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A construction of special self-orthogonal Latin squares based on frequency squares

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Abstract: Let $n = p^k$, where p is a prime and $k \geq 2$. In this paper, a construction for weakly pandiagonal strongly symmetric self-orthogonal diagonal Latin squares of order n is given by using frequency squares over finite field $of\ order\ p$. It is proved that there exists a weakly pandiagonal strongly symmetric self-orthogonal diagonal Latin square of order n for $n > 4$.

Keywords: Latin square, frequency square, self-orthogonal, strongly symmetric, weakly pandiagonal.

1. Introduction

A Latin square of order n is an $n \times n$ array such that every row and every column is a permutation of an n-set S. A transversal in a Latin square is a set of positions, one per row and one per column, among which the symbols occur precisely once each. A diagonal Latin square is a Latin square with the additional property that the main diagonal and back diagonal are both transversals.

Two Latin squares of order n are orthogonal if each symbol in the first square meets each symbol in the second square exactly once when they are superposed. A Latin square of order n is self-orthogonal if it is orthogonal to its transpose.

Let $I_n = \{0,1,\cdots,n-1\}$. A Latin square of order n over I_n, $L = (l_{i,j})$ is called strongly symmetrical if $l_{i,j} + l_{n-1-i,n-1-j} = n-1$ for all $i,j \in I_n$.

The investigation of the existence of a strongly symmetrical self-orthogonal diagonal LS(n) was started by Danhof et al [2]. They show that there exists a strongly symmetrical self-orthogonal diagonal LS(n) for each $n \in \{4,5,7,8,12\}$ and a strongly symmetrical self-orthogonal diagonal LS(n) does not exist for each $n \in \{2,3,6,10\}$. Du and Cao proved that a strongly symmetrical self-orthogonal diagonal LS (n) exists for all positive integers $n \equiv 0,1,3(mod 4)$ and $n \neq 3, 15$ in 2002 [3]. Cao and Li completely solved the existence of SSSODLS (n) [4]. They proved the following.

Lemma 1.1 ([4]) There exists strongly symmetrical self-orthogonal diagonal LS (n) if and only if $n \equiv 0,1,3(mod 4)$ and $n \neq 3$.

Let $A = (a_{i,j})$ be an $n \times n$ array, we index its rows and columns by $I_n = \{0,1,\cdots,n-1\}$. For $k \in I_n$, the set $\{a_{i,k+i} | \ i \in I_n\}$ and $\{a_{i,k-i} | \ i \in I_n\}$ are called k-th right diagonal and k-th left diagonal of A respectively, where the additions of the subscripts are all taken modulo n.

If A is a Latin square with the property that every right diagonal and every left diagonal is a transversal, then A is said to be a pandiagonal Latin square or a Knut Vik design, denoted by pandiagonal LS(n). It has been used in statistical designs to eliminate sources of variation along four dimensions ([10]) and in n-queens problems ([11, 12]) etc. Hedayat proved in [16] that a pandiagonal LS(n) and orthogonal pandiagonal LS(n) exist if and only if $n \equiv 1,5(mod 6)$.

Xu introduced a weak form of Knut Vik design to construct pandiagonal magic squares ([5]). A Latin square $A = (a_{i,j})$ of order n over I_n is called weakly pandiagonal, if the sum of n elements in each right diagonal and each left diagonal is the same, i.e. for each $w \in I_n$, $\sum_{i=0}^{n-1} l_{i,i+w} = \frac{n(n-1)}{2}$ and $\sum_{i=0}^{n-1} l_{i,w-i} = \frac{n(n-1)}{2}$, where the operations in the subscripts are all taken modulo n. Clearly, a pandiagonal LS(n) is necessarily a weakly pandiagonal LS(n). Xu proved in [5] that

Lemma 1.2 ([5]) An weakly pandiagonal self-orthogonal LS(n) exists if $n \equiv 0,1,3(mod\ 4)$ and $n \not\equiv 3,6(mod\ 9)$.

A weakly pandiagonal strongly symmetrical self-orthogonal diagonal LS (n) is denoted by *LS(n). The existence of *LS(n) is an intriguing problem itself and it is also an improvement question of Cao and Li's result.

