ACADEMIC
World Academic Union

ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 14, No. 3, 2019, *pp.*170-175

A conservative Galerkin finite element method for the Klein-Gordon equation in high dimensions

Huawei Zhao, Yue Cheng

School of Mathematics and Statistics, Nanjing University of Information Science & Technology, Nanjing, 210044, China

(Received January 06 2019, accepted May 20 2019)

Abstract. In this article, we design and analyze a Galerkin finite element method (FEM) to solve the nonlinear Klein-Gordon equation in $d(d = 1,2,3)$ dimensions. The scheme is proved to preserve well the total energy in the discrete sense, which is consistent with the conservative property possessed by the original problem. Numerical results are reported to show the high accuracy of the numerical methods and confirm the energy conservation.

Keywords: Klein-Gordon equation, Energy conservation, Galerkin FEM

1 Introduction

The Klein-Gordon (KG) equation is the basic equation in relativistic quantum mechanics and quantum field theory. The specific form of the KG equation is as follows

$$\begin{cases} \partial_{tt}u(\mathbf{x},t) - \Delta_h u(\mathbf{x},t) + u(\mathbf{x},t) + f(u) = 0, & \mathbf{x} \in \mathbb{R}^d, \ t > 0, \\ u(\mathbf{x},0) = \phi(\mathbf{x}), \qquad \partial_t u(\mathbf{x},t) = \psi(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^d. \end{cases} \tag{1.1}$$

Where $u = u(\boldsymbol{x},t)$ is the unknown real-valued filed,Δ is Laplace operator,ϕ,ψ and f are given real-valued function. In fact, the above KG equation is also known as the relativistic version of the Schrödinger equation under proper non-dimensionalization [13-15] and it is used to describe the motion of a spinless particle [3,17]. The KG equation is one of the important equations in solitons studies, particularly in the examination of solitons interactions for a collisionless plasma and the recurrence of initial states [6,21]. It is clear that the KG equation (1.1) is time symmetric or time reversible, it also preserves energy conservation as

$$\begin{aligned} \mathrm{E(t)} &= \int_{\mathbb{R}} [(\partial_t u(\mathbf{x},t))^2 + |\nabla u(\mathbf{x},t)|^2 + u^2(\mathbf{x},t) + 2F(u(\mathbf{x},t))]d\mathbf{x} \\ &= \int_{\mathbb{R}} [\psi^2(\mathbf{x}) + |\nabla \phi(\mathbf{x})|^2 + \phi^2(\mathbf{x}) + 2F(\phi(\mathbf{x}))]d\mathbf{x} := E(0), \ t \geq 0. \end{aligned} \tag{1.2}$$

where

$$F(u) = \int_0^u f(s)ds, \ u \in \mathbb{R}. \tag{1.3}$$

Extensive mathematical and numerical studies of the KG equation have been carried out in the literature. For instance, standard finite difference time domain (FDTD) methods such as energy conservative, semi-implicit and explicit finite difference discretization were proposed and analyzed in [8,16,18] Finite element or spectral discretization were also studied in [5,19]. Comparisons of different methods in this regime were carried out [7,16]. The Fourier collocation method was employed to obtain the numerical solution of the Klein-Gordon equation in the literature [1], a symplectic finite difference scheme was investigated in [4], S. Li and L. Vu-Quoc analyzed some conservative finite difference methods, and the related error estimates was also derived in the literatures [9,20].

In [10-12], in order to prove the unconditional convergence and establish the optimal error estimate of several traditional implicit FEMs for some important partial differential equations, Li and Sun creatively proposed a technical analysis method. They split the error into two parts, the temporal error and the spatial error. Since the spatial error is τ -independent, the numerical solution can be bounded in L^∞-norm by an inverse inequality unconditionally. Then, the optima L^2 error estimate can be obtained by a routine method.

In [22,23], Wang used Li-Sun's analysis method to establish the unconditional and optimal L^2 error estimates of several linearized FEMs for solving the general nonlinear Schrödinger equation and a nonlinear

Schrödinger-Helmholtz system. In [2], the authors also used Li-Sun's analysis method to build the unconditional and optimal L^2 error estimate of a linearized Euler FEM for the nonlinear Schrödinger equation equation.

