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Numerical Method for the Solution of Abel's Integral Equations using Laguerre Wavelet

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Abstract: Laguerre wavelet based numerical method is developed for the solution of Abel's integral equations. This method is based on Laguerre wavelets basis. Laguerre wavelet method is then utilized to reduce the Abel's Integral Equations into the solution of algebraic equations. Illustrative examples are shows that the validity, efficiency and applicability of the proposed technique. This algorithm provides high accuracy and compared with other existing methods.

Keywords: Abel's integral equations; Laguerre wavelets; Collocation method.

1. Introduction

Wavelets theory is a relatively new and an emerging tool in applied mathematical research area. It has been applied in a wide range of engineering disciplines; particularly, signal analysis for waveform representation and segmentations, time-frequency analysis and fast algorithms for easy implementation. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms [1, 2]. Since from 1991 the various types of wavelet method have been applied for numerical solution of different kinds of integral equation, a detailed survey on these papers can be found in [3]. Such as Lepik et al. [3] applied the Haar wavelets. Maleknejad et al. proposed Legendre wavelets [4], Rationalized haar wavelet [5], Hermite Cubic splines [6], Coifman wavelet as scaling functions [7]. Yousefi et al. [8] have introduced a new CAS wavelet. Shiralashetti and Mundewadi [9] applied the Bernoulli wavelet for the numerical solution of Fredholm integral equations.

Abel's integral equation finds its applications in various fields of science and engineering. Such as microscopy, seismology, semiconductors, scattering theory, heat conduction, metallurgy, fluid flow, chemical reactions, plasma diagnostics, X-ray radiography, physical electronics, nuclear physics [10-12].

In 1823, Abel, when generalizing the tautochrone problem derived the following equation:

$$\int_0^x \frac{y(t)}{\sqrt{x-t}}dt = f(x), \qquad 0 \le x,t \le 1 \tag{1.1}$$

where f(t) is a known function and y(t) is an unknown function to be determined. This equation is a particular case of a linear Volterra integral equation of the first kind. For solving Eq. (1.1) different numerical based methods have been developed over the past few years, such as product integration methods [13, 14], collocation method [15], homotopy analysis transform method [16]. The generalized Abel's integral equations on a finite segment appeared for the first time in the paper of Zeilon [17]. Baker [18] studied the numerical treatment of integral equations. Operational matrix method based on block-pulse functions for singular integral equations [19]. Baratella and Orsi [20] applied the product integration to solve the numerical solution of weakly singular volterra integral equations. Some of the author's, have solved for Abel's integral equations using the wavelet based methods, such as Legendre wavelets [21] and Chebyshev wavelets [22]. Shahsavaran et al [23] has solved Abel's integral equation of the first kind using piecewise constant functions and Taylor expansion by collocation method. Shiralashetti [24] Theoretical study on continuous polynomial wavelet bases through wavelet series collocation method for nonlinear Lane–Emden type equations. Shiralashetti [25] applied the Laguerre wavelets collocation method for the numerical solution of the Benjamina–Bona–Mohany equations. In this paper, we introduced the Laguerre wavelets based numerical method for solving Abel's integral equations of first and second kind.

The article is organized as follows: In Section 2, the basic formulation of Laguerre wavelets and the function approximation is presented. Section 3 includes the convergence and error analysis. Section 4 is

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devoted the method of solution. In section 5, numerical results are demonstrated the accuracy of the proposed method by some of the illustrative examples. Lastly, the conclusion is given in section 6.

2. Properties of Laguerre wavelet

2.1 Wavelets

In recent years, wavelets have found their way into many different fields of science and engineering. Wavelets constitute a family of functions constructed from dialation and translation of a single function called mother wavelet. When the dialation parameter a and translation parameter b varies continuously, we have the following family of continuous wavelets:

$$\psi_{a,b}(x) = |a|^{-1/2}\psi\left(\frac{x-b}{a}\right), \forall\, a, b\ \in R, a \neq 0.$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}, b = nb_0a_0^{-k}, a_0 > 1, b_0 > 0$. We have the following family of discrete wavelets

$$\psi_{k,n}(x) = |a|^{1/2}\psi\left(a_0^k x - nb_0\right), \forall\, a, b\ \in R, a \neq 0.$$

where $\psi_{k,n}(x)$ form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2\ and\ b_0 = 1$ then $\psi_{k,n}(x)$ forms an orthonormal basis.