The only known result of *LS(n) attributes to Zhang et al [6]. Although they proved that there exists a weakly pandiagonal strongly symmetrical self-orthogonal LS(n) provided $n \equiv 1{,}5(mod6), n \geq 5$, it is easy to verify that their result is also true for diagonal cases. So we have

Lemma 1.3 ([6]) There exists a *LS(n) provided $n \equiv 1{,}5(mod\ 6), n \geq 5$.

In this paper, we shall further investigate *LS(n) especially when n is a prime power. We shall use frequency squares to give a construction and prove the following.

Theorem 1.4 There exists a *LS(n) for $n > 4$ and n is a prime power.

A construction based on frequency squares will be discussed in section 2, and the proof of Theorem 1.4 will be given in section 3.

2. A construction for *LS(n) based on frequency squares

Frequency square will be used in our construction for *LS(n)s. Let $n = m\lambda$. An F$(n;\lambda)$ frequency square is an $n \times n$ array in which each of m distinct symbols occurs exactly λ times in each row and column. Moreover, two such squares are orthogonal if when superimposed, each of the m^2 possible ordered pairs occurs λ^2 times.

For $n = m\lambda$, it is known that the maximum number of mutually orthogonal frequency squares of the form F$(n;\lambda)$ is bounded above by $(n-1)^2/(m-1)$. Further, if q is any prime power and $i \geq 1$ is a positive integer, then using linear polynomials in $2i$ variables over the finite field F_q, a complete set of $F\left(q^i, q^{i-1}\right)$ mutually orthogonal frequency squares can be constructed. Specifically, take the polynomials $a_1x_1 + \cdots + a_{2i}x_{2i}$ where neither $(a_1, \cdots, a_i)$ nor $(a_{i+1}, \cdots, a_{2i})$ is the zero vector $(0, \cdots, 0)$ and no two of the vectors are nonzero F_q multiples of each other, i.e. $(a_1', \cdots, a_i') \neq e(a_1, \cdots, a_i)$ for any nonzero $e \in F_q$. Further details may be found in Chapter 4 of [8].

Let $V = V_k\big(GF(p)\big)$, $n = p^k$. Take

$$A_h = \left(a_{h,0}, a_{h,1}, \cdots, a_{h,k-1}\right), B_h = \left(b_{h,0}, b_{h,1}, \cdots, b_{h,k-1}\right),$$

$$X = (x_0, x_1, \cdots, x_{k-1}), Y = (y_0, y_1, \cdots, y_{k-1}),$$

where A_h, B_h are constant vectors in V, $h = 0,1, \cdots, k-1$, X, Y are variable vectors in V.

For any $i \in Z_n$, there exist a vector $R_i = \left(r_{i,0}, r_{i,1}, \cdots, r_{i,k-1}\right)$ such that

$$i = r_{i,0}p^{k-1} + r_{i,1}p^{k-2} + \cdots + r_{i,k-1}.$$

Let $V(1) = \{R_0, R_1, \cdots, R_{n-1}\}$, $V(2) = \{C_0, C_2, \cdots, C_{n-1}\}$, where $C_i = R_i$. Index the rows of an $n \times n$ array by $V(1)$ and the columns by $V(2)$.

Note that there are strongly symmetric property,

$$n - 1 - i = r_{n-1-i,0}p^{k-1} + r_{n-1-i,1}p^{k-2} + \cdots + r_{n-1-i,k-1},$$

$$n - 1 = (p-1)(p^{k-1} + p^{k-2} + \cdots + p + 1),$$

$$i + n - 1 - i = \left(r_{i,0}p^{k-1} + r_{i,1}p^{k-2} + \cdots + r_{i,k-1}\right)$$

$$+\left(r_{n-1-i,0}p^{k-1} + r_{n-1-i,1}p^{k-2} + \cdots + r_{n-1-i,k-1}\right)$$

$$= \left(r_{i,0} + r_{n-1-i,0}\right)p^{k-1} + \cdots + \left(r_{i,k-1} + r_{n-1-i,k-1}\right),$$

which forces $r_{i,0} + r_{n-1-i,0} = p - 1$ for any $i \in I_n$. Therefore

$$R_i + R_{n-1-i} = \left(r_{i,0}, r_{i,1}, \cdots, r_{i,k-1}\right) + \left(r_{n-1-i,0}, r_{n-1-i,1}, \cdots, r_{n-1-i,k-1}\right)$$

$$=(p-1, p-1, \cdots, p-1).$$

Let a, n be integers, $< a >_p$ be the smallest nonnegative integer such that $a \equiv < a >_p (mod\ n)$, i.e, $< a >_p = r$ if $a = pn + r$, where p, r are integers and $0 \leq r < n$.

