ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 14, No. 2, 2019, *pp*.103-114

Numerical Approach for Bagley- Torvik Fractional Differential Equations Using Haar Wavelets

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(*Received* February 10, *2019*, *accepted* March 15, *2019*)

Abstract: In this paper we consider Bagley-Torvik fractional differential equations, which are arising in the modeling of motion of rigid plate immersed in a Newtonian fluid. The main attribution of our content is that it transforms the fractional differential equations to a system of algebraic equations without any restrictions and assumptions. Theoretical results are authenticated by five numerical examples of both linear and nonlinear. To demonstrate the accuracy and efficiency of the Haar wavelet collocation method and results are compared with the existing methods.

Keywords: Fractional differential equations, Bagley-Torvik, Haar wavelets, Collocation method.

Mathematics Subject Classifications: 65T60, 65L60, 34A08.

1. Introduction

Many researchers have worked on fractional order Bagley-Torvik equations. Podlubny[1] had explained about fractional differential equations in his book. Diethelm and Ford [2] have presented the numerical solution of the Bagley-Torvik equation. Arvet and Tamme [3] have described the piecewise wise polynomial collocation method [PPCM] to solve Bagley-Torvik linear boundary value problems of fractional order. Arvet and Tamme [4, 5] have used a spline collocation method [SCM]. Jafari et al. have applied the Legendre wavelets [6], Pahdaman et al. have used the optimization technique based on training artificial neural network (ANN) to solve differential equations of fractional order [7].

The applications of fractional calculus in different fields of physics and engineering are namely dynamic of viscoelestic damped stricter [8], continuum and statistical mechanics[9], propagation of spherical flames, self similar protein dynamics[10], fluid dampers[11], bioscience, electromagnetism, signal processing control engineering[12], electrochemistry, diffusion processes, relaxation oscillation model[13], dynamics model of love, nonlinear oscillation of earthquake can be modeled with fractional derivatives[13, 14], one more important thing is a new mathematical concept to the solution of diverse problems in mathematics.

Wavelet is a mathematical tool to solve problems in science and engineering. There are many wavelets exists but Haar is the simplest wavelet in them. The graphical view of Haar scaling function is single block pulse and the mother wavelet of the Haar system is formed by two dilated unit block pulses stand by next to each other, where one of them is inverted. Haar wavelet has properties like compact support, orthogonality and simple applicability. Due to valuable properties and its simplicity Haar wavelets are using to solve problems in signal and image processing, in physics for characterization of Brownian motion, quantum field theory, numerical analysis viz. differential, integral, fractional differential equations. Over the years some researchers have worked on Haar wavelets by using different methods for numerical solutions. Lepik and Hein[15], Haariharan et al.[16], Majak et al.[17], Reddy et al.[18, 19] have given various applications of Haar wavelets with collocation method in the solution of higher order differential, integral and fractional differential equations.

Motion of an immersed Plate:

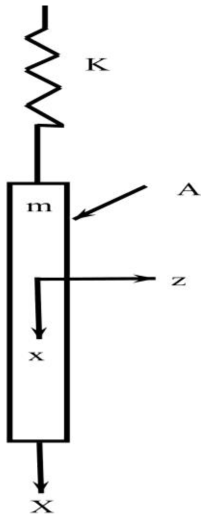

Figure. The immersed plate

Let m be the mass of the rigid plate immersed in a Newtonian fluid which is extended to infinity and connected to a fixed point by a mass less spring whose stiffness is K. Let us suppose that thefluid is not disturbed by the motion of the spring. Let A be the area of the plate which is sufficiently large in order to produce the fluid adjacent to the plate. The fluid velocity and stress are

$$\bar{v}(s,z) = \bar{v}_p(s)e^{\left(\frac{\rho s}{\mu}\right)^{0.5z}} \text{ and } \sigma(t,z) = \sqrt{\mu\rho}D_t^{0.5}[v(t,z)]. \quad (1)$$

Where, $\bar{v}_p(s)$ is the transform of the prescribed velocity of the plate. Net forces acting on the plate of displacement X, is given by

$$m\,\ddot{X} = F_x = -K\,X - 2A\,\sigma(t,0)\,. \quad (2)$$

Using equation (1) and $v_p(t,0) = \dot{X}(t)$, we obtained

$$m\frac{d^2X}{dt^2} + 2A\sqrt{\mu\rho}D^{1.5}{}_{(t)}X + K\,X = 0\,, \quad (3)$$

$$\text{Where, } D^{1.5}X = D^{0.5}\frac{dX}{dt} = \frac{d}{dt}D^{0.5}X. \quad (4)$$

Therefore, the presence of fractional derivative in the differential equation represents the motion of a simple physical system which includes familiar mechanical and fluid components. It may be predicted that in any systems its presence is characterized by localized motion in a viscous fluid [20].