In this paper, we introduce linearized Galerkin finite element method for solving nonlinear Klien-Gordon equation (1.1). In addition, we prove the conservation of the scheme.

The outline of this paper is arranged in the following way. In Section 2, we propose a new linearized Galerkin finite element method for solving the KG equation, and prove the conservation properties of FEM. In Section 3, we report several numerical results to show the high accuracy of the numerical methods and confirm the conservation lows.

2 Finite element scheme and energy conservation

In practical computation, the KG equation (1.1) is usually truncated on a bounded interval $\Omega = (a, b)$ in 1D, or a bounded rectangle $\Omega = (a, b) \times (c, d)$ in 2D, or a bounded box $\Omega = (a, b) \times (c, d) \times (e, f)$ in 3D. In this paper, we deal with the case in 2D, with homogeneous Dirichlet boundary conditions. Extension to the case in three dimensions or reduction to that in one dimension is direct with small modification. Then, we consider the initial-boundary value problem of the KG equation (1.1) truncated on the computational interval $\Omega = (a, b) \times (c, d)$ as follows

$$\partial_{tt}u(x,y,t) - \Delta_h u(x,y,t) + u(x,y,t) + f(u) = 0, \quad (x,y) \in \Omega, \ t > 0, \tag{2.1}$$

$$u(x,y,0) = \phi(x,y), \quad \partial_t u(x,y,0) = \psi(x,y), \quad (x,y) \in \bar{\Omega}, \tag{2.2}$$

$$u(x,y,t) = 0, \quad (x,y) \in \partial\Omega. \tag{2.3}$$

The weak formulation of the KG equation (2.1)-(2.3) can be specified as follows: to find $u \in H_0^1(\Omega)$ such that $\forall t > 0$,

$$\left(\partial_{tt}u, v\right) + \left(\nabla_h u, \nabla_h v\right) + \left(u, v\right) + \left(f(u), v\right) = 0, \quad \forall v \in H_0^1(\Omega), \tag{2.4}$$

with the initial condition $u_0 \in H_0^1$.

As usual, for any two complex-valued functions $u, v \in L^2(\Omega)$, we define the $L^2(\Omega)$ inner product as follows

$$(u, v) = \int_\Omega u(x)(v(x))^* dx,$$

where v^*denotes the conjugate of v. Let R_h:$H_0^1(\Omega) \to V_h$ be a Ritz projection operator defined by

$$(\nabla(v - R_h v), \nabla\omega) = 0, \quad \text{for all } \omega \in V_h. \tag{2.5}$$

For a positive integer N, choose time-step $\tau = T/N$ and denote time steps $t_n = n\tau, n = 0,1 \cdots N$, where $0 < T < T_{max}$ with T_{max} the maximal existing time of the solution. We denote $u^m = u(x, t_m)$. For a sequence of functions $\{\omega\}_{n=0}^N$, we define

$$\delta_t^+ \omega^n = \frac{\omega^{n+1} - \omega^n}{\tau}, \quad \delta_t^2 \omega^n = \frac{\omega^{n+1} - 2\omega^n + \omega^{n-1}}{\tau^2},$$

$$\hat{\omega}^n = \frac{1}{2}(\omega^{n+1} + \omega^{n-1}), \quad n = 1, 2, \ldots, N-1.$$

With above notations, a linearized three-level Galerkin FEM is: to seek $u_h^n \in V_h$ such that

$$\left(\delta_t^2 u_h^n, v\right) + \left(\nabla \hat{u}_h^n, \nabla v\right) + \left(\hat{u}_h^n, v\right) + \left(G(u_h^{n+1}, u_h^{n-1}), v\right) = 0, \quad n = 1, 2, \ldots, N-1, \tag{2.6}$$

where $G(z_1, z_2)$ is defined for $z_1, z_2 \in \mathbb{C}$ as

$$G(z_1, z_2) := \int_0^1 f(\theta z_1 + (1-\theta) z_2) d\theta = \frac{F(z_1) - F(z_2)}{z_1 - z_2}, \tag{2.7}$$

with $u_h^0 = \Pi_h u_0$.

