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A Modified F-expansion Method for Solving Nonlinear PDEs

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Abstract. In this paper, a modified F-expansion method is proposed by taking full advantages of F-expansion method and Riccati equation in seeking exact solutions of nonlinear PDEs. By the method, rich families of exact solutions of nonlinear PDEs have been obtained, including soliton-like solutions, trigonometric function solutions and rational solutions. The method can be applied to solve massive nonlinear PDEs (group), as well as helps us find new exact solutions. Furthermore, with the aid of computer symbolic systems (*Mathematica* or *Maple*), the method can be conveniently operated. Some illustrative equations are investigated by this method and some figures of partial solutions are provided for direct-viewing analysis.

Keywords: Modified F-expansion Method, Nonlinear PDEs, Riccati Equation, Exact Solution.

1. Introduction

Amounts of mathematical models can be described by nonlinear PDEs, especially some basic equations in physics and mechanics. As a result, the research on exact solutions of nonlinear PDEs becomes more and more important, such as the famous Inverse scattering method, Backlund transformation, Darboux transformation, Hirota bilinear method and Painleve method [1], He's Variational iteration method and Homotopy perturbation method [2, 3]. In recent years, directly searching for exact solutions of nonlinear PDEs has become more and more attractive partly due to the availability of computer symbolic systems like *Maple* or *Mathematica* which allows us to perform some complicated and tedious algebraic calculation on computer, as well as helps us find new exact solutions of PDEs, such as Homogeneous balance method [4, 5], Tanh-function method [6, 7], Sine-Cosine method [8], Jacobi elliptic functions method [9], Riccati equation method [10, 11], F-expansion and the extended F-expansion method [12-15] and so on.

In this paper, we put forward a modified F-expansion method by taking full advantages of F-expansion method and Riccati equation in seeking exact solutions of nonlinear PDEs. Before introducing the modified F-expansion method, we simply describe the F-expansion method as follows:

For the given NLPDE, say two variables x, t

$$P(u, u_t, u_{xx}, u_{xt}, u_{tt}, ...) = 0$$

We seek its traveling wave solution in the formal solution

$$u(\xi) = a_0 + \sum_{i=-N}^{N} a_i F^i(\xi), \ (a_N \neq 0)$$

With $F(\xi)$ satisfying the non-linear ODE

$$F'^2(\xi) = PF^4(\xi) + QF^2(\xi) + R$$

Where $' = \frac{d}{d\xi}$, P, Q, R are constants, which is more powerful than Jacobi elliptic functions method, but the method can just be well used to solve the nonlinear PDEs whose odd- and even-order derivatives terms do not coexist. In order to overcome this disadvantage, we substitute Riccati equation $\left(F'(\xi) = A + BF(\xi) + CF^2(\xi)\right)$ for the ODE $\left(F'^2(\xi) = PF^4(\xi) + QF^2(\xi) + R\right)$. In what follows we introduce the modified F-expansion method and apply it to some illustrative equations. Using the method, we

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obtain rich families of exact solutions, including soliton-like solutions, trigonometric function solutions and rational solutions, and some of the solutions are firstly derived by us. In fact, we can apply the method to amounts of nonlinear PDEs(group) and get many new exact solutions.

The rest of this paper is organized as follows: in Section 2, we give the description of the method; in Section 3, we apply the method to some illustrative equations. And we conclude the paper in the last section.

2. Summary of the method

Consider a given NLPDE with independent variables $x=(x_1,x_2,...,x_l,t)$ and dependent variable u

$$P(u,u_t,u_{x_1},u_{tt},...)=0 \tag{2.1}$$

Generally speaking, the left-hand side of Eq. (2.1) is a polynomial in u and its various partial derivatives. The main points of the modified F-expansion method for solving Eq. (2.1) are as follows

First, seek traveling wave solutions to Eq. (2.1) by taking

$$u(x_1,x_2,...,x_l,t)=u(\xi),\ \xi=k_1(x_1+k_2x_2+...+k_lx_l+\omega t) \tag{2.2}$$

where $k_1,k_2,\ldots,k_l,\omega$ are constants to be determined, inserting (2.2) into Eq.(2.1) yields an ODE for $u(\xi)$

$$P(u,u^{/},u^{//},...)=0 \tag{2.3}$$

Second, suppose that $u(\xi)$ can be expressed as

$$u(\xi)=a_0+\sum_{i=-N}^{N}a_iF^i(\xi),\ \left(a_N\neq 0\right) \tag{2.4}$$

where a_0,a_i are constants to be determined, $F(\xi)$ satisfies Riccati equation

$$F^{/}(\xi)=A+BF(\xi)+CF^2(\xi),\ \left(C\neq 0\right) \tag{2.5}$$

where A,B,C are constants to be determined, integer N can be determined by considering the homogeneous balance between the governing nonlinear term(s) and highest order derivatives of $u(\xi)$ in Eq.(2.3). And