2.2 Laguerre Wavelets

Laguerre wavelets $\psi_{n,m}(x) = \psi(k,n,m,x)$ have four arguments; $n = 1,2,3,\dots,2^{k-1}$, k can assume any positive integer, m is the order of the Laguerre polynomials and x is the normalized time. They are defined on the interval [0, 1) as:

$$\psi_{n,m}(x) = \begin{cases} 2^{k/2}\overline{L}_m\left(2^k x - 2n + 1\right), \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}}, \\ 0, \qquad otherwise \end{cases} \tag{2.1}$$

where $\overline{L_m}(x) = \frac{L_m}{m!}$, m = 0,1,2,...,M-1. In Eq. (2.1) the coefficients are used for orthonormality.
Here, $L_m(x)$ are the well-known Laguerre polynomial of order m with respect to the weight function w(x) = 1 on the interval [0, ∞) and satisfy the following recursive formula,

$$L_0(x) = 1,$$

$$L_1(x) = 1 - x,$$

$$L_{m+2}(x) = \frac{(2m+3-x)L_{m+1}(x) - (m+1)L_m(x)}{m+2}, m = 0,1,2 \dots$$

2.3 Function Approximation

A function $f(x)$ defined over [0, 1) can be expanded as a Laguerre wavelet series as follows:

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(x), \tag{2.2}$$

where, $C_{n,m}$ denotes inner product of $f(x)$ and $\psi_{n,m}(x)$

$$\text{i.e., } C_{n,m} = \left(f(x), \psi_{n,m}(x)\right). \tag{2.3}$$

If the infinite series in (2.2) is truncated, then (2.2) can be rewritten as:

$$f(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x) = C^T\Phi(x), \tag{2.4}$$

where C and $\Phi(x)$ are $2^{k-1}M \times 1$ matrices given by:

$$\mathrm{C} = \left[\mathrm{c}_{10}, \mathrm{c}_{11}, \dots, \mathrm{c}_{1,\mathrm{M}-1}, \mathrm{c}_{20}, \dots, \mathrm{c}_{2,\mathrm{M}-1}, \dots, \mathrm{c}_{2^{\mathrm{k}-1},0}, \dots, \mathrm{c}_{2^{\mathrm{k}-1},\mathrm{M}-1}\right]^{\mathrm{T}}$$
$$= [\mathrm{c}_1, \mathrm{c}_2, \dots, \mathrm{c}_{2^{\mathrm{k}-1},\mathrm{M}}]^{\mathrm{T}}, \tag{2.5}$$

$$\Phi(x) = [\psi_{10}(x), \psi_{11}(x), \dots, \psi_{1,M-1}(x), \psi_{20}(x), \dots, \psi_{2,M-1}(x), \quad \dots, \psi_{2^{k-1},0}(x), \dots, \psi_{2^{k-1},M-1}(x)]^T$$
$$= [\psi_1(x), \psi_2(x), \dots, \psi_{2^{k-1},M}(x)]^T. \tag{2.6}$$

3. Convergence and Error Analysis

Theorem 3.1. A Continuous function $y(x)$ in $L^2[0,1)$ be bounded then the Laguerre wavelets expansion of $y(x)$ converges to it.