We use $\cdot$ to denote the inner product in V. Define a linear function from $V(1) \times V(2)$ to $GF(p)$.

Let $F_h = \left(F_h\left(R_i, C_j\right)\right)_{n\times n}$, where

$$F_h(R_i, C_j) =< A_h \cdot R_i + B_h \cdot C_j + d_h >_p, h = 0,1,\cdots,k-1.$$

The following Lemma gives the sufficient condition when F_h becomes a frequency square.

Lemma 2.1 ([7]) If $A_h \neq (0,\cdots,0)$ and $B_h \neq (0,\cdots,0)$, then F_h is a $F(p^k; p^{k-1})$.

Let $F = \left(F(R_i, C_j)\right)_{n\times n}$, where

$$F(R_i, C_j) = p^{k-1}F_0(R_i, C_j) + p^{k-2}F_1(R_i, C_j) + \cdots + F_{k-1}(R_i, C_j).$$

Note that $F(R_i, C_j)$ can also be represented as

$$F(R_i, C_j) = \left(F_0(R_i, C_j), F_1(R_i, C_j), \cdots, F_{k-1}(R_i, C_j)\right).$$

Let

$$D = \begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

Where

$$A = \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_{k-1} \end{pmatrix}, B = \begin{pmatrix} B_0 \\ B_1 \\ \vdots \\ B_{k-1} \end{pmatrix}.$$

Now we check that when F is a Latin square.

Lemma 2.2 If A, B are inverse, then F is a Latin square.

Proof Fix R_i, let C_j run over V, we want to show no two are the same. Otherwise, suppose that

$$\left(F_0(R_i, C_j), F_1(R_i, C_j), \cdots, F_{k-1}(R_i, C_j)\right) = \left(F_0(R_i, C_l), F_1(R_i, C_l), \cdots, F_{k-1}(R_i, C_l)\right).$$

Then

$$< A_h \cdot R_i + B_h \cdot C_j + d_h >_p =< A_h \cdot R_i + B_h \cdot C_l + d_h >_p$$

for fixed i and $h = 0,1,\cdots,k-1$. Thus $B_h(C_j - C_l)^T = 0$. Since $C_j - C_l$ is not zero vector, B should have rank less than k, a contradiction. This shows that when the upper right $k \times k$ submatrix B in D is inversive, F has the row Latin property.

Similarly, when the upper left $k \times k$ submatrix A in D is inversive, F has the column Latin property.

Lemma 2.3 If D is inverse, then F is self-orthogonal.

Proof Suppose that $\left(F(R_i, C_j), F(R_j, C_i)\right) = \left(F(R_{i'}, C_{j'}), F(R_{j'}, C_{i'})\right)$, then

$$F(R_i, C_j) = F(R_{i'}, C_{j'}), F(R_j, C_i) = F(R_{j'}, C_{i'})$$

which forces the following.

$$F_h(R_i, C_j) = F_h(R_{i'}, C_{j'}),\ F_h(R_j, C_i) = F_h(R_{j'}, C_{i'}),\ h = 0,1,\cdots,k-1.$$

Since $R_i = C_i$, we have

$$F_h(R_i, C_j) = F_h(R_{i'}, C_{j'}),\ F_h(C_j, R_i) = F_h(C_{j'}, R_{i'}),\ h = 0,1,\cdots,k-1.$$

Therefore for any $h = 0,1,\cdots,k-1$ we have

$$A_h \cdot R_i + B_h \cdot C_j = A_h \cdot R_{i'} + B_h \cdot C_{j'},$$

$$B_h \cdot R_i + A_h \cdot C_j = B_h \cdot R_{i'} + A_h \cdot C_{j'},$$

which can also be written as

$$(A_h, B_h)\begin{pmatrix} (R_i - R_{i'})^T \\ \left(C_j - C_{j'}\right)^T \end{pmatrix} = 0,\quad (B_h, A_h)\begin{pmatrix} (R_i - R_{i'})^T \\ \left(C_j - C_{j'}\right)^T \end{pmatrix} = 0$$

for $h = 0,1,\cdots,k-1$. i.e.