In the present analysis we construct a simple Haar wavelet collocation method (HWCM) for the numerical solution of second order Bagley-Torvik equations of fractional order(0.5 or 1.5) initial and boundary value problems. We mainly focus on the following initial and boundary conditions carried out to confirm and certify reliability of the algorithm.

Consider the general form of Bagley-Torvik equation as follows

$$PD^2y(x) + QD^{\alpha}y(x) + R[y(x)]^k = f(x), \text{ where } \alpha = 0.5 \text{ or } 1.5,\ 0 \le x \le 1, \quad (5)$$

where, $P \neq 0,\ Q,\ R$ are constant coefficients, where k is the nonlinear integer of the equation.

Here the initial conditions given as

$$D^n y(0) = \delta_n,\ n = 0,1.$$

Whereas the boundary condition at

$$x = x_0, \text{ for } 0 < x_0 \le 1,\ D^n y(x_0) = \eta_n,\ n = 0,1.$$

$\eta_n,\ \delta_n$ are real constants.

This paper is organized as in section 2 fractionally integrated Haar wavelets are introduced. In section 3, steps related to application of the method are given. Numerical problems are solved in section4. Obtained results are discussed in the section 5. In section 6, conclusion of the work is inserted.

2. Haar wavelets and their integrals

The Haar function defined on the interval $[a,\ b]$ is defined using two parameters that is the dilation parameter $j = 0,\ 1,\ 2,\ \ldots.,J$ and $k = 0,\ 1,\ 2,\ \ldots.,2^j - 1$, where $J$is the maximal level of resolution. with these parameters i^{th}Haar wavelet[15] in Haar family is defined as

$$h_i(x) = \begin{cases} 1, & \text{for } x \in [\zeta_1(\text{i}),\ \zeta_2(\text{i})), \\ -1, & \text{for } x \in [\zeta_2(\text{i}),\ \zeta_3(\text{i})), \\ 0, & \text{otherwise}, \end{cases} \tag{6}$$

here $i = 2^j + k + 1$, $\zeta_1(i) = \frac{k}{m}$, $\zeta_2(i) = \frac{k+0.5}{m}$, $\zeta_3(i) = \frac{k+1}{m}$,
where$m = 2^j$, Eq.(6) is valid for $i > 2$. $h_1(x)$ and $h_2(x)$ are called father and mother wavelets in Haar wavelet family and are given by

$$h_1(x) = \begin{cases} 1, & \text{for } x \in [a, b), \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

$$h_2(x) = \begin{cases} 1, & \text{for } x \in \left[a,\ p\ \right), \\ -1, & \text{for } x \in \left[p, b\right), \\ 0, & \text{otherwise}, \end{cases} \tag{8}$$

where, $p = \dfrac{a+b}{2}$.

The integrals of Haar functions$h_i(x)$ can be evaluated as

$$P_{1,i}(x) = \int_0^1 h_i(x)\, dx, \tag{9}$$

$$P_{\upsilon,i}(x) = \int_0^1 P_{\upsilon-1,i}(x)\, dx, \quad \upsilon = 2, 3, \ldots. \tag{10}$$

If υ is a fractional and for $i = 1$, then using gamma function Eq.(10) becomes

$$P_{\upsilon,1}(x) = \frac{1}{\Gamma\upsilon+1}(x-a)^{\upsilon}, \tag{11}$$

for $i \geq 2$, we have

$$P_{\upsilon,i}(x) = \begin{cases} 0, & \text{for } x \in [a, \zeta_1(\text{i})\ ) \\ T_1(x), & \text{for } x \in [\zeta_1(\text{i}),\ \zeta_2(\text{i})) \\ T_2(x), & \text{for } x \in [\zeta_2(\text{i}),\ \zeta_3(\text{i})) \\ T_3(x), & \text{for } x \in [\zeta_3(\text{i}),\ b) \end{cases} \tag{12}$$

where,

$$T_1(x) = \frac{1}{\Gamma\upsilon+1}(x-\zeta_1(\text{i}))^{\upsilon},$$

$$T_2(x) = \frac{1}{\Gamma\upsilon+1}\left\{(x-\zeta_1(\text{i}))^{\upsilon} - 2(x-\zeta_2(\text{i}))^{\upsilon}\right\},$$