(a) when $N=\dfrac{p}{q}$ is fraction, let $u(\xi)=v^{\frac{p}{q}}(\xi)$;

(b) when N is negative integer, let $u(\xi)=v^N(\xi)$,

we change Eq.(2.3) into another ODE for $v(\xi)$, whose balancing number will be a positive integer.

Third, substitute (2.4) into Eq.(2.3), and using (2.5), then the left-hand side of Eq.(2.3) can be converted into a finite series in $F^p(\xi)$ ($p=-N,\ldots,-1,0,1,\ldots,N$), equating each coefficient of $F^p(\xi)$ to zero yields a system of algebraic equations for $a_i\left(i=-N,...,-1,0,1,...,N\right)$, $k_\lambda\left(\lambda=2,...,l\right)$, ω.

Fourth, solve the system of algebraic equations, probably with the aid of *Mathematica* or *Maple*, a_i, k_λ, ω can be expressed by A, B, C (or the coefficients of ODE(2.3)) and k_1. Substituting these results into (2.4), we can obtain the general form of travelling wave solutions to Eq.(2.3).

Fifth, with the aid of Appendix, from the general form of travelling wave solutions, we can give a series of soliton-like solutions, trigonometric function solutions and rational solutions of Eq.(2.1).

3. Applications of the method

3.1. Gardner equation

$$u_t+uu_x-\alpha u^2u_x+\beta u_{xxx}=0, \tag{3.1.1}$$

where α, β are real constants.

(i) we assume that Eq.(3.1.1) has travelling wave solution in the form

$$u(x,t)=u(\xi),\ \xi=k(x+\omega t)\quad (k\neq 0) \tag{3.1.2}$$

Substituting (3.1.2) into (3.1.1), we have

$$\omega u' + uu' - \alpha u^2 u' + \beta k^2 u^{(3)} = 0 \qquad (3.1.3)$$

(ii) Balancing $u^{(3)}$ with u^2u' yields N=1.Therefore we may choose

$$u(\xi) = a_0 + a_{-1}F^{-1}(\xi) + a_1 F(\xi). \qquad (3.1.4)$$

Substituting (3.1.4) into Eq.(3.1.3), and using (2.5), the left-hand side of Eq.(3.1.3) can be converted into a finite series in $F^p(\xi)$ （p=-4,…,-1,0,1,…,4）, equating each coefficient of $F^p(\xi)$ to zero yields a system of algebraic equations for a_{-1}, a_0, a_1, ω.

$$F^4: 6C^3k^3\beta a_1 - Ck\alpha a_1^3 = 0 \qquad (3.1.5.1)$$

$$F^3: 12BC^2k^3\beta a_1 + Cka_1^2 - 2Ck\alpha a_0 a_1^2 - Bk\alpha a_1^3 = 0 \qquad (3.1.5.2)$$

$$F^2: \begin{array}{l} 7B^2Ck^3\beta a_1 + 8AC^2k^3\beta a_1 + Ck\omega a_1 + Cka_0a_1 - Ck\alpha a_0^2 a_1 + \\ Bka_1^2 - Ck\alpha a_{-1}a_1^2 - 2Bk\alpha a_0 a_1^2 - Ak\alpha a_1^3 = 0 \end{array} \qquad (3.1.5.3)$$

$$F: \begin{array}{l} B^3k^3\beta a_1 + 8ABCk^3\beta a_1 + Bk\omega a_1 + Bka_0a_1 - Bk\alpha a_0^2 a_1 + \\ Aka_1^2 - Bk\alpha a_{-1}a_1^2 - 2Ak\alpha a_0 a_1^2 = 0 \end{array} \qquad (3.1.5.4)$$

