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Fractions and Gorenstein flat modules

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Abstract. In this paper, we give some equivalent conditions on when the fractions of rings can preserve Gorenstein flat modules.

Keywords: Gorenstein flat module, exact sequence, fraction of a ring

1. Introduction

In the 1940s, due to the needs of the theory of ring theory and the rise of homology algebra, the module theory has been further developed. Projective modules, injective modules and flat modules are three main types of modules in homological algebra [3]. Gorenstein homology algebra [2] which was found by Enochs and Jenda has been very popular since 1990s. Many algebraists work on this topic which make the theory fruitful.

Commutative algebra [1] and homology algebra [3] are intertwined and promote each other for very long time. We have showed when the fraction of Z_n can be a field [9]. It was showed by Enochs and Jenda that the fractions of Gorenstein rings preserve Gorenstein injective modules. In this paper, we study the relationship between fractions of rings and Gorenstein flat. For more details of commutative algebra and Gorenstein homological algebra, we refer to [1],[2] and [3].

Throughout this paper, R is a commutative ring with unity.

2. Preliminaries

In this section, we recall some basic preliminaries.

First of all, we give the definition of Gorenstein flat modules [7].

Definition 2.1 A R -module M is said to be Gorenstein flat if there exists an exact sequence

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

of flat modules such that $M = \mathrm{Ker}(F^0 \to F^1)$ and $\mathrm{Tor}_i^R(E, M) = 0$ for all $i \geq 1$ and any injective module E.

Definition 2.2 Let R be a commutative ring with unit 1 and S be a multiplicatively set of a ring R. Defining equivalence relation on the set $R \times S$:(a, s) if and only if $\exists u \in S$ so that $u(at - bs) = 0$. Then we use a/s and $S^{-1}R$ to denote the equivalence class of $(a, s) \in R \times S$ and fractional ring, respectively.

We need the following property of fractions of rings.

Proposition 2.3 [1] Let S be a multiplicatively set of a ring R, then $S^{-1}R$ is a flat R-module.

By Proposition 2.3, one gets that

Proposition 2.4 [1] Let S be a multiplicatively set of a ring R. Then the functor $S^{-1}R \otimes \_ : Mod\ R \to Mod\ S^{-1}R$ is exact.

For any prime ideal p, take $S = R - p$, we have the following proposition.

Proposition 2.5 [2] Let R, p be as above. For any R-modules M, N, then $\mathrm{Tor}_i^R(M, N)_p = \mathrm{Tor}_i^{R_p}(M_p, N_p)$ for any $i \geq 1$.

3. Main results

In this section, we show that the fraction preserves Gorenstein flat modules.

Firstly, we show the following properties of flat modules.

Proposition 3.1 Let R be a commutative ring and Q be a flat R-module. If M is a flat R-module, then $M \otimes_R Q$ is flat.

Proof. It suffices to show that the functor $(M \otimes_R Q) \otimes\_$ preserves monomorphism.
For any monomorphism$0 \to L \to N$, applying the functor$Q \otimes_{R_}$, one gets an exact sequence $0 \to Q \otimes_R L \to Q \otimes_R N$ (*) since Q is flat. Then applying the functor $M \otimes_{R\_}$ to the sequence (*), then one gets the following exact sequence $0 \to M \otimes_R Q \otimes_R L \to M \otimes_R Q \otimes_R N$ since M is also flat. The assertion follows.

Theorem 3.2 Let M be a Gorenstein flat R-module and S any multiplicatively closed subset of R. Then $S^{-1}M$ is Gorenstein flat if and only if $\mathrm{Tor}_i^{S^{-1}R}(F, S^{-1}M) = 0$ for any injective $S^{-1}R$-module F and $i \geq 1$.

Proof. The necessity follows from Definition 2.1. We only need to show the sufficiency.
Since M is Gorenstein flat, by Definition 2.1, one gets the following exact sequence of flat R-modules:

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

such that$M = \mathrm{Ker}(F^0 \to F^1)$ and$\mathrm{Tor}_i^R(E, M) = 0$ hold for any injective module R and $i \geq 1$. By Propositions 2.3 and 2.4, applying the functor $S^{-1}R \otimes\_$, we get the following exact sequence of flat $S^{-1}R$-module:

$$\cdots \to S^{-1}F_1 \to S^{-1}F_0 \to S^{-1}F^0 \to S^{-1}F^1 \to \cdots$$

with $S^{-1}M = \mathrm{Ker}(S^{-1}F^0 \to S^{-1}F^1)$. Then one gets the assertion easily.

Corollary 3.3 Let $p$be an arbitrary prime ideal of R and M a Gorenstein flat R-module. Then M_p is a Gorenstein flat R_P-module if and only if $\mathrm{Tor}_i^{R_p}(F, M_p) = 0$ for any injective R_p-module F and $i \geq 1$.

Proof. This is a straight result of Theorem 3.2.

In particular, we have the following corollary.

Corollary 3.4 Let p be an arbitrary prime ideal of R and M a Gorenstein flat R-module. If every injective R_P-module F is of the form E_p for some injective R-module E, then M_p is a Gorenstein flat R_P-module.

Proof. By Corollary 3.3, we only have to show $\mathrm{Tor}_i^{R_p}(F, M_p) = 0$ for any injective R_p-module F and $i \geq 1$. That is, $\mathrm{Tor}_i^{R_p}(E_p, M_p) = 0$ for $i \geq 1$. Then by Proposition 2.5, one gets the assertion.

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References

[1] M. F. Atiyah, I. G. Macdonald. Introduction to Commutative Algebra. Massachusetts：Addison Wesley Publishing Company, 1996.

[2] H. Matsumura. Commutative algebra, Reading, Massachusetts: The Benjamin/Cummings Publishing Company, 1980 .

[3] E. Enochs, O. Jenda. Relative Homological Algebra. Massachusetts：Walter de Gruyter · Berlin · New York, 2000.

[4] W. T. Tong. An introduction to homology algebra. Higher Education Press, 1996.

[5] J. J. Rotman. An Introduction to Homological Algebra. Berlin: Springer, 2008.

[5] M. B. Nathanson. Elementary Methods in Number Theory. Berlin：Springer.

[6] X. Y. Yang, Gorenstein injective and projective and flat modules. Journal of Northwest Normal University, 2009:400-405.

[7] E. Enochs, O. Jenda. Gorenstein injective and projective modules. Mathematische Zeitschrift, 1995, 220(1): 611-633.

[8] Z. K. Liu, Y. P. Liu. Strongly n-gorenstein c-projective modules and injective modules . Journal of Northwest Normal University, 2013, 49(4):1-5.

[9] Y. Zhuang. X. J. Zhang. Z. Y. Zhu. The Determination of when the Fractions of Z_n are Fields. Journal of Anhui Normal University (Natural Science), 2019, 42(3): 227-229.