$$T_3(x) = \frac{1}{\Gamma\upsilon+1}\left\{(x-\zeta_1(\text{i}))^{\upsilon} - 2(x-\zeta_2(\text{i}))^{\upsilon} + (x-\zeta_3(\text{i}))^{\upsilon}\right\},$$

if υ is a natural number and for $i = 1$Eq.(10) becomes

$$P_{\upsilon,1}(x) = \frac{1}{\upsilon!}(x-a)^{\upsilon}, \tag{13}$$

for $i \geq 2,\ \upsilon = 1,$ we have

$$P_{\upsilon,i}(x) = \begin{cases} x-\zeta_1(\text{i}), & \text{for } x \in [\zeta_1(\text{i}), \zeta_2(\text{i})), \\ \zeta_2(\text{i})\text{-}x, & \text{for } x \in [\zeta_2(\text{i}), \zeta_3(\text{i})), \\ 0, & \text{for otherwise.} \end{cases} \tag{14}$$

For $i \geq 2,\ \upsilon = 2,$ we have

$$P_{\upsilon,i}(x) = \begin{cases} 0, & \text{for } x \in [0, \zeta_1(\text{i})), \\ \dfrac{(x-\zeta_1(\text{i}))^2}{2}, & \text{for } x \in [\zeta_1(\text{i}), \zeta_2(\text{i})), \\ \dfrac{1}{4m^2} - \dfrac{(\zeta_2(\text{i})-x)^2}{2}, & \text{for } x \in [\zeta_2(\text{i}), \zeta_3(\text{i})), \\ \dfrac{1}{4m^2}, & \text{for } x \in [\zeta_3(\text{i}), 1). \end{cases} \tag{15}$$

Any function which is having finite energy on $[a,\ b]$ and square integrable i.e. $y \in L^2[a,\ b]$ can be decomposed as infinite sum of Haar wavelets:

$$y(x) = \sum_{i=1}^{\infty} d_i h_i(x), \tag{16}$$

where $d_i's$ are called Haar coefficients. If $y$is either piecewise constant or wish to approximate by piecewise constant on each subinterval then the above infinite series will be terminated at a finite number of terms as

$$y(x) = \sum_{i=1}^{2^{J+1}} d_i h_i(x). \tag{17}$$

3. Method of solution

Haar wavelet collocation method:

Approximate highest order derivative by piecewise constant on each subinterval for given resolution $J \in N$in Eq.(5)

$$D^{(2)}y(x) = \sum_{i=1}^{2^{J+1}} d_i h_i(x). \tag{18}$$

Express fractional order derivative of $y$for $\alpha = 0.5$ or 1.5in terms of Haar wavelets

$$D^{\alpha}y(x) = \sum_{i=1}^{2^{J+1}} d_i P_{2-\alpha,i}(x) \quad \text{for } \alpha = 1.5, \tag{19}$$

$$Dy(x) = \sum_{i=1}^{2^{J+1}} d_i P_{1,i}(x) + \delta_1, \tag{20}$$

$$D^{\alpha}y(x) = \sum_{i=1}^{2^{J+1}} d_i P_{2-\alpha,i}(x) + \delta_1, \quad \text{for } \alpha = 0.5, \tag{21}$$

$$\therefore \qquad y(x) = \sum_{i=1}^{2^{J+1}} d_i P_{2,i}(x) + \delta_1 x + \delta_0. \tag{22}$$

Decomposed $y(x)$ in terms of integrated Haar functions and replace (16-20) into the given fractional differential Eq. (5).

Discritize equation obtained in above at collocation points $x_l = \frac{\tilde{x}_{l-1} - \tilde{x}_l)}{2}$, $l = 1,2, \ldots 2^{J+1}$, where $\tilde{x}_n$ is the grid point given by $\tilde{x}_n = a + n\frac{(a-b)}{2^{J+1}}$. $n = 0,1,2, \ldots, 2^{J+1}$. Resulting into $2^{J+1} \times 2^{J+1}$ algebraic system.

Calculate the Haar wavelet coefficients $d_i's$ and obtain the Haar solution for unknown function y.

4. Numerical experiments

Example 1: Consider the Bagley–Torvik equation that governs the motion of a rigid plate immersed in a Newtonian fluid [6, 7, and 21]:

$$MD^2 y(x) + 2S\sqrt{\mu\rho} D^{3/2} y(x) + Ky(x) = f(x), \quad x \in (0,1).$$

For the sake of comparison with some of numerical methods [6, 7], we chose $M = 2S\sqrt{\mu\rho} = K = 1$ and $f(x) = 1 + x$, with $y(0) = 1$, $y'(0) = 1$.

