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The bifurcation control for a Lorenz system with time delay

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Abstract. The main aim of this manuscript is to investigate the control issue for a Lorenz system with time delay applying a hybrid control method based on state feedback and parameter perturbation. By choosing the time delay as the bifurcation parameter and utilizing stability theory for delay differential equation, the local stability in two different cases with time delay equal to 0 and not equal to 0 is discussed. With the help of Hopf bifurcation theorem, the beingness of Hopf bifurcation is established by combining the distribution results of the characteristic roots. And the hybrid control method can availably postpone the Hopf bifurcation by numerical simulations.

Keywords: Lorenz system, time delay, hybrid control, stability, Hopf bifurcation.

1. Introduction

Chaos and bifurcation control have been extensively studied in the domains of physics, mathematics, biology, and engineering in the last years. When chaos has deleterious consequences in some engineering applications, then chaos can be eliminated by control, which shows great potential in many practical use. For example, information processing, power system protection, biomedical systems, encryption and communication, etc [1-6]. The mainly explored emphasis of the bifurcation control is how to delay or eliminate the bifurcation phenomenon, in order to avoid negative consequences and purposefully establish or strengthen beneficial bifurcations for people to employ.

At present, the commonly utilized control method is feedback control method [7-9]. Cheng and Cao [7] shown the control issue of Hopf bifurcation of a delayed complex networks system. In [8], Ou et al. discussed a linear feedback controller with state variables, which is used to control the equilibrium point and periodic orbit of Lorenz system. In 2003, Luo et al. [10] firstly put forward a completely new control measure, which described by the state feedback and parameter perturbation are linked to govern the period-doubling bifurcation and chaos of discrete nonlinear systems. Liu [11] further analyzed the continuous system without time delay by utilizing a hybrid control strategy. Peng and Zhang [12-13] presented the Hopf bifurcation control of two predator-prey models by applying hybrid control method.

2. The delayed Lorenz system with control

In [14], Lian et al. proposed a Lorenz system with a time delay:

$$\begin{cases} \dot{x} = a(y-x), \\ \dot{y} = bx(t-\tau) - xz, \\ \dot{z} = -cz + dx^2, \end{cases} \tag{1}$$

where x, y, z are state variables, a, b, c, d are parameters of system above, τ denotes time delay, which can be understood as the hunting delay of predator to prey or delay time of signal transmission, etc. They discussed the corresponding bifurcation of system (1).

In this manuscript, in the light of the discussions above and inspired by Lian et al. [14], we design a controller which is devoted to delay the Hopf bifurcation for system (2), the corresponding mathematical model is described as:

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$$\begin{cases} \dot{x} = p[a(y-x)] + qx, \\ \dot{y} = p[bx(t-\tau) - xz] + qy, \\ \dot{z} = p[-cz + dx^2] + qz, \end{cases} \tag{2}$$

where $p > 0$ and $q \in R$ is control parameter.

This manuscript is arranged as below. In the next Section, the beingness and local stability of equilibrium point are qualitatively analyzed and the intrinsic bifurcation is postponed by discussing the relevant characteristic equation. In Section 4, we check the effectiveness and correctness of theoretical analysis by using numerical simulations. Finally, a concise conclusion is provided.

3. Existence, stability of equilibrium and Hopf bifurcation analysis

3.1. Existence of equilibrium

When $x^2 = \dfrac{abcp^3 + (ac-ab)p^2q - (a+c)pq^2 + q^3}{adp^3} \geq 0$, the system (2) has following three equilibrium: $E^*(0,0,0)$, $E^1(x,(1-\dfrac{q}{ap})x,\dfrac{pdx^2}{pc-q})$, $E^2(-x,-(1-\dfrac{q}{ap})x,\dfrac{pdx^2}{pc-q})$.

When $x^2 = \dfrac{abcp^3 + (ac-ab)p^2q - (a+c)pq^2 + q^3}{adp^3} < 0$, the system (2) has unique equilibrium $E^*(0,0,0)$.