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix}\begin{pmatrix} (R_i - R_{i'})^T \\ (C_j - C_{j'})^T \end{pmatrix} = 0.$$

Since $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ is inversive, $\begin{pmatrix} (R_i - R_{i'})^T \\ (C_j - C_{j'})^T \end{pmatrix} = 0$. It follows that $R_i = R_{i'}$ and $C_j = C_{j'}$. Therefore F is self orthogonal.

Remark If D, A, B are inverse, the $A + B, A - B$ are both inverse.

Lemma 2.4 If $< 2d_h - (A_h + B_h) \cdot (1, \cdots, 1) + 1 >_p = 0$, then F_h is strongly symmetric.

Proof We show that F_h is strongly symmetric. In fact,

$$F_h(R_s, C_t) + F_h(R_{n-1-s}, C_{n-1-t})$$
$$=< A_h \cdot R_s + B_h \cdot C_t + d_h >_p + < A_h \cdot R_{7-s} + B_h \cdot C_{7-t} + d_h >_p$$
$$= A_h \cdot (R_s + R_{7-s}) + B_h \cdot (C_t + C_{7-t}) + 2d_h$$
$$= A_h \cdot (p-1, \cdots, p-1) + B_h \cdot (p-1, \cdots, p-1) + 2d_h$$
$$= 2d_h - \left(A_h \cdot (1, \cdots, 1) + B_h \cdot (1, \cdots, 1)\right)$$
$$= p - 1.$$

Lemma 2.5 If $A_h + B_h \neq 0$, then F is right diagonal; If $A_h - B_h \neq 0$, then F is right diagonal.

Proof We have

$$F_h(R_i, C_i) =< A_h \cdot R_i + B_h \cdot R_i + d_h >_p =< (A_h + B_h) \cdot R_i + d_h >_p$$

and

$$F_h(R_i, C_{n-1-i}) =< A_h \cdot R_i + B_h \cdot R_{n-1-i} + d_h >_p$$
$$=< A_h \cdot R_i + B_h \cdot (R_{n-1} - R_i) + d_h >_p$$
$$=< (A_h - B_h) \cdot R_i + B_h \cdot R_{n-1} + d_h >_p.$$

If $F(R_i, C_i) = F(R_j, C_j)$, then for any $h = 0, 1, \cdots, k-1$, we have $F_h(R_i, C_i) = F_h(R_j, C_j)$. That is

$$< A_h \cdot R_i + B_h \cdot R_i >_p =< A_h \cdot R_j + B_h \cdot R_j >_P.$$

i.e.

$$< (A_h + B_h) \cdot R_i >_p =< (A_h + B_h) \cdot R_j >_p.$$

Since $< A_h + B_h >_p \neq 0$, we have $< R_i >_p =< R_j >_p$, which indicates that F is right diagonal. Similarly, by $< A_h - B_h >_p \neq 0$, we have F is left diagonal.

Lemma 2.6 If $a_h + b_h \neq 0$, then

$$\{F_h(R_i, C_j), F_h(R_{i+p^{k-1}}, C_{j+p^{k-1}}), \cdots, F_h(R_{i+(p-1)p^{k-1}}, C_{j+(p-1)p^{k-1}})\} = GF(p),$$

If $a_h - b_h \neq 0$, then

$$\{F_h(R_i, C_j), F_h(R_{i-p^{k-1}}, C_{j-p^{k-1}}), \cdots, F_h(R_{i-(p-1)p^{k-1}}, C_{j-(p-1)p^{k-1}})\} = GF(p).$$