The exact solution of this problem is $y(x) = 1 + x$.

In Figure 1 Haar and exact solution for J=3 is inserted. In Table 1 numerical results are compared to the other numerical methods such as ANN [7], GA-Genitc algorithm based technique (GA-based), GP-Genetic pattern based technique (GA-PS based)[21].

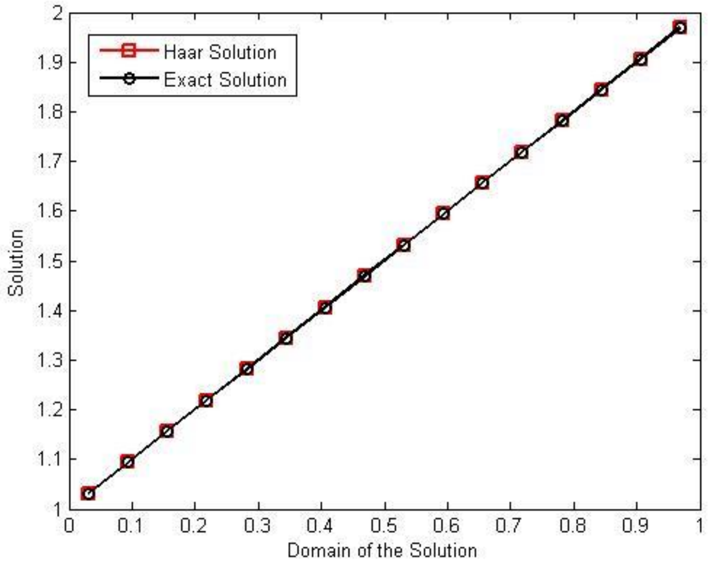

Figure 1: Comparison of Haar and exact solution with J=3.

Table 1 : Comparison of numerical results for Example 1.

X	Exact	ANN [7]	GA-based [22]	GA-PS based [22]	E_{ANN}[7]	E_{GA}[22]	E_{GP}[22]	HWCM	E_{HWCM}
0	1.00	1.0000	1.0248	1.0160	0	2.30 E-02	1.60 E-06	1.00	0
0.2	1.20	1.1999	1.2208	1.1998	2.18E-06	3.13 E-02	1.95 E-06	1.20	0
0.4	1.40	1.3999	1.4269	1.4016	3.24 E-06	3.45 E-02	1.62 E-06	1.40	0
0.6	1.60	1.5999	1.6345	1.6074	2.91 E-06	1.36 E-02	7.42 E-06	1.60	0
0.8	1.80	1.7999	1.8287	1.7999	2.71 E-06	2.30 E-02	1.27 E-06	1.80	0
1	2.00	1.9999	1.9850	1.9537	4.03 E-06	3.13 E-02	4.62 E-06	2.00	0

Example 2: Consider the Cauchy problem of the Bagley- Torvik equation of a rigid plate [4]:

$$D^2 y(x) + D^{1.5} y(x) + y(x) = f(x), \quad 0 < x < 1,$$

where, $f(x) = \frac{15}{4}\sqrt{x} + \frac{15}{8}\sqrt{\pi x} + x^2\sqrt{x}$,
subject to the initial conditions $y(0) = 0,\ y'(0) = 0$.
The exact solution of this problem is $y(x) = x^2\sqrt{x}$.
Comparison of Haar and exact solution with J=4 is drawn in Figure 2. Absolute errors for various values of J are shown in Figure 3. In Table 2 comparison of maximum absolute errors for HWCM with SCM [4] are inserted.

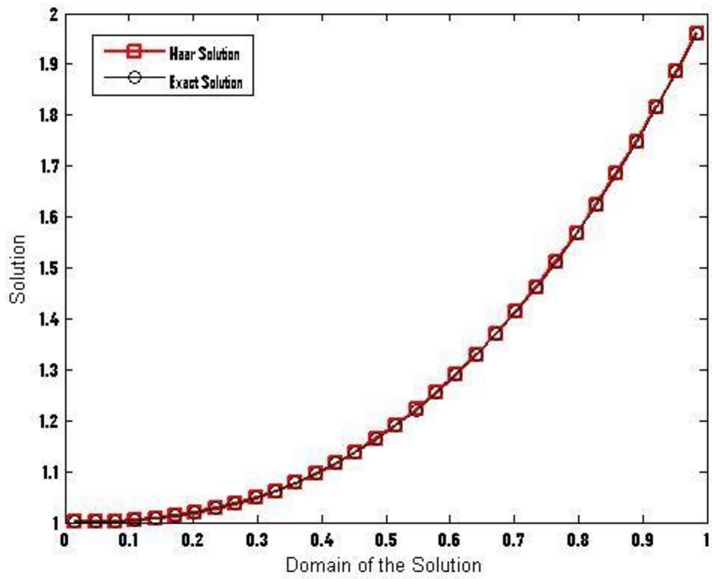

Figure 2:Comparison of Haar and exact solution of Example 2 with J=4.

