig. 2.

For $\tau_1 = 0.8 \in [0, \tau_{10})$, $\tau_2 > 0$, we have $\omega_{20} = 0.3817$, $\tau_{20} = 0.8065$. According to Theorem 2, E^* is asymptotically stable when $\tau_2 \in [0, \tau_{20})$ and unstable when $\tau_2 > \tau_{20}$, which is illustrated in Fig.3 and Fig.4.

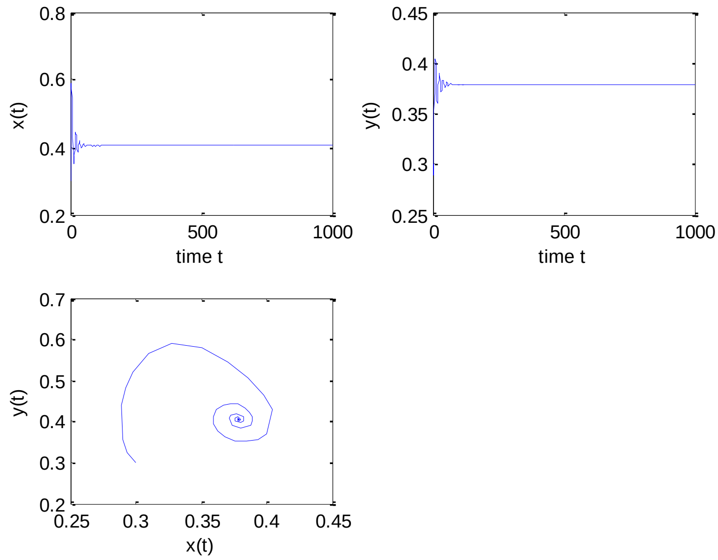

Fig. 1: When $\tau_2 = 0$, E^* is asymptotically stable for $\tau_1 = 0.95 < \tau_{10} = 1.5277$.

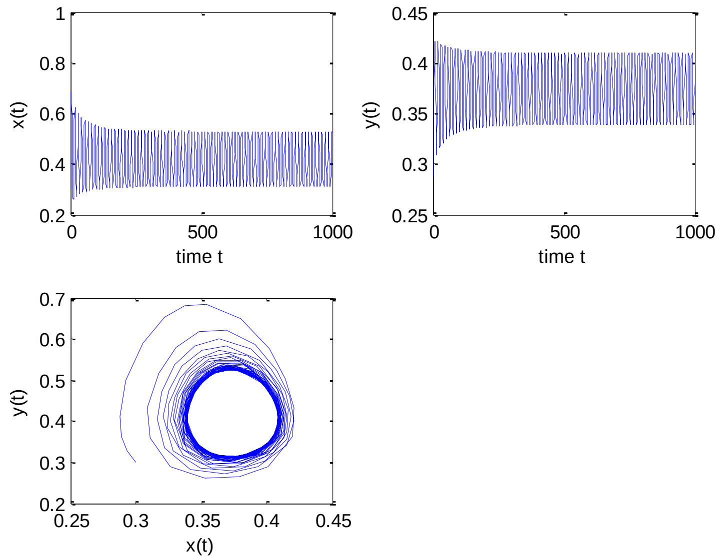

Fig. 2: When $\tau_2 = 0$, E^* undergoes a Hopf bifurcation for $\tau_1 = 1.55 > \tau_{10} = 1.5277$

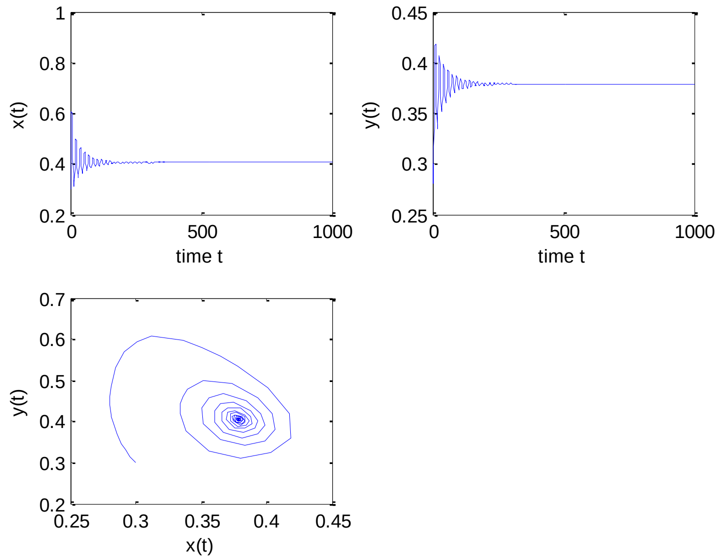

Fig. 3: E^* is asymptotically stable for $\tau_2 = 0.55 < \tau_{20} = 0.8065$ and $\tau_1 = 0.8$.

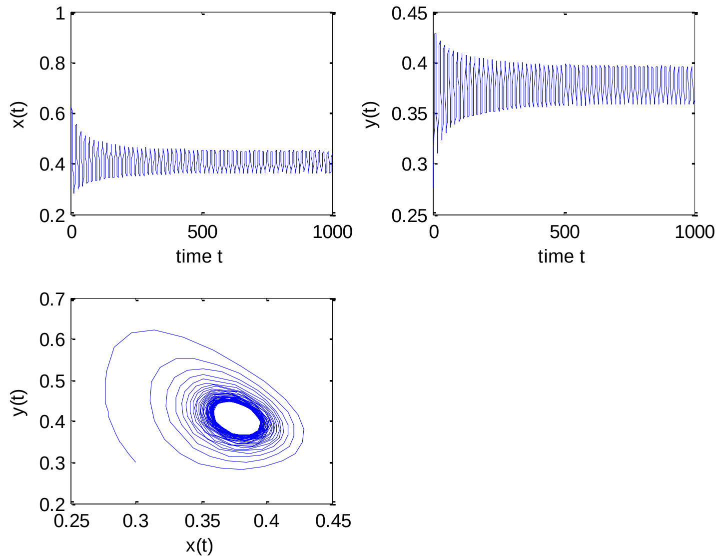

Fig. 4: E^* undergoes a Hopf bifurcation for $\tau_2 = 0.85 > \tau_{20} = 0.8065$ and $\tau_1 = 0.8$.

5. Conclusion

In this paper, we have investigated the Hopf bifurcation analysis in a delayed predator-prey model with square root response function. By setting a same group of parameter values, according to the existing two time delays and discussing three different cases, we know that the positive equilibrium will lose its originally stability and a Hopf bifurcation occurs, and a family of periodic solutions bifurcate E^* when the time delay pass though some critical values. The results of our numerical simulations are in accordance with the theoretical analysis. Through this study, We further found that the time delay is able to cause a periodic evolution of the prey and predator populations and alter the dynamics of system (2) significantly. This lays a foundation for next research on the control problem about ecological models.

6. Acknowledgements

The authors would like to thank the editors and referees for the careful reading of the manuscript and valuable suggestions. This work was supported by National Natural Science Foundation of China (No. 11872189, 11472116) and Postgraduate Research & Practice Innovation Program of Jiangsu Province (KYCX17_1784) .

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