3.2. Stability of equilibrium and Hopf bifurcation analysis

Here, we analyze the stability of system (2) at the equilibrium $E^*(0,0,0)$.

By linearizing the system (2), then we obtain

$$\begin{cases} \dot{x} = apy + (q-ap)x, \\ \dot{y} = pbx(t-\tau) + qy, \\ \dot{z} = (q-pc)z. \end{cases} \tag{3}$$

The Jacobian matrix of linearized system (3)can be written by

$$\det \begin{vmatrix} q-ap-\lambda & ap & 0 \\ pbe^{-\lambda\tau} & q-\lambda & 0 \\ 0 & 0 & q-pc-\lambda \end{vmatrix} = 0.$$

Then the characteristic equation is given by

$$\lambda^3 + m_1\lambda^2 + m_2\lambda + m_3 + (n_1\lambda + n_2)e^{-\lambda\tau} = 0, \tag{4}$$

with $m_1 = ap - 3q + cp$, $m_2 = acp^2 - 2apq - 2cpq + 3q^2$,

$m_3 = -acp^2q + apq^2 + cpq^2 - q^3$, $n_1 = -abp^2$, $n_2 = abp^2q - abcp^3$.

Next, we will discuss the local stability of the equilibrium and the conditions when Hopf bifurcation occurs. Owing to the existence of time delay in the system (2), the following two cases are considered.

Case 1: $\tau = 0$. The characteristic equation (4) becomes

$$\lambda^3 + m_1\lambda^2 + (m_2+n_1)\lambda + m_3 + n_2 = 0. \tag{5}$$

If the condition (H1) $m_1 > 0, m_3+n_2 > 0, m_1(m_3+n_2) > m_2+n_1$ is satisfied, then the total roots of Eq.(5) possess negative real parts.

Hence, when the condition (H1) holds, the equilibrium point $E^*(0,0,0)$ is locally asymptotically stable on the basis of Routh-Hurwitz criteria.

Case 2: $\tau \neq 0$.

Let iw_1 ($w_1 > 0$) be the root of Eq. (4), substituting it into the Eq.(4) and separating the real and imaginary part, we obtain

$$\begin{cases} n_1 w \cos w\tau - n_2 \sin w\tau = w^3 - m_2 w, \\ n_1 w \sin w\tau + n_2 \cos w\tau = m_1 w^2 - m_3. \end{cases} \tag{6}$$

Taking square of both sides of Eq. (6), this implies that

$$w^6 + e_1 w^4 + e_2 w^2 + e_3 = 0, \tag{7}$$

with $e_1 = m_1^2 - 2m_2, e_2 = m_2^2 - 2m_1 m_3 - n_1^2, e_3 = m_3^2 - n_2^2.$

Make $w^2 = v$, Eq.(7) is represented the following form:

$$v^3 + e_1 v^2 + e_2 v + e_3 = 0. \tag{8}$$

And, define

$$f(v) = v^3 + e_1 v^2 + e_2 v + e_3, \tag{9}$$

According to Eq. (9), and it is easy to know that

$$f'(v) = 3v^2 + 2e_1 v + e_2.$$

Since $f(0) = e_3$, $\lim_{v \to +\infty} f(v) = +\infty$, if the condition $e_3 < 0$ holds, then Eq.(9) possesses at least one positive root.

Utilizing the analysis of zero distribution of the transcendental characteristic equation by Ruan and Wei [15], the relevant consequences are obtained as below.

Lemma 1 For the polynomial equation (4), we get the following conclusions:

(1) If (H21) $e_3 \geq 0, \Delta = e_1^2 - 3e_2 \leq 0$ holds, no positive root exists about Eq.(4);

(2) If (H22) $e_3 \geq 0, \Delta = e_1^2 - 3e_2 > 0, v^* = \dfrac{-e_2 + \sqrt{\Delta}}{3} > 0,\ f(v) \leq 0$ or (H23) $e_3 < 0$ is satisfied, then Eq. (4) possesses positive root.