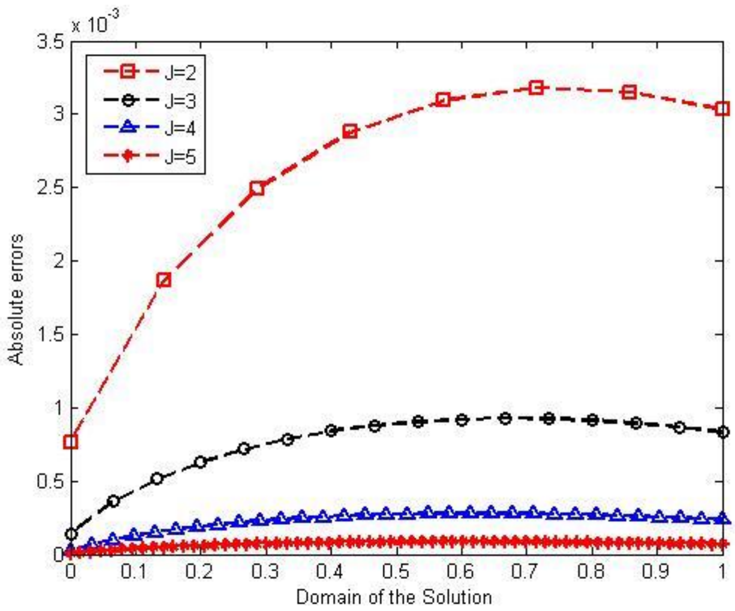

Figure 3: Comparison of absolute errors for Example 2.

Table 2 : Comparison of maximum absolute errors for Example 2.

J	**HWCM**	**SCM[4]**		
		$r=1$	$r=1.8$	$r=2.6$
4	8.7E-05	3.1E-02	1.1E-02	6.6E-03
8	2.8E-05	1.6E-02	3.2E-03	1.2E-03
16	9.2E-06	8.1E-03	9.2E-04	2.2E-04
32	3.1E-06	4.1E-03	2.7E-04	3.8E-05
64	1.0E-06	2.1E-03	7.8E-05	6.9E-06

Example 3: Consider the boundary value problem for Bagley–Torvik equation [3]:

$$D^2 y(x) + D^{1.5} y(x) + y(x) = f(x),\ \ 0 < x < 1,$$

where $f(x) = \frac{15}{4}x^{0.5} + \frac{15}{8}\sqrt{\pi x} + x^{2.5} + 1,$ subject to $y(0) = 1,\ y(1) = 2.$
Exact solution is given by $x^{2.5} + 1.$
Haar and exact solution with J=5 is shown in Figure 4. Comparison of absolute errors for different values of J is drawn in Figure 5. Maximum absolute errors are compared to the PPCM [3] in Table 3.

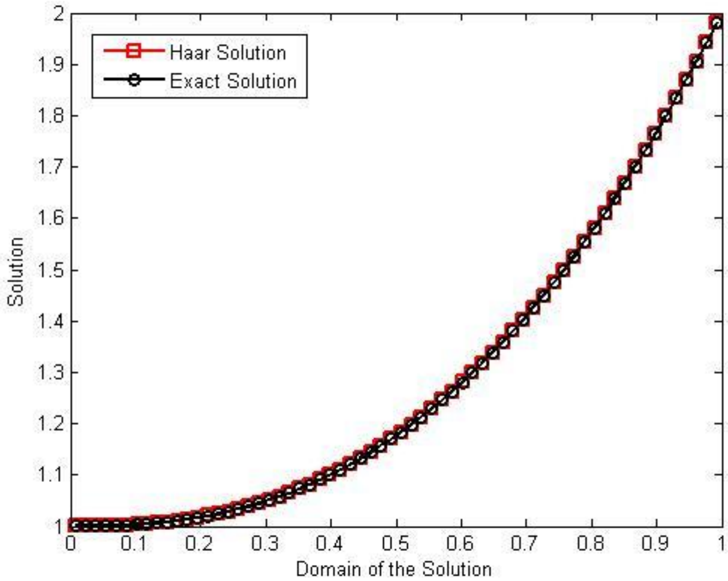

Figure 4: Comparison of Haar and exact solution of Example 3 with J=5.