Proof: Let $y(x)$ be a bounded real valued function on [0,1) the Laguerre coefficients of continuous functions $y(x)$ is defined as,

$$c_{n,m} = \int_0^1 y(x)\psi_{n,m}(x)\,dx$$

$$c_{n,m} = \int_I \; y(x)\frac{2^{\frac{k}{2}}}{m!}L_m(2^k x - 2n + 1)\,dx, \quad \text{where } I = \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right).$$

Put $2^k x - 2n + 1 = z$.

$$c_{n,m} = \int_{-1}^{1} \frac{2^{\frac{k}{2}}}{m!} y\left(\frac{z-1+2n}{2^k}\right) L_m(z) 2^{-k}\,dz$$

$$= \frac{2^{\frac{-k}{2}}}{m!}\int_{-1}^{1} y\left(\frac{z-1+2n}{2^k}\right) L_m(z)\,dz$$

by GMVT(Generalized Mean Value Theorem) for integrals

$$= \frac{2^{\frac{-k}{2}}}{m!} y\left(\frac{w-1+2n}{2^k}\right)\int_{-1}^{1} L_m(z)dz, \qquad for\ some\ w\epsilon(-1,1).$$

$$put \int_{-1}^{1} L_m(z)dz = h.$$

$$|c_{n,m}| = \left|\frac{2^{\frac{-k}{2}}}{m!}\right| \left|y\left(\frac{w-1+2n}{2^k}\right)\right| h$$

Since y is bounded, therefore $\sum_{n,m=0}^{\infty} C_{n,m}$ is absolutely convergent.
Hence the Laguerre series expansion of $y(x)$ converges Uniformly.

Theorem 3.2. Suppose that $y(x) \in C^m[0,1]$ and $C^T\Phi(x)$ is the approximate solution using Laguerre wavelet. Then the error bound would be given by,

$$\|E(x)\| \le \left\|\frac{2}{m!4^m 2^{m(k-1)}} \max_{x\in[0,1]} |y^m(x)|\right\|.$$

Proof: Applying the definition of norm in the inner product space, we have,

$$\|E(x)\|^2 = \int_0^1 [y(x) - C^T\Phi(x)]^2\,dx.$$

Divide interval [0, 1] into 2^{k-1} subintervals $I_n = \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right), n = 1,2,3,\dots,2^{k-1}$.

$$\|E(x)\|^2 = \sum_{n=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} [y(x) - C^T\Phi(x)]^2\,dx.$$

$$\|E(x)\|^2 = \sum_{n=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} [y(x) - P_m(x)]^2\,dx.$$

Where $P_m(x)$ is the interpolating polynomial of degree m which approximates $y(x)$ on I_n.
By using the maximum error estimate for the polynomial on I_n, then

$$\|E(x)\|^2 \le \sum_{n=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left[\frac{2}{m!\,4^m 2^{m(k-1)}} \max_{x\in I_n} |y^m(x)|\right]^2 dx.$$

$$\|E(x)\|^2 = \int_0^1 \left[\frac{2}{m!\,4^m 2^{m(k-1)}} \max_{x\in[0,1]} |y^m(x)|\right]^2 dx.$$

$$\|E(x)\|^2 \le \left\|\frac{2}{m!\,4^m 2^{m(k-1)}} \max_{x\in[0,1]} |y^m(x)|\right\|$$

which, we have used the well-known maximum error bound for the interpolation.

4. Method of Solution

Consider the Abel integral equation of the form,

$$\lambda y(x) = f(x) + \int_0^x \frac{y(t)}{\sqrt{x-t}} dt, \quad 0 \le x, t \le 1 \tag{4.1}$$

where $\lambda = 0$ or λ=1. We first approximate y(x) as truncated series defined in Eq. (2.4). That is,

$$y(x) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x) = C^T \phi(x) \tag{4.2}$$

where C and $\phi(x)$ are defined similarly to Eqs. (2.5) and (2.6). Then substituting Eq. (4.2) in Eq. (4.1), we get

$$\lambda C^T \phi(x) = f(x) + \int_0^x \frac{C^T \phi(t)}{\sqrt{x-t}} dt \tag{4.3}$$

Now assume Eq.(4.3) is precise at following collocation points $x_i = \frac{2i-1}{2^k M}, i = 1,2, \dots, 2^{k-1}M$. Then we obtain,

$$\lambda C^T \phi(x_i) = f(x_i) + \int_0^{x_i} \frac{C^T \phi(t)}{\sqrt{x_i - t}} dt. \tag{4.4}$$

Now, we get system of algebraic equations with unknown coefficients. By solving this system of equations, we get Laguerre wavelet coefficients and then substituting these coefficients in Eq. (4.2), we get the approximate solution of Eq. (4.1).