This is a three-level scheme which could not start by itself, so we need another two-level scheme to compute u_h^1. One method to compute u_h^1 is applying Taylor's expansion to u^1 at the point $(\boldsymbol{x}, 0)$, we obtain

$$\begin{aligned} u_h^1 \approx\ & u(\mathbf{x},0)+\tau\partial_t u(\mathbf{x},0)+\frac{\tau^2}{2}\partial_{tt}u(\mathbf{x},0) \\ =\ & \phi(\mathbf{x})+\tau\psi(\mathbf{x})+\frac{\tau^2}{2}\left(\Delta_h\phi(\mathbf{x})-\phi(\mathbf{x})-f\left(\phi(\mathbf{x})\right)\right). \end{aligned} \tag{2.8}$$

Theorem 2.1 (conservation properties of FEM) The FEM scheme (2.6) with (2.7) and (2.8) conserves the discrete energy as

$$\begin{aligned} E^n =& \left\|\delta_t^+ u_h^n\right\|_{L^2}^2+\frac{1}{2}\left(|u_h^n|_{H^1}^2+|u_h^{n+1}|_{H^1}^2\right)+\frac{1}{2}\left(\left\|u_h^n\right\|_{L^2}^2+\left\|u_h^{n+1}\right\|_{L^2}^2\right) \\ &+\int_\Omega [F(u_h^n)+F(u_h^{n+1})]d\mathbf{x}\equiv E^0,\ \ n=0,1,2,\cdots,N-1. \end{aligned}$$

Proof Choosing $v=\delta_t^+(u_h^{n+1}-u_h^{n-1})$ as the test function at (2.6) , we get

$$\begin{aligned} &\left(\delta_t^2 u_h^n,\delta_t^+(\mathrm{u}_h^{n+1}-\mathrm{u}_h^{n-1})\right)+\left(\nabla\hat{u}_h^n,\nabla\delta_t^+(\mathrm{u}_h^{n+1}-\mathrm{u}_h^{n-1})\right)+\left(\hat{u}_h^n,\delta_t^+(\mathrm{u}_h^{n+1}-\mathrm{u}_h^{n-1})\right) \\ &+\left(G(u_h^{n+1},u_h^{n-1}),\delta_t^+(\mathrm{u}_h^{n+1}-\mathrm{u}_h^{n-1})\right)=0,\ \ \mathrm{n}=1,2,\cdots N-1. \end{aligned}$$

Then, we expend the above equation combine with integration by part method, it is easy to see the each part as

$$\begin{aligned} \int_\Omega \delta_t^2 u_h^n\cdot\delta_t^+(\mathrm{u}_h^{n+1}-\mathrm{u}_h^{n-1})d\mathbf{x} &= \int_\Omega \frac{\mathrm{u}_h^{n+1}-2\,\mathrm{u}_h^n+\mathrm{u}_h^{n-1}}{\tau^2}\cdot\delta_t^+(\mathrm{u}_h^{n+1}-\mathrm{u}_h^n+\mathrm{u}_h^n-\mathrm{u}_h^{n-1})d\mathbf{x} \\ &= \int_\Omega(\delta_t^+\,\mathrm{u}_h^n-\delta_t^+\,\mathrm{u}_h^{n-1})\cdot\delta_t^+(\delta_t^+\,\mathrm{u}_h^n+\delta_t^+\,\mathrm{u}_h^{n-1})d\mathbf{x} \\ &= \delta_t^+\int_\Omega(\delta_t^+\,\mathrm{u}_h^n)^2-(\delta_t^+\,\mathrm{u}_h^{n-1})^2\,d\mathbf{x}, \end{aligned}$$

$$\begin{aligned} \int_\Omega \nabla\hat{u}_h^n\cdot\nabla\delta_t^+(\mathrm{u}_h^{n+1}-\mathrm{u}_h^{n-1})d\mathbf{x} &= \int_\Omega \frac{1}{2}\nabla(\mathrm{u}_h^{n+1}+\mathrm{u}_h^{n-1})\cdot\delta_t^+\nabla(\mathrm{u}_h^{n+1}-\mathrm{u}_h^{n-1})d\mathbf{x} \\ &= \frac{1}{2}\delta_t^+\int_\Omega\nabla((\mathrm{u}_h^{n+1})^2-(\mathrm{u}_h^{n-1})^2)d\mathbf{x}, \end{aligned}$$