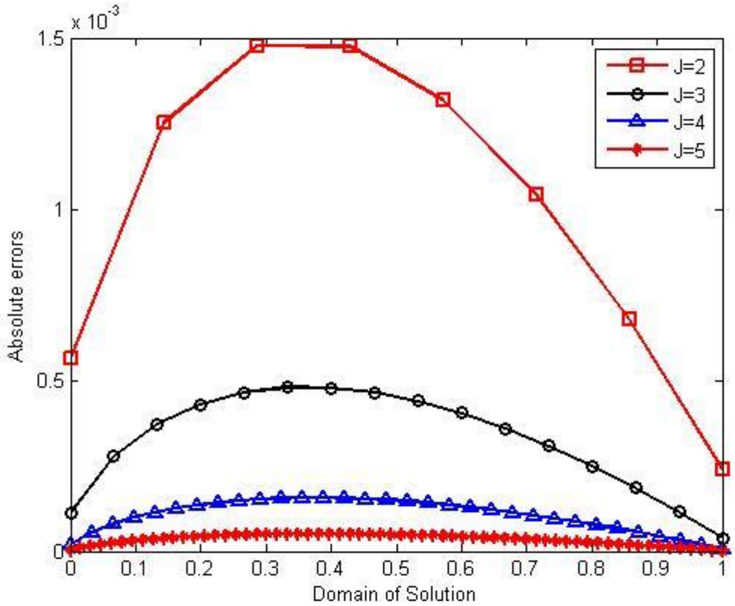

Figure 5: Comparison of absolute errors for Example 3.

Table 3 : Comparison of maximum absolute errors for Example 3.

J	**HWCM**	**PPCM[3]**
4	4.7E-03	1.7E-02
8	1.5E-03	9.3E-03
16	4.7E-04	4.9E-03
32	1.5E-04	2.5E-03
64	5.1E-05	1.3E-03
128	1.7E-05	6.7E-04
256	5.8E-06	3.4E-04

Example 4: Consider the fractional order Cauchy problem [5]:

$$D^{1.5}y(x)+2Dy(x)+3\sqrt{x}D^{0.5}y(x)+(1-x)y(x)=f(x),\ \ 0<x<1,$$

where, $f(x)=\frac{2}{\Gamma(1.5)}x^{0.5}+4x+\frac{4}{\Gamma(1.5)}x^2+(1-x)x^2,$ with $y(0)=y'(0)=0$.

Its analytic solution is $y(x)=x^2$.

Comparison of Haar and exact solution with J=4 is displayed in Figure 6. Absolute errors are compared in the Figure 7 for different J values. Comparison of maximum absolute errors of HWCM with SCM [5] is tabulated in Table 4.

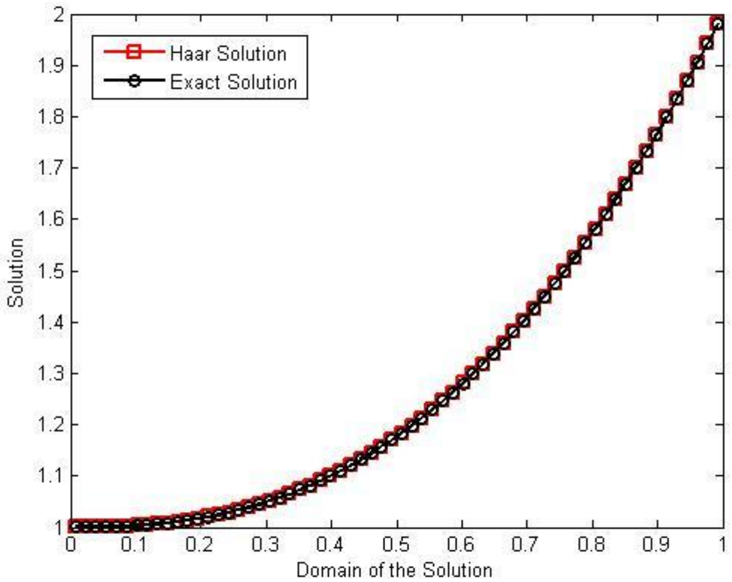

Figure 6: Comparison of Haar and exact solution with J=4 for Example 4.