Proof For any $w \in I_n$, we have

$$F_h\left(R_{i+tp^{k-1}}, C_{j+tp^{k-1}}\right)$$
$$=< A_h \cdot R_{i+tp^{k-1}} + B_h \cdot C_{j+tp^{k-1}} + d_h >_p$$
$$=< \left(a_{h,0}, \cdots, a_{h,s} \cdots, a_{h,k-1}\right) \cdot \left(r_{i,0} + t, \cdots, r_{i,s} \cdots, r_{i,k-1}\right)$$
$$+ \left(b_{h,0}, \cdots, b_{h,s} \cdots, b_{h,k-1}\right) \cdot \left(r_{j,0} + t, \cdots, r_{j,s} \cdots, r_{j,k-1}\right) + d_h >_p$$
$$=< A_h \cdot R_i + B_h \cdot B_j + \left(a_{h,0} + b_{h,0}\right)t + d_h >_p.$$

runs over $0, 1, \cdots, p-1$ when t runs over $0, 1, \cdots, p-1$. Similarly we have (2).

Lemma 2.7 If $a_h + b_h \neq 0$, and $a_h - b_h \neq 0$, then F_h is weakly pandiagonal.

Proof For any $i, w \in I_n$, we have

$$\sum_{i=0}^{n-1} F_h\left(R_i, C_{i+w}\right) = \sum_{i=0}^{p^{k-1}-1} \sum_{t=0}^{p-1} F_h\left(R_{i+tp^{k-1}}, C_{i+tp^{k-1}+w}\right)$$

$$= \sum_{i=0}^{p^{k-1}-1} \left(\frac{p(p-1)}{2}\right)$$

$$= p^k \frac{p-1}{2}.$$

$$\sum_{i=0}^{n-1} F_h(R_i, C_{w-i}) = \sum_{i=0}^{p^{k-1}-1} \sum_{t=0}^{p-1} F_h\left(R_{i-tp^{k-1}}, C_{w-i-tp^{k-1}}\right)$$

$$= \sum_{i=0}^{p^{k-1}-1} \left(\frac{p(p-1)}{2}\right)$$

$$= p^k \frac{p-1}{2}.$$

So F is weakly pandiagonal.

Consequently, we have

Theorem 2.8 F is a *LS(n) if the following conditions hold.

(1) $A_h \neq (0,\cdots,0)$ and $B_h \neq (0,\cdots,0), h \in I_k$;

(2) A, B, D are inverse in I_p;

(3) $< 2d_h - (A_h + B_h) \cdot (1,\cdots,1) + 1 >_p = 0, h \in I_k$;

(4) $a_h + b_h \neq 0$, and $a_h - b_h \neq 0, h \in I_k$.

3. Families of *LS(n) based on main construction

Lemma 3.1 Let $k = 2, p > 2$. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & p-1 \\ 1 & 0 \end{pmatrix}.$$

Let $d_0 = \frac{p-1}{2}$, $d_1 = \frac{p+1}{2}$, then F is a *LS(n).

Proof $|A| = 1, |B| = 1, |D| = 4 - 4p + p^2 = 4$. By Theorem 2.8, F is a *LS(n).

Example 1. *LS(9)

Let $d_0 = 1, d_1 = 2$ and

$$D = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to check that D, d_1, d_2 satisfy the conditions in Theorem 2.8. We have

$$A_1 = \begin{pmatrix} 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 \\ 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 \\ 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 \\ 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 \\ 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 \\ 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 \end{pmatrix}$$

The A_1, A_2 are both frequency squares. Let $A = 3A_1 + A_2$, then

$$A=\begin{pmatrix}5&2&8&3&0&6&4&1&7\\3&0&6&4&1&7&5&2&8\\4&1&7&5&2&8&3&0&6\\8&5&2&6&3&0&7&4&1\\6&3&0&7&4&1&8&5&2\\7&4&1&8&5&2&6&3&0\\2&8&5&0&6&3&1&7&4\\0&6&3&1&7&4&2&8&5\\1&7&4&2&8&5&0&6&3\end{pmatrix}$$

One can check that A is a *LS(9).