$$\begin{aligned} \int_\Omega \hat{u}_h^n\cdot\delta_t^+(\mathrm{u}_h^{n+1}-\mathrm{u}_h^{n-1})d\mathbf{x} &= \int_\Omega \frac{1}{2}(\mathrm{u}_h^{n+1}+\mathrm{u}_h^{n-1})\cdot\delta_t^+(\mathrm{u}_h^{n+1}-\mathrm{u}_h^{n-1})d\mathbf{x} \\ &= \frac{1}{2}\delta_t^+\int_\Omega(\mathrm{u}_h^{n+1})^2-(\mathrm{u}_h^{n-1})^2\,d\mathbf{x}, \end{aligned}$$

$$\begin{aligned} \int_\Omega G(u_h^{n+1},u_h^{n-1})\cdot\delta_t^+(\mathrm{u}_h^{n+1}-\mathrm{u}_h^{n-1})d\mathbf{x} &= \int_\Omega \frac{F(u_h^{n+1})-F(u_h^{n-1})}{\mathrm{u}_h^{n+1}-\mathrm{u}_h^{n-1}}\cdot\delta_t^+(\mathrm{u}_h^{n+1}-\mathrm{u}_h^{n-1})d\mathbf{x} \\ &= \delta_t^+\int_\Omega F(u_h^{n+1})-F(u_h^{n-1})d\mathbf{x}. \end{aligned}$$

Combine the above four parts , we have

$$\begin{aligned} &\delta_t^+\int_\Omega(\delta_t^+\,\mathrm{u}_h^n)^2-(\delta_t^+\,\mathrm{u}_h^{n-1})^2+\frac{1}{2}\nabla((\mathrm{u}_h^{n+1})^2-(\mathrm{u}_h^{n-1})^2)+\frac{1}{2}((\mathrm{u}_h^{n+1})^2-(\mathrm{u}_h^{n-1})^2) \\ &+F(u_h^{n+1})-F(u_h^{n-1})d\mathbf{x}=0. \end{aligned}$$

Then, taking summation for $1,2,\cdots,n$, we conclude that

$$\begin{aligned} E^n =& \left\|\delta_t^+ u_h^n\right\|_{L^2}^2+\frac{1}{2}\left(|u_h^n|_{H^1}^2+|u_h^{n+1}|_{H^1}^2\right)+\frac{1}{2}\left(\left\|u_h^n\right\|_{L^2}^2+\left\|u_h^{n+1}\right\|_{L^2}^2\right) \\ &+\int_\Omega [F(u_h^n)+F(u_h^{n+1})]d\mathbf{x}\equiv E^0,\ \ n=0,1,2,\cdots,N-1. \end{aligned}$$

3 Numerical results

In this section, we report numerical results to show the high accuracy of the numerical methods and conservation laws in 2D.

Example 4.1 To test the high accuracy of the proposed scheme, we consider the following KG equation with homogeneous Dirichlet boundary.

$$\begin{aligned} &\partial_{tt}u(x,y,t)-\Delta_h u(x,y,t)+u(x,y,t)+f(u)=g, \quad (x,y)\in\Omega=(0,2)\times(0,2),\ t>0,\\ &u(x,y,0)=\sin\pi x\cdot\sin\pi y, \qquad \partial_t u(x,y,0)=0, \quad (x,y)\in\bar{\Omega}=[0,2]\times[0,2],\\ &u(x,y,t)=0, \qquad\qquad\qquad\qquad (x,y)\in\partial\Omega. \end{aligned}$$