5. Illustrative examples

In this section, we present Laguerre wavelet based method for the numerical solution of Abel's integral equations to demonstrate the capability of the present method.

Example 1. Consider the Abel's integral equation of first kind [22],

$$\frac{2}{105}\sqrt{x}(105 - 56x^2 + 48x^3) = \int_0^x \frac{y(t)}{\sqrt{x-t}} dt \tag{5.1}$$

We apply the present method to solve Eq. (5.1) with k = 1 and M = 4. Then we get truncating approximate solution with unknowns as,

$$y(x) \approx \sum_{m=0}^{3} c_{1,m}\psi_{1,m}(x) = C^T \Psi(x) \tag{5.2}$$

Then applying the procedure discussed in the section 3. We get a system of four algebraic equations with four unknowns and solving this system, we obtain the Laguerre wavelet coefficients as, $c_{1,0} = \frac{-43\sqrt{2}}{8}, c_{1,1} = \frac{-21\sqrt{2}}{16}, c_{1,2} = \frac{5\sqrt{2}}{4}, c_{1,3} = \frac{-9\sqrt{2}}{4}$and substituting in Eq. (5.2), we obtain:

$$y(x) = \frac{-43\sqrt{2}}{8}\psi_{10}(x) - \frac{21\sqrt{2}}{16}\psi_{11}(x) + \frac{5\sqrt{2}}{4}\psi_{12}(x) - \frac{9\sqrt{2}}{4}\psi_{13}(x)$$

On simplifying, we get $y(x) = x^3 - x^2 + 1$, which is exact solution of Eq. (5.1).

Example 2. Consider the Abel's integral equation of the first kind [22],

$$x = \int_0^x \frac{y(t)}{\sqrt{x-t}}\, dt. \tag{5.2}$$

which has the exact solution $y(x) = \frac{2}{\pi}\sqrt{x}$. We solved the Eq. (5.2) using the present method and obtained approximate solution is compared with exact and other existing methods which reflects in table 1 and figure 1. Error analysis is shown in table 2 and figure 2.

Table 1. Numerical results for Example 2.

x	Exact solution	Present method (k = 1, M = 8)	Method [22] (k = 1, M = 8)	Method [23] (m = 16)
0.1	0.201317	0.199701	0.200128	0.200460
0.2	0.284705	0.284883	0.286092	0.297987
0.3	0.348691	0.348550	0.347394	0.337588
0.4	0.402634	0.402578	0.404161	0.405769
0.5	0.450158	0.450155	0.449568	0.464014
0.6	0.493124	0.493043	0.492704	0.490550
0.7	0.532634	0.532679	0.532315	0.539721
0.8	0.569410	0.569339	0.569156	0.562698
0.9	0.603951	0.603611	0.603742	0.606044

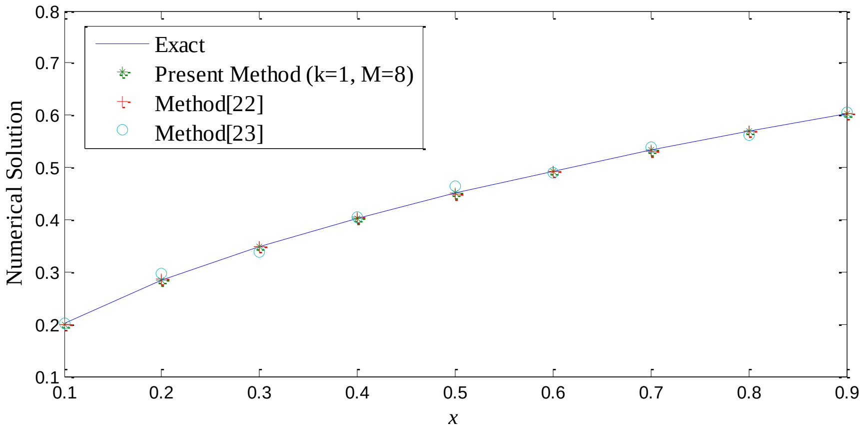

Figure 1. Comparison of numerical solutions with exact solutions of example 2.