$$F^0: \begin{array}{l} -B^2Ck^3\beta a_{-1} - 2AC^2k^3\beta a_{-1} - Ck\omega a_{-1} - Cka_{-1}a_0 + Ck\alpha a_{-1}a_0^2 + AB^2k^3\beta a_1 + \\ 2A^2Ck^3\beta a_1 + Ak\omega a_1 + Ck\alpha a_{-1}^2 a_1 + Aka_0a_1 - Ak\alpha a_0^2 a_1 - Ak\alpha a_{-1}a_1^2 = 0 \end{array} \qquad (3.1.5.5)$$

$$F^{-1}: \begin{array}{l} -B^3k^3\beta a_{-1} - 8ABCk^3\beta a_{-1} - Bk\omega a_{-1} - Cka_{-1}^2 - Bka_{-1}a_0 + \\ +2Ck\alpha a_{-1}^2 a_0 + Bk\alpha a_{-1}a_0^2 + Bk\alpha a_{-1}^2 a_1 = 0 \end{array} \qquad (3.1.5.6)$$

$$F^{-2}: \begin{array}{l} -7AB^2k^3\beta a_{-1} - 8A^2Ck^3\beta a_{-1} - Ak\omega a_{-1} - Bka_{-1}^2 + Ck\alpha a_{-1}^3 - \\ Aka_{-1}a_0 + 2Bk\alpha a_{-1}^2 a_0 + Ak\alpha a_{-1}a_0^2 + Ak\alpha a_{-1}^2 a_1 = 0 \end{array} \qquad (3.1.5.7)$$

$$F^{-3}: -12A^2Bk^3\beta a_{-1} - Aka_{-1}^2 + Bk\alpha a_{-1}^3 + 2Ak\alpha a_{-1}^2 a_0 = 0 \qquad (3.1.5.8)$$

$$F^{-4}: -6A^3k^3\beta a_{-1} + Ak\alpha a_{-1}^3 = 0 \qquad (3.1.5.9)$$

(iii) Solving the algebraic equations(3.1.5), we have the following solutions of a_{-1}, a_0, a_1, ω

*Case*1: when A=0, we have

$$BCk\alpha\beta \neq 0,\ a_0 = \frac{1 \pm \sqrt{6}Bk\sqrt{\alpha}\sqrt{\beta}}{2\alpha},\ a_{-1} = 0,\ a_1 = \pm\sqrt{6}Ck\sqrt{\frac{\beta}{\alpha}},\ \omega = \frac{-1 + 2B^2k^2\alpha\beta}{4\alpha} \qquad (3.1.6)$$

*Case*2: when B=0, we have

$$Ak\alpha\beta \neq 0,\ a_0 = \frac{1}{2\alpha},\ a_{-1} = \pm\sqrt{6}Ak\sqrt{\frac{\beta}{\alpha}},\ a_1 = \pm\sqrt{6}Ck\sqrt{\frac{\beta}{\alpha}},\ \omega = -\frac{1}{4\alpha} + 4ACk^2\beta; \qquad (3.1.7)$$

$$ACk\alpha\beta \neq 0,\ a_0 = \frac{1}{2\alpha},\ a_{-1} = 0,\ a_1 = \pm\sqrt{6}Ck\sqrt{\frac{\beta}{\alpha}},\ \omega = -\frac{1}{4\alpha} - 2ACk^2\beta \qquad (3.1.8)$$

*Case*3: when A=B=0, we have

$$Ck\alpha\beta \neq 0,\ a_0 = \frac{1}{2\alpha},\ a_{-1} = 0,\ a_1 = \pm\sqrt{6}Ck\sqrt{\frac{\beta}{\alpha}},\ \omega = -\frac{1}{4\alpha} \qquad (3.1.9)$$

Substituting these solutions into (3.1.4),from Appendix,we can obtain many soliton-like solutions, trigonometric function solutions and rational solutions of Eq.(3.1.1)(where we left the same type solutions out):

(I) when $A=0$, $B=1$, $C=-1$; from Appendix, then $F(\xi) = \frac{1}{2} + \frac{1}{2}\tanh(\frac{1}{2}\xi)$. By *case*1, we have soliton-like solutions of Eq.(3.1.1)

$$u_1 = \frac{1}{2\alpha} \mp \frac{\sqrt{6}}{2}k\sqrt{\frac{\beta}{\alpha}}\tanh[\frac{1}{2}k(x + \frac{2k^2\alpha\beta - 1}{4\alpha}t)]$$