Here, we choose $f(u)=u^2$ and g is chosen correspondingly to the exact solution

$$u=\sin\pi x\cdot\sin\pi y\cdot\cos t\,.$$

We solve the above system by the linearized three-level Galerkin FEM in (2.6)-(2.8) with a linear FE approximation and a quadratic FE approximation. A uniform triangular partition with $M+1$ nodes in each direction is used, where $h=\sqrt{2}/M$. we choose $\tau=1/M$ for the linear FEM and $\tau=(1/M)^{3/2}$ for the quadratic FEM. We define Rate(j)=log(Error(j-1)/Error(j))/log(h(j-1)/h(j)) , where $j=2,3,4$,and Rate(1) doesn't exist. We present numerical results in Table 1 and Table 2 at time $t=1.0$.To show the energy conservation of the FEM, we take $g=0$ at the example and present numerical results in Figure 1.It is clear to see our methods have high accuracy and energy is conservative.

Table 1: L^2 errors of FEM at t=1.0 (Example 4.1)

L-FEM	$\tau=1/M$	$\|u(\cdot,t_n)-u_h^n\|_{L^2}$		
	M=10	M=20	M=40	M=80
Error	2.3809e-01	6.4910e-02	1.6674e-02	4.2065e-03
Rate	-	1.8750	1.9608	1.9869

Table 2: L^2 errors of quadratic FEM at t=1.0 (Example 4.1)

Q-FEM	$\tau=(1/M)^{3/2}$	$\|u(\cdot,t_n)-u_h^n\|_{L^2}$		
	M=32	M=40	M=64	M=80
Error	4.2312e-05	2.0959e-05	5.2213e-06	2.5632e-06
Rate	-	3.1482	2.9570	3.1885

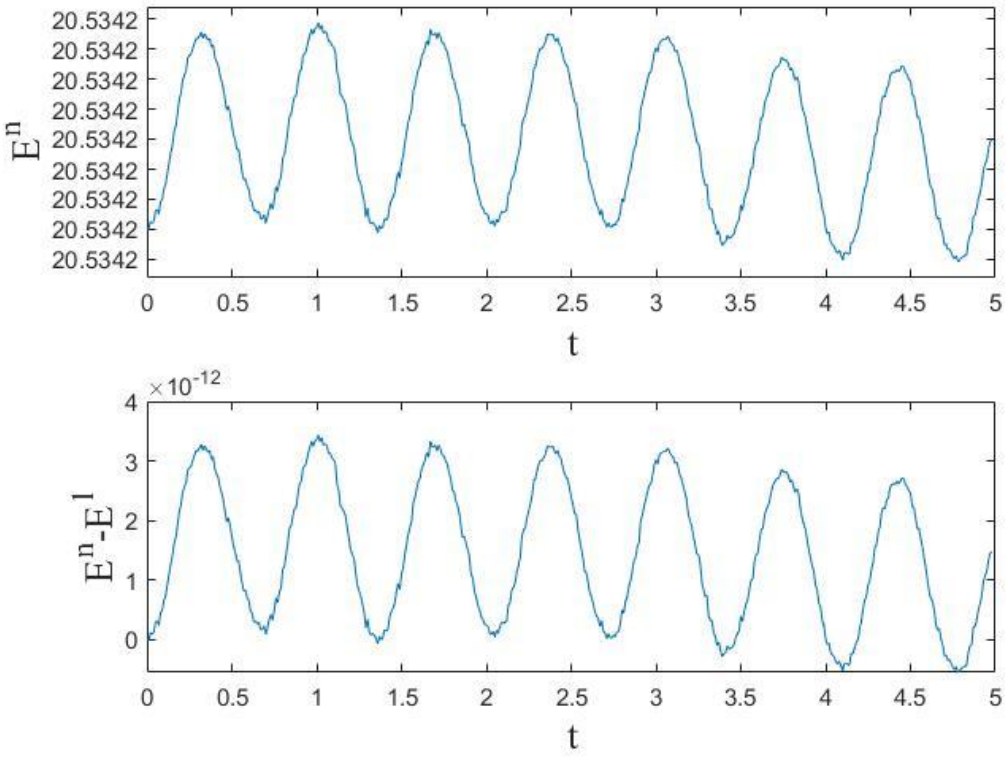

Figure 1:Total energy and its difference from the initial value computed by FEM for Example 4.1 with $g = 0, t = 5$

References

[1] W. Cao, B. Guo, Fourier collocation method for solving nonlinear Klein-Gordon equation, J. Comput. Phys., 108 (1993) 296-305.