Table 2. Error analysis of example 2.

x	Present method (k = 1, M = 8)	Method [22] (k = 1, M = 8)	Method [23] (m = 16)
0.1	1.61e-03	1.18e-03	8.576e-04
0.2	1.78e-04	1.38e-03	1.32e-02
0.3	1.40e-04	1.29e-03	1.11e-02
0.4	5.50e-05	1.52e-03	3.13e-03
0.5	3.14e-06	5.90e-04	1.38e-02
0.6	8.00e-05	4.19e-04	2.57e-03
0.7	4.56e-05	3.19e-04	7.08e-03
0.8	7.10e-05	2.54e-04	6.71e-03
0.9	3.38e-04	2.08e-04	2.09e-03

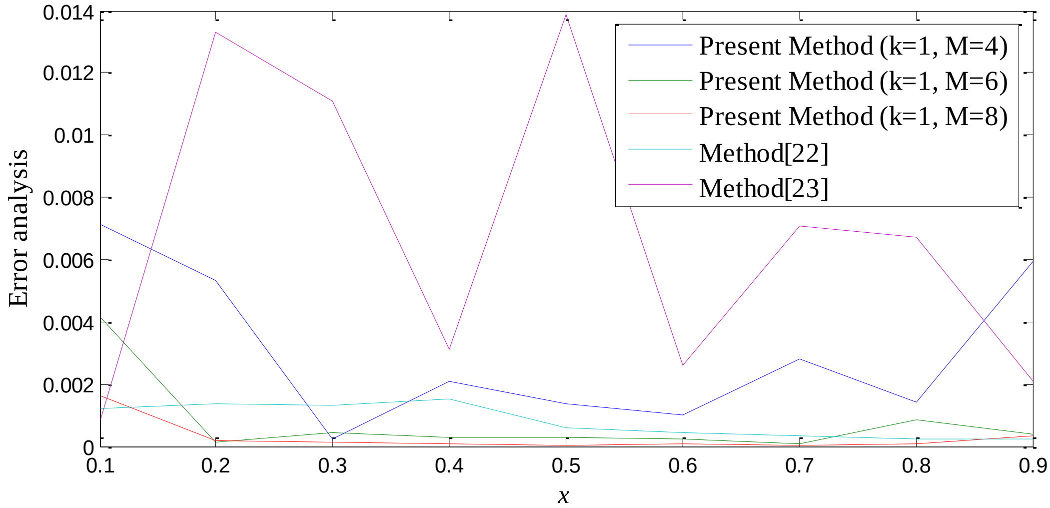

Figure 2. Comparison of error analysis of example 2.

Example 3. Consider the following Abel's integral equation of the second kind [22],

$$4y(x) = \frac{4}{\sqrt{x+1}} - arcsin\left(\frac{1-x}{1+x}\right) + \frac{\pi}{2} - \int_0^x \frac{y(t)}{\sqrt{x-t}}dt,\ \ 0 \leq x < 1. \tag{5.3}$$

which has the exact solution $y(x) = \frac{1}{\sqrt{x+1}}$. Applying the Laguerre wavelet method for solving Eq. (5.3), then obtained approximate solution is compared with the exact solution and method in [23] as shown in table 3 and figure 3. Error analysis is shown in table 4.

Table 3: Numerical results for example 3.

x	Exact solution	Present method ($k = 1, M = 8$)	Method [23] ($m = 16$)
0.1	0.953462589245592	0.953464128446307	0.95646081381695
0.2	0.912870929175277	0.912871888201732	0.90601007037324
0.3	0.877058019307029	0.877058847823016	0.88361513925322
0.4	0.845154254728517	0.845154959209069	0.84340093819493
0.5	0.816496580927726	0.816497101668559	0.80822420481499
0.6	0.790569415042095	0.790569829421686	0.79221049469412
0.7	0.766964988847370	0.766965234716791	0.76284677221990
0.8	0.745355992499930	0.745356093607430	0.74933888037055
0.9	0.725476250110012	0.725476461420005	0.72434536240934