For $k \geq 3$, let

$$A=\begin{pmatrix}y&0&0&0&\cdots&0&0\\0&1&0&0&\cdots&0&0\\0&0&1&0&\cdots&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&0&0&\cdots&1&0\\0&0&0&0&\cdots&0&1\end{pmatrix},$$

$$B=\begin{pmatrix}0&1&0&0&\cdots&0&0\\1&0&1&0&\cdots&0&0\\1&0&0&1&\cdots&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\1&0&0&0&\cdots&0&1\\1&0&0&0&\cdots&0&0\end{pmatrix},\quad B'=\begin{pmatrix}0&1&0&0&\cdots&0&1\\1&0&1&0&\cdots&0&0\\1&0&0&1&\cdots&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\1&0&0&0&\cdots&0&1\\1&0&0&0&\cdots&0&0\end{pmatrix}.$$

Lemma 3.2 Let $k \geq 3$, then

$$|E+B|=\begin{cases}y, & k\ odd,\\ y-1, & k\ \ even,\end{cases}\quad |E+B'|=\begin{cases}y-1, & k\ odd,\\ y-2, & k\ \ even,\end{cases}$$

and

$$<|E-B|>_p=<y-(k-1)>_p,<|E-B|>_p=<y-k>_p.$$

Lemma 3.3 Let $k \geq 3$, $p=3$, there exists a *LS(n).

Proof Let $k \geq 3$, $p=3$, $d_1=\cdots,d_{k-2}=1, d_{k-1}=\frac{p+1}{2}$. For $k\equiv 1,2(mod3)$, let $y=2$, $d_0=1$ then A,B satisfy the conditions in Theorem 2.8; For $k\equiv 0(mod3)$ and k is odd, let $y=1$ and $d_0=\frac{p+1}{2}$. Then A,B,d_i satisfy the conditions in Theorem 2.8. For $k\equiv 0(mod3)$ and k is even, let $y=1$, $d_0=\frac{p+1}{2}$. Then A,B',d_isatisfy the conditions in Theorem 2.8. So there exists a *LS(n) by Theorem 2.8.

Example 2. *LS(27).

For $k=3$, let

$$A=\begin{pmatrix}2&0&0\\0&1&0\\0&0&1\end{pmatrix},B=\begin{pmatrix}0&1&0\\1&0&1\\1&0&0\end{pmatrix}$$

and $d_0=d_1=1, d_2=2$, then by Theorem 2.8 F is a *LS(27).

Lemma 3.4 Let $k\geq 3$, $p>3$, there exists a *LS(n).

Proof Let $k\geq 3$, $p>3$, $d_1=\cdots,d_{k-2}=1,d_{k-1}=\frac{p+1}{2}$. If $k\equiv 1,2,\cdots p-1\ (mod\ p)$, choose y such that $2\leq y<p,<2d-y>_p=0$, and $y\neq<k>_p-1$, then A,B satisfy the conditions in Theorem 2.8; If $k\equiv 0(mod\ p)$, let $y=2, d_0=1$, then A,B satisfy the conditions in Theorem 2.8. So there exists a *LS(n) by Theorem 2.8.

Lemma 3.5 There exist a *LS(2^t) when $t>2$.

Proof For $p=2$, let

$$A=\begin{pmatrix}1&1&0&0&\cdots&0&0&0\\0&1&1&0&\cdots&0&0&0\\0&0&1&1&\cdots&0&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&0&0&\cdots&0&1&1\\0&0&0&0&\cdots&0&0&1\end{pmatrix}_{k\times k},\ B=\begin{pmatrix}0&1&0&0&\cdots&0&0&0\\0&0&1&0&\cdots&0&0&0\\0&0&0&1&\cdots&0&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&0&0&\cdots&0&0&1\\1&1&0&0&\cdots&0&0&0\end{pmatrix}_{k\times k}$$

and $d_0 = d_1 = \cdots = d_{k-2} = 0, d_{k-1} = 1$. Then $A, B, d_i, i = 0, \cdots, k-1$ satisfy the conditions (1)-(3) in Theorem 2.8. F is also weakly pandiagonal.

By Lemma 1.3, Lemma 3.1, Lemma 3.3-3.5 we have

Theorem 1.4 There exist a *LS(n) when n is an prime power and $n > 4$.

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