We assume Eq. (8) exists three positive roots, which described by v_1, v_2 and v_3, then Eq. (7) also possesses three positive roots $w_k = \sqrt{v_k}$, $k = 1, 2, 3$.

In accordance with Eq. (6), we define

$$\tau_k^{(j)} = \frac{1}{w_k} \arccos\left\{ \frac{n_1 w^4 + m_1 n_2 w^2 - m_2 n_1 w^2 - m_3 n_2}{n_1^2 w^2 + n_2^2} \right\} + \frac{2\pi j}{w_k},\ k = 1, 2, 3;\ j = 0, 1, 2, \ldots, \tag{10}$$

Thus $\pm i w_k$ is a pair of purely imaginary roots of Eq. (4) with $\tau = \tau_k^{(j)}$, and define $\tau = \min_{k \in \{1,2,3\}} \{\tau_k^{(0)}\}$, $w_0 = w_{k_0}$.

Let $\lambda(\tau) = \alpha(\tau) + i w(\tau)$ be a root of Eq. (4) satisfying $\alpha(\tau_0) = 0$, $w(\tau_0) = w_0$, and the transversal condition is given below and proved.

Lemma 2 If (H3) $AC + BD \neq 0$ holds, then

$$\left(\frac{d(\mathrm{Re}\,\lambda)}{d\tau} \right)_{\lambda = i w_0} \neq 0 .$$

Proof Substituting $\lambda(\tau)$ into Eq.(4), and differentiating the two sides of Eq.(4) about τ , we get

$$(3\lambda^2 + 2m_1\lambda + m_2 + n_1 e^{-\lambda\tau} - n_1 \tau \lambda e^{-\lambda\tau} - n_2 \tau e^{-\lambda\tau}) \frac{d\lambda}{d\tau} = n_1 e^{-\lambda\tau} \lambda^2 + n_2 e^{-\lambda\tau} \lambda ,$$

then, it implies that

$$(\frac{d\lambda}{d\tau})^{-1} = \frac{3\lambda^2 + 2m_1\lambda + m_2 + n_1 e^{-\lambda\tau}}{n_1 \lambda^2 e^{-\lambda\tau} + n_2 \lambda e^{-\lambda\tau}} - \frac{\tau}{\lambda}. \tag{11}$$

Because of $\lambda^3 + m_1\lambda^2 + m_2\lambda + m_3 = -(n_1\lambda + n_2) e^{-\lambda\tau}$, we can derive that

$$(\frac{d\lambda}{d\tau})^{-1}=\frac{3\lambda^2+2m_1\lambda+m_2+n_1e^{-\lambda\tau}}{-\lambda(\lambda^3+m_1\lambda^2+m_2\lambda+m_3)}-\frac{\tau}{\lambda}.$$

Substituting $\lambda=iw_0$ into above formula, and separating the real part, we obtain

$$\mathrm{Re}(\frac{d\lambda}{d\tau})^{-1}_{\lambda=i\omega_0}=\mathrm{Re}(\frac{A+Bi}{C+Di})=\frac{AC+BD}{C^2+D^2}\ , \tag{12}$$

where

$$A=-3\omega_0{}^2+m_2+n_1\cos\omega_0\tau_0,\ B=2m_1\omega_0-n_1\sin\omega_0\tau_0,$$
$$C=-\omega_0{}^4+m_2\omega_0^2,\ D=m_1\omega_0^3-m_3\omega_0.$$

Noting that $\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=iw_0}$ and $\left(\frac{d(\mathrm{Re}\lambda)}{d\tau}\right)_{\lambda=iw_0}$ have the same sign, then

$$sign\left\{\left(\frac{d(\mathrm{Re}\lambda)}{d\tau_1}\right)\right\}_{\lambda=iw_{10}}=sign\left\{\mathrm{Re}\left(\frac{d\lambda}{d\tau_1}\right)^{-1}\right\}_{\lambda=iw_{10}}\neq 0\,.$$

Therefore, $\left(\frac{d(\mathrm{Re}\lambda)}{d\tau}\right)_{\lambda=iw_0}\neq 0$ if (H3) $AC+BD\neq 0$ satisfies. The proof about the transversal condition is complete.