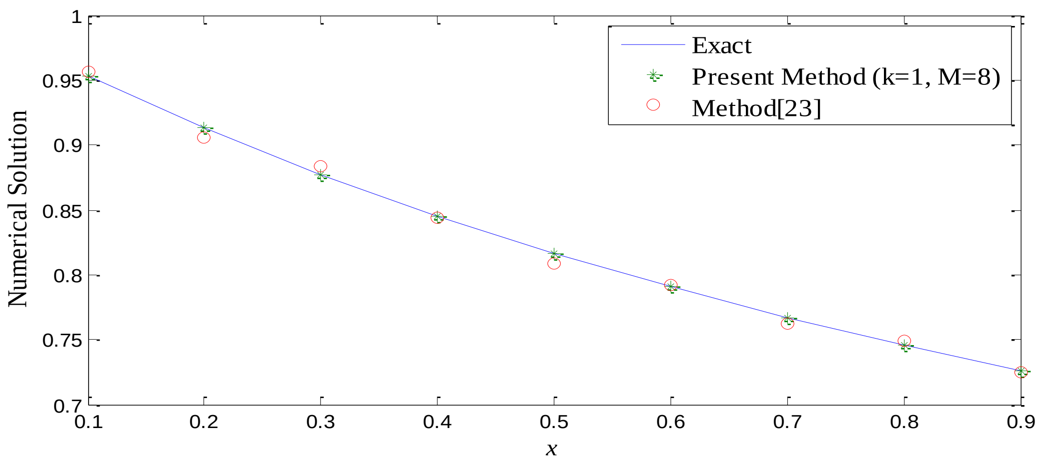

Figure 3. Comparison of Numerical solutions for example 3.

Table 4. Error analysis of example 3.

x	Present method (k = 1, M = 8)	Method [23] (m = 16)
0.1	1.53e-06	2.99e-03
0.2	9.59e-07	6.86e-03
0.3	8.28e-07	6.55e-03
0.4	7.04e-07	1.75e-03
0.5	5.20e-07	8.27e-03
0.6	4.14e-07	1.64e-03
0.7	2.45e-07	4.11e-03
0.8	1.01e-07	3.98e-03
0.9	2.11e-07	1.13e-03

Example 4. Consider the Abel's integral equations of the second kind [22],

$$y(x) = 2\sqrt{x} - \int_0^x \frac{y(t)}{x-t} dt \tag{5.4}$$

which has the exact solution $y(x) = 1 - \exp(\pi x)\, erfc(\sqrt{\pi x})$. We solved the Eq. (5.4) by the present method, we get the approximate solution and is compared with exact and other existing methods as shown in table 5 and figure 4. Error analysis is shown in table 6 and figure 5.

Table 5: Numerical results for example 4.

x	Exact solution	Present method (k = 1, M = 8)	Method [22] (k = 0, M = 16)	Method [23] (m = 16)
0.1	0.414059	0.408765	0.415689	0.402472
0.2	0.508352	0.507275	0.505528	0.519751
0.3	0.564309	0.563442	0.566205	0.554755
0.4	0.603347	0.602653	0.601908	0.605031
0.5	0.632868	0.632420	0.634188	0.640487
0.6	0.656323	0.655868	0.655109	0.654785
0.7	0.675601	0.675301	0.676588	0.678700
0.8	0.691842	0.691583	0.691596	0.688860
0.9	0.705787	0.705067	0.704377	0.706495

Table 6: Error analysis of example 4.

x	Present method (k = 1, M = 8)	Method [22] (k = 0, M = 16)	Method [23] (m = 16)
0.1	5.29e-03	1.62e-03	1.15e-02
0.2	1.07e-03	2.82e-03	1.13e-02
0.3	8.65e-04	1.89e-03	9.55e-03
0.4	6.93e-04	1.43e-03	1.68e-03
0.5	4.47e-04	1.32e-03	7.61e-03
0.6	4.55e-04	1.21e-03	1.53e-03
0.7	2.99e-04	9.86e-04	3.09e-03
0.8	2.58e-04	2.45e-04	2.98e-03
0.9	7.19e-04	1.40e-03	7.08e-04

