ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 14, No. 1, 2019, *pp*.003-009

Projective Synchronization of a Hyperchaotic Lorenz System

Li Xin[1] , Xuerong Shi[2*], Mingjie Xu[3]
[1]School of Information Engineering, Yancheng Teachers University,
Yancheng, 224002, China
[2]School of Mathematics and Statistics, Yancheng Teachers University,
Yancheng, 224002, China, E-mail: sxryctc@163.com (Xuerong Shi) .
[3]School of Electronic Information, Jiangsu University of Science and Technology,
Zhenjiang, 212000, China
(Received November 26, 2018, accepted December 29, 2018)

Abstract: In this paper, the dynamical behaviors and projective synchronization of a five-dimensional hyperchaotic Lorenz system are investigated. First of all, a hyperchaotic system is constructed by introducing two state variables into the Lorenz chaotic system. Secondly, the dynamical behaviors of the proposed system, such as the dissipative property and equilibrium point, are discussed. Thirdly, based on the stability theory, the projective synchronization of the systems can be achieved. Finally, some numerical simulations are given to verify the projective synchronization scheme.

Keywords: Lorenz system, Hyperchaotic, Projective synchronization

1 Introduction

Chaos is a very interesting nonlinear phenomenon. In 1963, Lorenz discovered the famous Lorenz chaotic system [1]. After that, chaotic systems have been researched extensively, such as the Lü system [2-4], the Chen system [5] and the Rössler system [6]. Recently, much work has been done in constructing hyperchaotic models [7-9]. However, there is no universal method to get hyperchaotic systems. Compared with chaotic systems, hyperchaotic systems must have at least two positive Lyapunov exponents, and the dimension must be four or more [10]. Hyperchaotic systems can be obtained by adding one or more state variables to a three-dimensional chaotic system [11, 12]. Hyperchaotic systems have more abundant dynamical characteristics and complex behaviors than chaotic systems [13, 14]. So they are better suitable for some engineering applications, such as chemical reactions, electric circuits [15], cryptography [16, 17], fluid dynamics and secure communication [18-20].

Chaos synchronization is another fascinating concept. Pecora and Carroll proposed a drive-response chaotic synchronization scheme in 1990 [21], and realized the synchronization of two chaotic systems in the circuit, which promoted the theoretical study of chaotic synchronization and chaos control. Since then, many effective chaotic synchronization methods have emerged, such as complete synchronization [22, 23], generalized synchronization [24], phase synchronization [25], lag synchronization [26, 27], projective synchronization [28], anticipating synchronization [29] and exponential synchronization [30]. In recent years, the synchronization of chaotic fractional differential systems [31, 32] has attracted more and more attention because of its potential applications in secure communication and control processing [33-35].

The research of projection synchronization has received extensive attention from experts at home and abroad in recent years. Projection synchronization is that under certain conditions, the output of the coupled drive system and the response system state is not only phase locked, but the amplitude of each corresponding state also evolves according to a certain scale factor relationship. The method has been widely observed and discussed in coupled integer order chaotic systems.

The other parts of article is organized as follows. In section 2, a new five-dimensional hyperchaotic Lorenz system is constructed and the dynamical behaviors of the hyperchaotic system are discussed, such as attractor, dissipativity and equilibrium point. In section 3, the projective synchronization scheme of the hyperchaotic system is designed and some numerical simulations are completed. In section 4, some conclusions are given.

2 System description

2.1 A new hyperchaotic Lorenz system

The famous Lorenz chaotic system can be represented by the following autonomous differential equations

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1) \\ \dot{x}_2 = cx_1 - x_2 - x_1x_3 \\ \dot{x}_3 = x_1x_2 - bx_3 \end{cases}, \tag{1}$$

where a, b and c are real constants. When the parameters are chosen as $a = 10$, $b = 8/3$ and $c = 28$, the system (1) is chaotic.

A new five-dimensional system is constructed by adding two variables into Lorenz chaotic system. In the first equation of the system (1), x_4 is introduced and the rate of change of x_4 is $\dot{x}_4 = -x_2x_3 + dx_4$. In the second equation of the system (1), another state x_5 is introduced and the rate of change of x_5 is $\dot{x}_5 = rx_1$. The new five-dimensional system can be described as

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1) + x_4 \\ \dot{x}_2 = cx_1 - x_2 - x_1x_3 + x_5 \\ \dot{x}_3 = x_1x_2 - bx_3 \\ \dot{x}_4 = -x_2x_3 + dx_4 \\ \dot{x}_5 = rx_1 \end{cases}, \tag{2}$$

where a, b, c, d and r are real constants. When the parameters are chosen as $a = 10$, $b = 8/3$, $c = 28$, $d = -6$ and $r = -5$, the system (2) is hyperchaotic. The chaotic attractors of the system (2) are plotted in Fig. 1 with the initial state $(x_1, x_2, x_3, x_4, x_5) = (1,\ 1,\ 1,\ 1,\ 1)$.