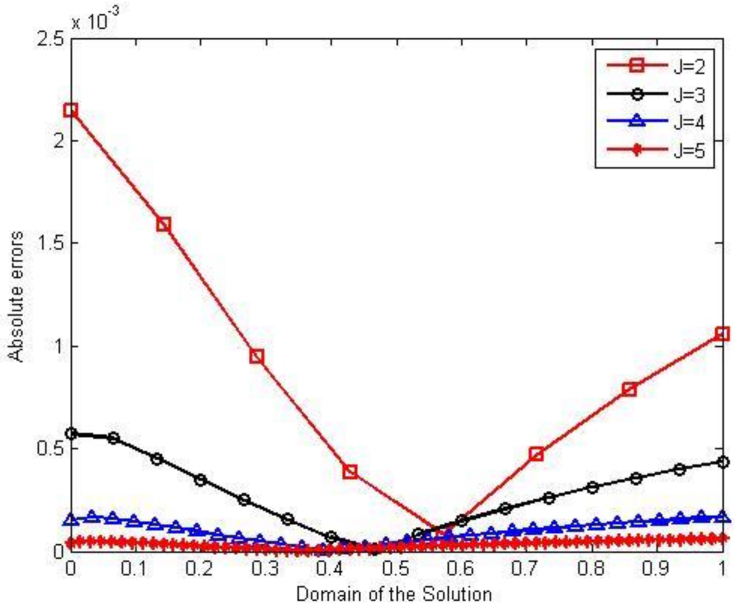

Figure 7: Comparison of absolute errors for Example 4.

Table 4 : Comparison of maximum absolute errors for Example 4.

J	**HWCM**	**SCM[5]**		
		$r=1$	$r=1.8$	$r=2.6$
4	1.6E-05	1.9E-02	5.4E-03	5.3E-03
8	6.1E-05	1.0E-02	1.5E-03	9.6E-04
16	2.2E-06	5.4E-03	4.0E-04	1.7E-04
32	8.1E-06	2.8E-03	1.0E-04	3.2E-05
64	2.9E-06	1.5E-03	2.0E-05	5.7E-06
128	1.0E-06	7.7E-04	6.6E-06	1.0E-06

Example 5 : Consider the nonlinear fractional differential equation[21, 23] :

$$D^{\alpha} y(x) + y^{2}(x) = 1, \ \ 0 < \alpha \le 1, 0 < x < 1,$$

subject to the initial condition $y(0) = 0$.
The exact solution when $\alpha = 1$is $y(x) = \frac{e^{2x}-1}{e^{2x}+1}$.
Comparison of Haar, exact and adomian decomposition method (ADM) [23] solutions for $\alpha = 1$ is graphically shown in Figure 8. Comparison of HWCM and exact solution for $\alpha = 1$ and HWCM for $\alpha = 0.5$with Legendre$\alpha = 0.5$ [22] is shown in Figure 9. In Figure 10 absolute errors for various values of J are drawn.

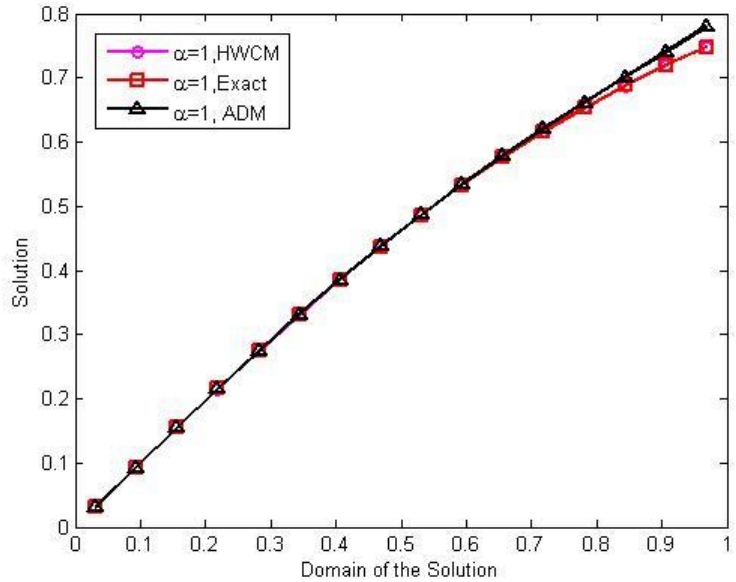

Figure 8: Comparison of Haar, ADM [23] and exact solution for Example 5.