[2] W. Cai, J. Li, Z. Chen, Unconditional convergence and optimal error estimates of the Euler semi-implicit scheme for a generalized nonlinear Schrödinger equation, Adv. Comput. Math., 42 (2016) 1311-1330.

[3] A.S. Davydov, Quantum Mechanics, 2nd edn. Pergamon, Oxford, (1976).

[4] D. Duncan, Symplectic finite difference approximations of the nonlinear Klein-Gordon equation, SIAM J., 34 (1997) 1742-1760.

[5] E. Deeba, S. Khuri, A decomposition method for solving the nonlinear Klein-Gordon equation, J. Comput. Phys., 124 (1996) 442–448.

[6] S.M. El-Sayed, The Decomposition Method for Studying the Klein-Gordon Equation, 18 (2003) 1025-1030.

[7] S. Jiménez, L. Vázquez, Analysis of four numerical schemes for a nonlinear Klein-Gordon equation. Comput., 35 (1990) 61–94.

[8] S. Li, L. Vu-Quoc, Finite difference calculus invariant structuer of a class of algorithms for the nonlinear Klein Gordon equation. SIMA J., Numer. Anal. 32 (1995) 1839-1875.

[9] S. Li, L. Vu-Quoc, Finite difference calculus invariant structure of a class of algorithms for the nonlinear Klein-Gordon equation, SIAM J., 32 (1995) 1839–1875.

[10] B. Li, Mathematical modeling, analysis and computation for some complex and nonlinear flow problems, PhD Thesis, City University of Hong Kong, Hong Kong, July (2012)

[11] B. Li, W. Sun, Error analysis of linearized semi-implicit Galerkin finite element methods for the nonlinear parabolic equations, Int. J. Numer. Anal. Model., 10 (2013) 622-633.

[12] B. Li, W. Sun, Unconditional convergence and optimal error estimates of a Glerkin-mixed FEM for incompressible miscible flow in porous media, SIAM J. Numer. Anal., 51 (2013) 1959-1977.

[13] N. Masmoudi, K. Nakanishi, From nonlinear Klein-Gordon equation to a system of coupled nonlinear Schrödinger equations. Math. Ann., 324 (2002) 359–389.

[14] S. Machihara, The nonrelativistic limit of the nonlinear Klein-Gordon equation. Funkcial. Ekvac., 44 (2001) 243

252.

[15] S. Machihara, K. Nakanishi, T. Ozawa, Nonrelativistic limit in the energy space for nonlinear Klein-Gordon equations. Math. Ann., 322 (2002) 603–621.

[16] P.J. Pascual, S. Jiménez, L. Vázquez, Numerical Simulations of a Nonlinear Klein-Gordon Model. Applications. Computational Physics, (Granada, 1994). Lecture Notes in Physics, 448 (1995) 211–270.

[17] J.J. Sakurai, Advanced Quantum Mechanics. Addison Wesley, New York, (1967).

[18] W. Strauss, L. Vazquez, Numerical solution of a nonlinear Klein-Gordon equation. J. Comput.Phys., 28 (1978) 271 278.

[19] Y. Tourigny, Product approximation for nonlinear Klein-Gordon equations. SIMA J., 9 (1990) 449–462.

[20] L. Vu-Quoc, S. Li, Invariant-conserving finite difference algorithms for the nonlinear Klein-Gordon equation. Comput., 107 (1993) 341–391.

[21] A. Wazwaz, Compactons, Solitons and Periodic Solutions for Some Forms of Nonlinear Klein-Gordon Equations, 28 (2006) 1005-1013.

[22] J. Wang, A new error analysis of Crank-Nicolson Galerkin FEMs for a generalized nonlinear Schrödinger equation, J. Sci. Comput., 60 (2014) 390-407.

[23] J. Wang, Unconditional stability and convergence of Crank-Nicolson Galerkin FEMs for a nonlinear Schrödinger Helmholtz system, Numer. Math., DOI：10.1007/s00211-017-0944-0.}