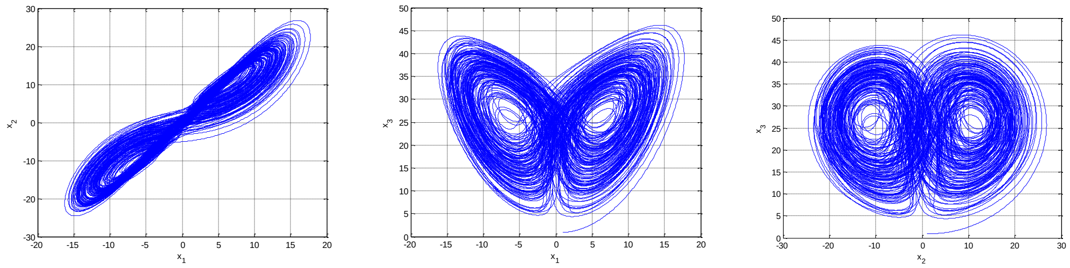

Fig. 1. Chaotic attractors of the system (2) in 2D spaces with $a = 10$, $b = 8/3$, $c = 28$, $d = -6$ and $r = -5$.

2.2 Dissipativity

The divergence of system (2) is calculated as

$$\nabla v = \frac{\partial \dot{x}_1}{\partial x_1} + \frac{\partial \dot{x}_2}{\partial x_2} + \frac{\partial \dot{x}_3}{\partial x_3} + \frac{\partial \dot{x}_4}{\partial x_4} + \frac{\partial \dot{x}_5}{\partial x_5} = -19.667\,. \tag{3}$$

When $\nabla v < 0$, the system (2) is a dissipative system and the exponential shrinkage is -19.667. That is, in the dynamical system (2), when $t \to +\infty$, each volume containing the dynamical system trajectory shrinks to zero at an exponential rate of -19.667. The orbit of the dynamical system is ultimately limited to a specific subset of zero volume, and the asymptotic motion is located on the attractors of the system (2).

2.3 Equilibrium point and stability

Let

$$\begin{cases} a(x_2 - x_1) + x_4 = 0 \\ cx_1 - x_2 - x_1x_3 + x_5 = 0 \\ x_1x_2 - bx_3 = 0 \\ -x_2x_3 + dx_4 = 0 \\ rx_1 = 0 \end{cases}. \tag{4}$$

The only equilibrium point $E_0(0,\ 0,\ 0,\ 0,\ 0)$ of system (2) is available. Then the Jacobian matrix of the system (2) at the equilibrium point E_0 is described as

$$J_0 = \begin{bmatrix} -a & a & 0 & 1 & 0 \\ c - x_3 & -1 & -x_1 & 0 & 1 \\ x_2 & x_1 & -b & 0 & 0 \\ 0 & -x_3 & -x_2 & d & 0 \\ r & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 & 1 & 0 \\ 28 & -1 & 0 & 0 & 1 \\ 0 & 0 & -8/3 & 0 & 0 \\ 0 & 0 & 0 & -6 & 0 \\ -5 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The corresponding eigenvalues of the Jacobian matrix J_0 are $\lambda_1 = 11.704$, $\lambda_2 = 0.1866$, $\lambda_3 = -2.667$, $\lambda_4 = -6$ and $\lambda_5 = -22.890625$, respectively. Here λ_1 and λ_2 are two positive real numbers, λ_3, λ_4 and λ_5 are three negative real numbers. Therefore, the equilibrium point E_0 is a saddle point and unstable.