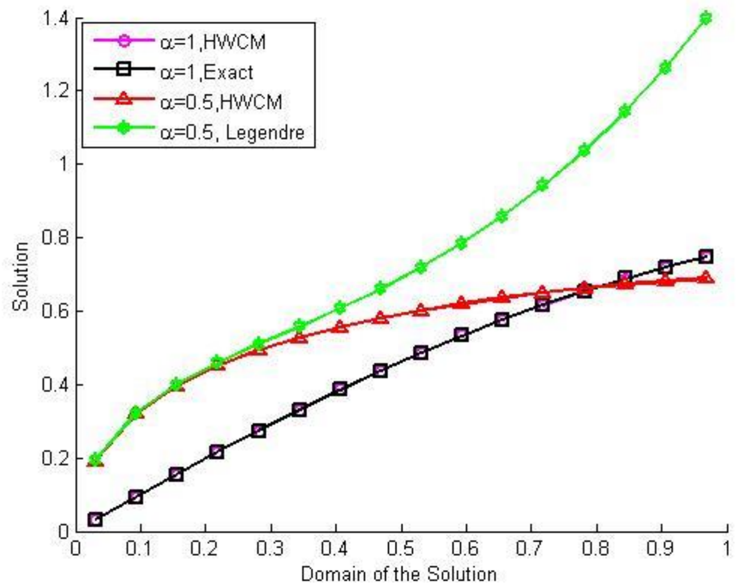

Figure 9: Comparison of Haar, exact and Legendre [23] solutions for Example 5.

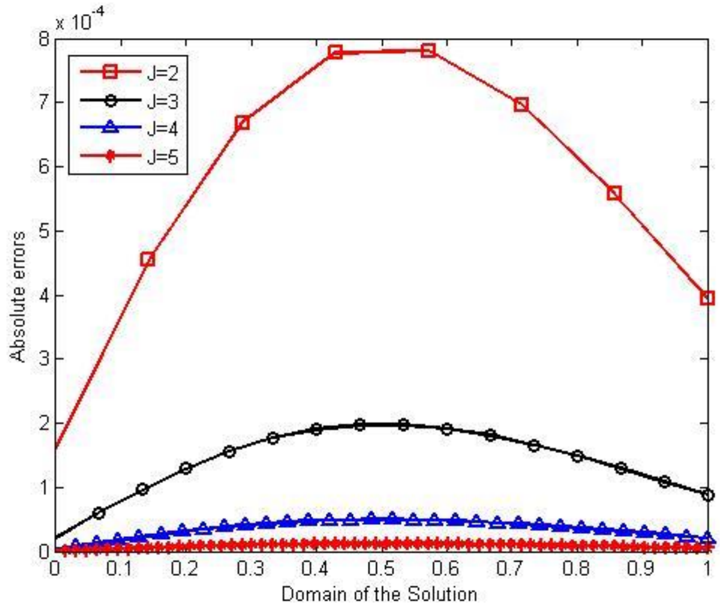

Figure 10: Comparison of absolute errors for Example 5.

5. Results and Discussions

The comparison of Haar and Exact solution at collocation points for Examples 1-4 have been shown in Figures 1, 2, 4 and 6 respectively. These figures ensured that the numerical solution obtained by proposed method is in excellent agreement with the exact solution each problem. Same scenario is continued for the last example and one more interesting point is that proposed method is having good approximation property then ADM when $\alpha = 1$, these findings are shown in Figure 8. In Figure 9 comparison of Haar solution and Legendre solution for different values of $\alpha = 1,\ 0.5$ is represented. The Haar solution curve for $\alpha = 0.5$ is closer to Haar and exact solution with$\alpha = 1$, whereas Legendre solution goes faraway. Moreover absolute errors obtained to Examples 2-5 for different resolutions are drawn in Figures 3, 5, 7, 10 by this we concluded that as the resolution value increases absolute error curve approaches towards x-axis (where the absolute errors are zero). Comparison of numerical results obtained to Examples 1-4 are shown in Tables 1-4. Here the proposed method exhibited the precise results compared to other existing numerical methods such as ANN, GA-based, GA-P based, SCM, PPCM. From this observation we concluded that a very good improvement in the accuracy is achieved by increasing the level of resolution.

6. Conclusions

This work exhibited the applicability of the Haar wavelet collocation method to solve Bagley-Torvik equations arising in the modeling of rigid plate immersed in a Newtonian fluid. We had given several examples (linear and nonlinear) to examine the accuracy of the HWCM. The numerical results obtained by this method are compared with other methods we concluded that HWCM is the unbeaten numerical technique for above said problems.

Acknowledgments

Author A. Padmanabha Reddy is grateful to Vision Group on Science and Technology, Govt. of Karnataka, India, for financial assistance under the scheme “Seed Money to Young Scientists for Research (SMYSR-FY-2015-16/GRD-497)”.

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