Therefore, on the basis of Lemma 1.2 and Hopf bifurcation theory [16], the following results can be summarized.

Theorem 1 For the system (2), suppose that (H1) is valid, we have the conclusions as below.

(1) If (H21) is satisfied, then the equilibrium E^* is asymptotically stable for total $\tau\geq 0$.

(2) If (H22) or (H23) holds, then the equilibrium E^* is asymptotically stable for total $\tau\in[0,\tau_0)$.

(3) If condition (2) above and (H3) hold, the equilibrium E^* is unstable for $\tau>\tau_0$. Furthermore, a Hopf bifurcation happens at the equilibrium E^* when $\tau=\tau_0$.

4. Numerical simulation

In this part, for the sake of to prove the availability of the theoretical analysis above, the numerical example by Matlab software is presented.

To easy to compare, choosing a same group of parameters in [14] $a=10, b=-8, c=2.5, d=4$. When $p=1.0, q=0$, the system (2) becomes uncontrolled system, the corresponding bifurcation analysis has been studied in [14].

We take the parameters $p=0.3, q=0.1$, then the system (2) is represented by the following form:

$$\begin{cases}\dot{x}=0.3[10(y-x)]+0.1x,\\ \dot{y}=0.3[-8x(t-\tau)-xz]+0.1y,\\ \dot{z}=0.3[-2.5z-4x^2]+0.1z.\end{cases} \tag{13}$$

By calculation, we obtain $x^2=\frac{abcp^3+(ac-ab)p^2q-(a+c)pq^2+q^3}{adp^3}=-4.1588<0$, then system (13) has unique equilibrium $E^*(0,0,0)$. When $\tau=0$, condition (H2) is satisfied and then the equilibrium E^* of the system (13) is asymptotically stable. For $\tau\neq 0$, we get $w_0=2.0311, \tau_0=0.4483$. According to the Theorem 1, when $\tau_0=0.40$, the equilibrium E^* is asymptotically stable; for $\tau_0=0.4483$, the stability of system is disappear, and the Hopf bifurcation occurs at the equilibrium E^*; for $\tau_0=0.46$, the equilibrium E^* is unstable, which are shown in Figure 1, 2, 3, respectively.

Remark: For the convenience of observing the limit cycles, Fig. 2 gives the xoy plane phase diagram when $\tau_0 = 0.4483$.

In the light of the hybrid control method, and comparing Figures 1, 2 and 3 in this manuscript with Figures 2, 3 and 4 in [14], it is clearly discovered that the Hopf bifurcation is postponed. And, we find that the time delay increases from $\tau_0 = 0.1477$ to $\tau_0 = 0.4483$via computation.

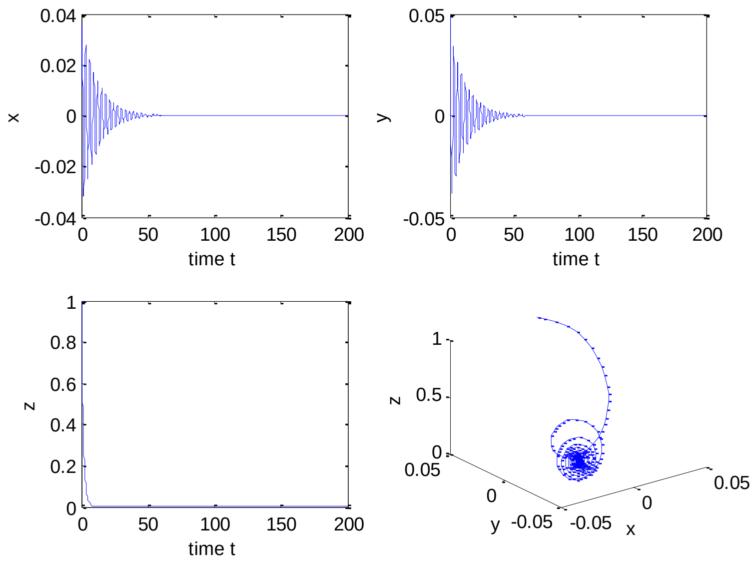

Fig. 1: The equilibrium E^* of the controlled system (13) is locally asymptotically stable when $\tau_0 = 0.40 < 0.4483$.