3 Projective synchronization of the new 5D hyperchaotic system

3.1 Projection synchronization theory of linear separation

Let chaotic system be

$$x(t) = f(x(t), t)\,, \tag{5}$$

where $x(t) \in R^n$ is the n-dimensional state vector of the system, $f: R^n \to R^n$ defines a vector field of n-dimensional vector space. The function $f(x(t), t)$ is decomposed into $f(x(t), t) = g(x(t)) + h(x(t), t)$. Where $g(x(t)) = Ax(t)$ is the linear part of $f(x(t), t)$, A is a constant full rank matrix, and all the real parts of its eigenvalues are negative. So $h(x(t), t) = f(x(t), t) - g(x(t))$ is the non-linear part of $f(x(t), t)$. In this way, the system (5) can be rewritten as

$$x(t) = g(x(t)) + h(x(t), t)\,. \tag{6}$$

Construct a new system

$$y(t) = g(y(t)) + h(x(t), t)/\alpha\,, \tag{7}$$

where $y(t) = R^n$ is the n-dimensional state vector of the system (7). α is a preset synchronization scale factor. The synchronization error between system (6) and system (7) is defined as $e(t) = x(t) - \alpha y(t)$, and its solution is determined by the following equation.

$$\dot{e}(t) = \dot{x}(t) - \alpha\dot{y}(t) = g\big(x(t)\big) - \alpha g\big(y(t)\big) = A\big(x(t) - \alpha y(t)\big) = Ae(t)\,. \tag{8}$$

The zero of $e(t)$ is the equilibrium point of $\dot{e}(t)$, because all the real parts of the eigenvalues of A are negative. According to the stability criterion of linear systems, synchronization errors are asymptotically stable at zero, i.e. $\lim_{t\to+\infty} e(t) = 0$. That is to say that state vector $x(t)$ of system (6) and the state vector $y(t)$ of system (7) achieve projective synchronization according to the given synchronization scale factor α.

3.2 Realization of projection synchronization

Projective synchronization is interesting in view of its proportionality between the synchronized dynamical status. It can be used for digital communication especially in secure communications. In this section, we focus on researching projective synchronization of the new 5D hyperchaotic system (2).

According to the projection synchronization theory of linear separation, the system (2) can be rewritten as

$$f(x_1, x_2, x_3, x_4, x_5) = g(x_1, x_2, x_3, x_4, x_5) + h(x_1, x_2, x_3, x_4, x_5)\,, \tag{9}$$

$$g(x_1, x_2, x_3, x_4, x_5) = \begin{pmatrix} -a & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -b & 0 & 0 \\ 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}, \tag{10}$$

$$h(x_1, x_2, x_3, x_4, x_5) = \begin{pmatrix} ax_2 + x_4 \\ cx_1 - x_1x_3 + x_5 \\ x_1x_2 \\ -x_2x_3 \\ rx_1 + x_5 \end{pmatrix}. \tag{11}$$

Then the drive system can be written as

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ \dot{x}_5\end{pmatrix}=\begin{pmatrix}-a&0&0&0&0\\0&-1&0&0&0\\0&0&-b&0&0\\0&0&0&d&0\\0&0&0&0&-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}+\begin{pmatrix}ax_2+x_4\\cx_1-x_1x_3+x_5\\x_1x_2\\-x_2x_3\\rx_1+x_5\end{pmatrix}, \quad (12)$$

And the response system is defined as

$$\begin{pmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\\ \dot{y}_4\\ \dot{y}_5\end{pmatrix}=\begin{pmatrix}-a&0&0&0&0\\0&-1&0&0&0\\0&0&-b&0&0\\0&0&0&d&0\\0&0&0&0&-1\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\\y_4\\y_5\end{pmatrix}+\frac{1}{\alpha}\begin{pmatrix}ax_2+x_4\\cx_1-x_1x_3+x_5\\x_1x_2\\-x_2x_3\\rx_1+x_5\end{pmatrix}, \quad (13)$$

where α is a projection factor.
Let

$$\begin{cases}e_1(t)=x_1(t)-\alpha y_1(t)\\e_2(t)=x_2(t)-\alpha y_2(t)\\e_3(t)=x_3(t)-\alpha y_3(t)\\e_4(t)=x_4(t)-\alpha y_4(t)\\e_5(t)=x_5(t)-\alpha y_5(t)\end{cases}, \quad (14)$$

and the error system can be obtained as

$$\begin{pmatrix}\dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\\ \dot{e}_4\\ \dot{e}_5\end{pmatrix}=\begin{pmatrix}-a&0&0&0&0\\0&-1&0&0&0\\0&0&-b&0&0\\0&0&0&d&0\\0&0&0&0&-1\end{pmatrix}\begin{pmatrix}e_1\\e_2\\e_3\\e_4\\e_5\end{pmatrix}. \quad (15)$$

Then the solution can be solved as

$$\begin{cases}e_1(t)=m_1e^{-at}\\e_2(t)=m_2e^{-t}\\e_3(t)=m_3e^{-bt}\\e_4(t)=m_4e^{dt}\\e_5(t)=m_5e^{-t}\end{cases}, \quad (16)$$

where m_1, m_2, m_3, m_4 and m_5 are real numbers with the parameter values $a = 10, b = 8/3, c = 28, d = -6$ and $r = -5$. According to the stability criterion of linear system, the synchronization error evolution converges to zero as $t \to +\infty$. Namely, the drive system (12) and the response system (13) will get projective synchronization according to the predetermined synchronization scale factor.