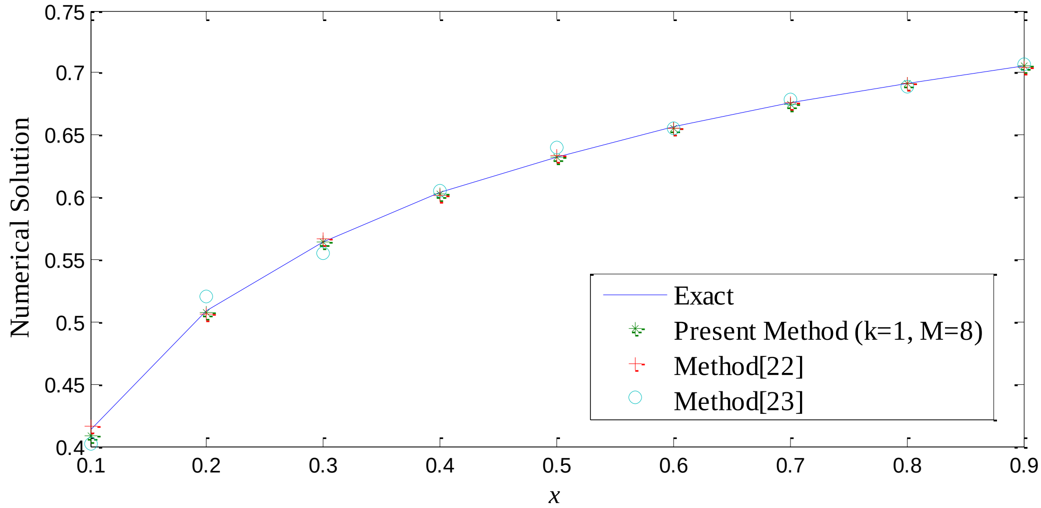

Figure 4. Comparison of Numerical solutions of example 4.

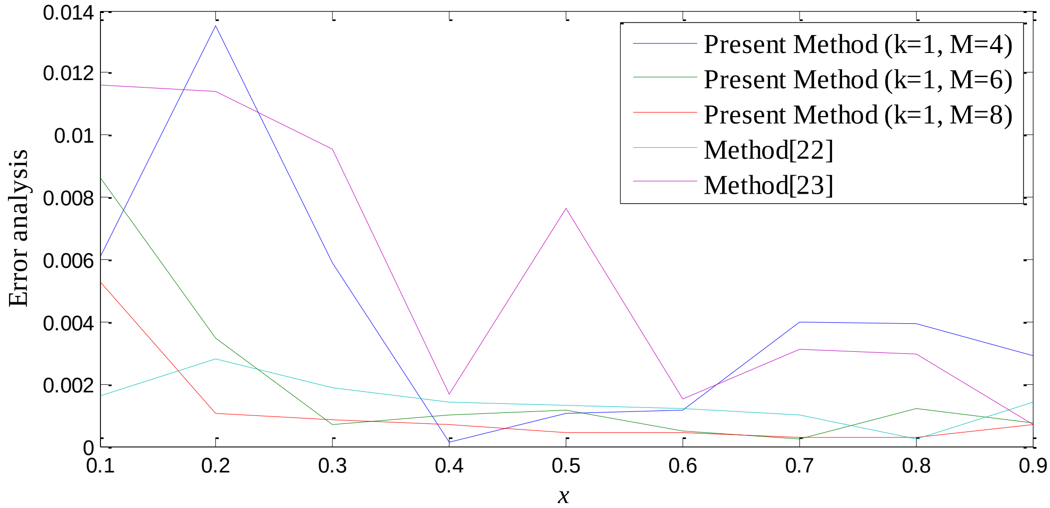

Figure 5. Comparison of Error analysis of example 4.

6. Conclusion

The Laguerre wavelet method is applied for the numerical solution of Abel's integral equations. The present method reduces an integral equation into a set of algebraic equations. Obtained results are higher accuracy with exact ones and existing methods [22, 23], which can be observed in section 4. The numerical result shows that the accuracy improves with increasing the values of M for better accuracy. Convergence theorem reveals that existence of solution.

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