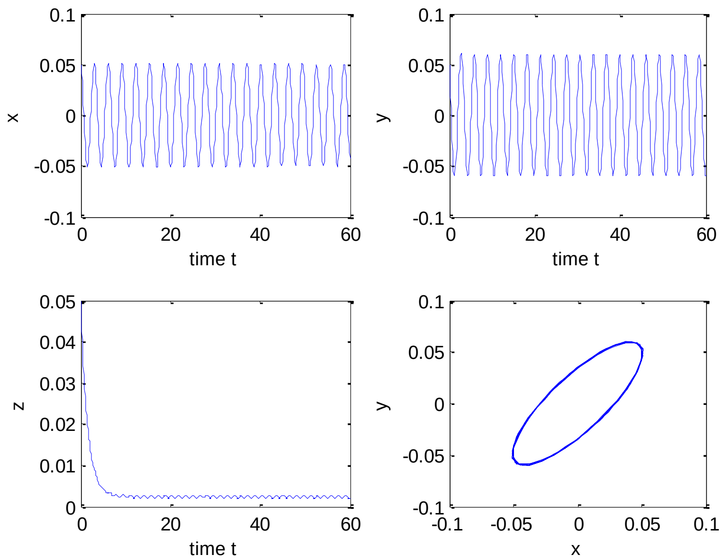

Fig. 2: The equilibrium E^* of the controlled system (13) undergoes a Hopf bifurcation when $\tau_0 = 0.4483$.

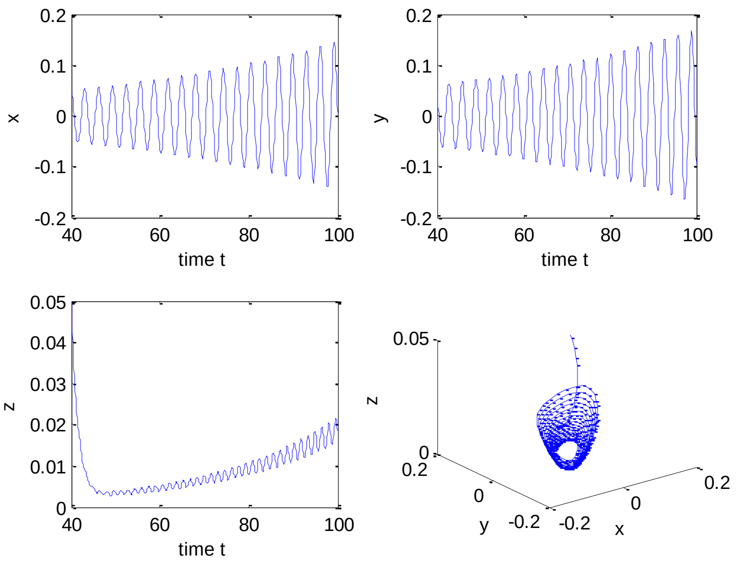

Fig. 3: The equilibrium E^* of the controlled system (13) is unstable when $\tau_0 = 0.46 > 0.4483$.

5. Conclusion

In this manuscript, the bifurcation control of a delayed Lorenz system is explored. By utilizing the hybrid control strategy and comparing Reference [14] with this work, it is observed that the unstable equilibrium of uncontrolled system in [14] turns into asymptotically stable in controlled system (13). That is to say, the hybrid control method can postpone the bifurcation. The simulation numerical results check the effectiveness of theoretical analysis. Compared to the regular state feedback method, hybrid control method obviously further explores the control effect of parameter perturbation. This method can postpone the bifurcation point of original system, so as to better control the bifurcation.

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