3.3 Numerical simulations

In order to verify the validity of the proposed projective synchronization method, the simulation results have been carried out. In the following numerical simulations, the parameters are always chosen as $a = 10$, $b = 8/3$, $c = 28$, $d = -6$ and $r = -5$. The initial values of the addressed system are set as $(x_1, x_2, x_3, x_4, x_5)$=(1, 2, 3, 4, 5) and $(y_1, y_2, y_3, y_4, y_5)$= $(-1, -2, 3, 2, 1)$. The step size is selected as 0.001. The scale factor is chosen as $\alpha = -2$ and $\alpha = 2$, respectively.

Simulation results of two-dimensional projection synchronization phase diagram ($\alpha = -2$ and $\alpha = 2$) are shown in Fig. 2 and Fig. 3. The three-dimensional projection synchronization phase diagram ($\alpha = -2$ and $\alpha = 2$) is shown in Fig. 4. Reverse synchronization error curve($\alpha = -2$) is shown in Fig. 5. Phase synchronization error curve($\alpha = 2$) is shown in Fig. 6.

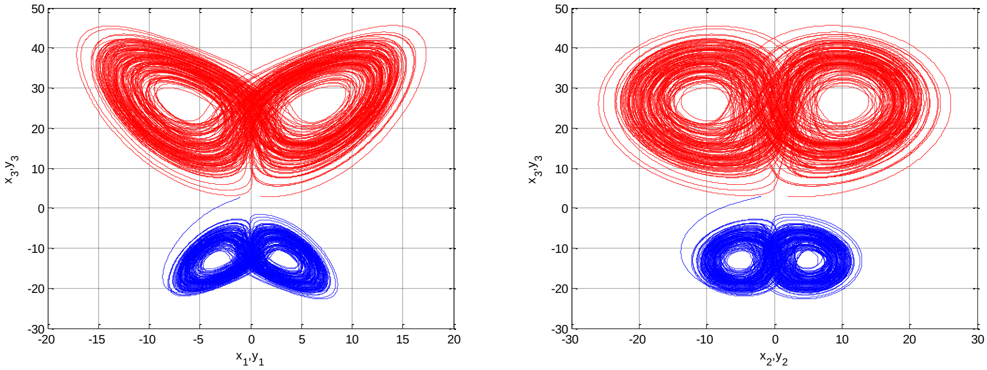

Fig. 2. The phase diagram of two-dimensional reverse projection synchronization ($\alpha = -2$)

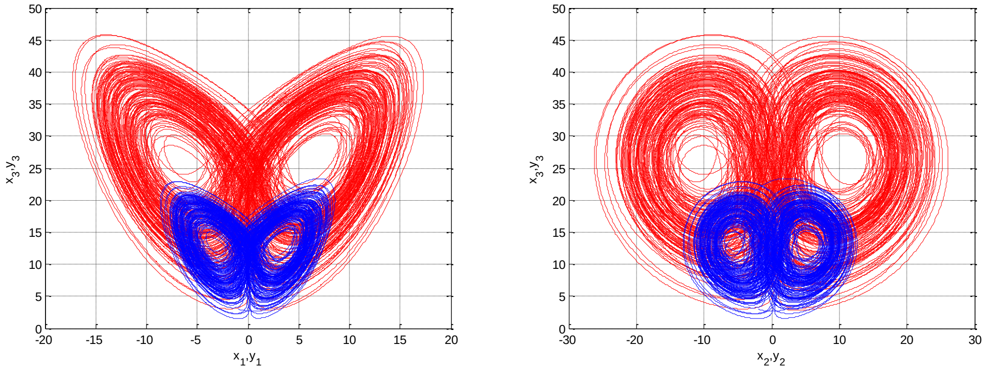

Fig. 3. The phase diagram of two-dimensional projection synchronization($\alpha = 2$)

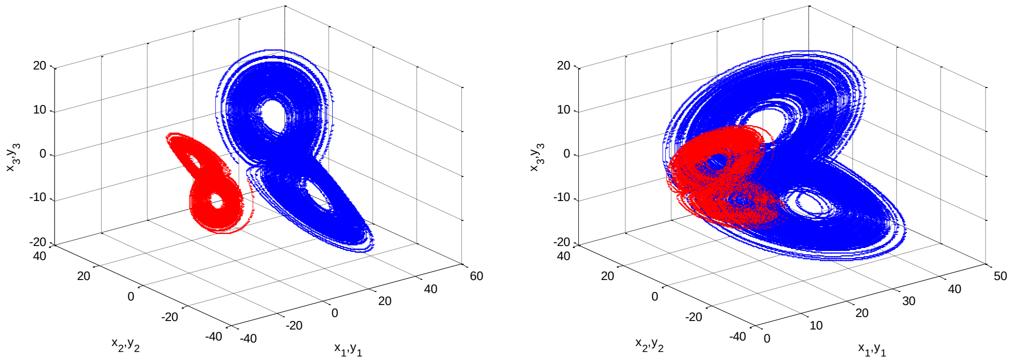

Fig. 4. The three-dimensional projection synchronization phase diagram ($\alpha = -2$ and $\alpha = 2$)

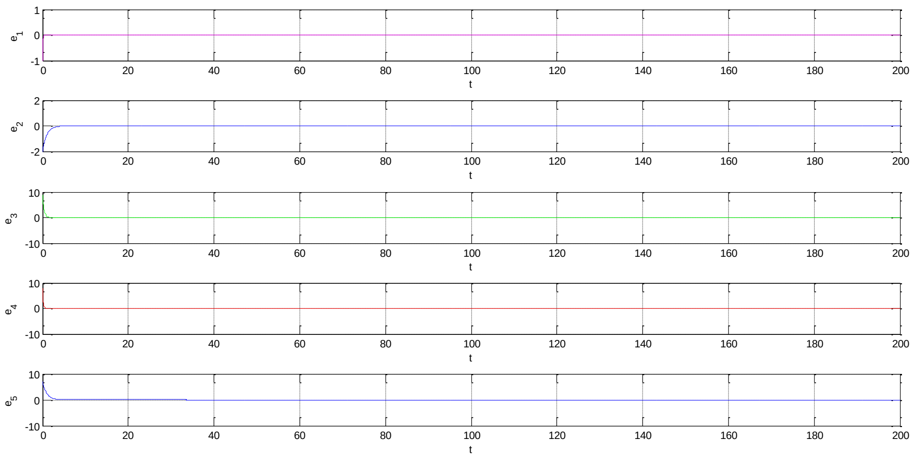

Fig. 5. Reverse synchronization error curve($\alpha = -2$)

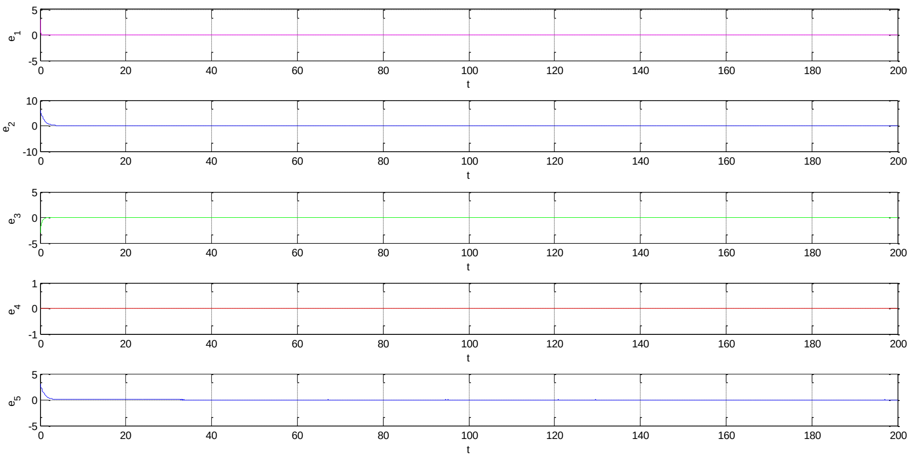

Fig. 6. Phase synchronization error curve($\alpha = 2$)

4 Conclusions

In this paper, a new 5D autonomous hyperchaotic system is presented based on Lorenz chaotic system. The hyperchaotic system has abundant and complex dynamical behaviors. The attractors of the system are shown by some phase diagrams. Furthermore, according to the stability theory, the projective synchronization of the new 5D hyperchaotic system is studied. The proposed projection synchronization scheme is simple and does not require the calculation of the Lyapunov exponent. Detailed numerical simulation results have been given to verify the effectiveness of the projection synchronization scheme.

Acknowledgments

Our work is supported by National Natural Science Foundation of China (Grant Nos. 11872327, 51777180) and the Qing Lan Project of the Jiangsu Higher Education Institutions of China.

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