(II) when $A=0$, $B=-1$, $C=1$; from Appendix, then $F(\xi)=\frac{1}{2}-\frac{1}{2}\coth(\frac{1}{2}\xi)$. By $case1$, we have soliton-like solutions of Eq.(3.1.1)

$$u_2=\frac{1}{2\alpha}\mp\frac{\sqrt{6}}{2}k\sqrt{\frac{\beta}{\alpha}}\coth[\frac{1}{2}k(x+\frac{2k^2\alpha\beta-1}{4\alpha}t)]$$

(III) when $A=\frac{1}{2}$, $B=0$, $C=-\frac{1}{2}$; from Appendix, then $F(\xi)=\coth\xi\pm\text{csch}\xi$ or $\tanh\xi\pm i\text{sech}\xi$. By $case2$,we have soliton-like solutions of Eq.(3.1.1)

$$u_3=\frac{1}{2\alpha}\mp\sqrt{6}k\sqrt{\frac{\beta}{\alpha}}\text{csch}[k(x-\frac{4k^2\alpha\beta+1}{4\alpha}t)]$$

$$u_4=\frac{1}{2\alpha}\mp\sqrt{6}k\sqrt{\frac{\beta}{\alpha}}i\,\text{sech}[k(x-\frac{4k^2\alpha\beta+1}{4\alpha}t)]$$

$$u_5=\frac{1}{2\alpha}\mp\frac{\sqrt{6}}{2}k\sqrt{\frac{\beta}{\alpha}}\{\coth[k(x+\frac{2k^2\alpha\beta-1}{4\alpha}t)]\pm\text{csch}[k(x+\frac{2k^2\alpha\beta-1}{4\alpha}t)]\}$$

$$u_6=\frac{1}{2\alpha}\mp\frac{\sqrt{6}}{2}k\sqrt{\frac{\beta}{\alpha}}\{\tanh[k(x+\frac{2k^2\alpha\beta-1}{4\alpha}t)]\pm i\,\text{sech}[k(x+\frac{2k^2\alpha\beta-1}{4\alpha}t)]\}$$

(IV) when $A=1$, $B=0$, $C=-1$; from Appendix, then $F(\xi)=\tanh\xi$ or $\coth\xi$. By $case2$,we have combined soliton-like solutions of Eq.(3.1.1)

$$u_7=\frac{1}{2\alpha}\pm\sqrt{6}k\sqrt{\frac{\beta}{\alpha}}\coth[k(x-\frac{1+16k^2\alpha\beta}{4\alpha}t)]\mp\sqrt{6}k\sqrt{\frac{\beta}{\alpha}}\tanh[k(x-\frac{1+16k^2\alpha\beta}{4\alpha}t)]$$

(V) when $A=C=\frac{1}{2}$, $B=0$; from Appendix, then $F(\xi)=\sec\xi+\tan\xi$ or $\csc\xi-\cot\xi$. By $case2$, we have trigonometric function solutions of Eq.(3.1.1)

$$u_8=\frac{1}{2\alpha}\pm\sqrt{6}k\sqrt{\frac{\beta}{\alpha}}\sec[k(x+\frac{4k^2\alpha\beta-1}{4\alpha}t)]$$

$$u_9=\frac{1}{2\alpha}\pm\sqrt{6}k\sqrt{\frac{\beta}{\alpha}}\csc[k(x+\frac{4k^2\alpha\beta-1}{4\alpha}t)]$$

$$u_{10}=\frac{1}{2\alpha}\pm\frac{\sqrt{6}}{2}k\sqrt{\frac{\beta}{\alpha}}\sec[k(x-\frac{2k^2\alpha\beta+1}{4\alpha}t)]\pm\frac{\sqrt{6}}{2}k\sqrt{\frac{\beta}{\alpha}}\tan[k(x-\frac{2k^2\alpha\beta+1}{4\alpha}t)]$$

$$u_{11}=\frac{1}{2\alpha}\pm\frac{\sqrt{6}}{2}k\sqrt{\frac{\beta}{\alpha}}\csc[k(x-\frac{2k^2\alpha\beta+1}{4\alpha}t)]\mp\frac{\sqrt{6}}{2}k\sqrt{\frac{\beta}{\alpha}}\cot[k(x-\frac{2k^2\alpha\beta+1}{4\alpha}t)]$$

(VI) when $A=C=-\frac{1}{2}$, $B=0$; from Appendix, then $F(\xi)=\sec\xi-\tan\xi$ or $\csc\xi+\cot\xi$. By $case2$, we have trigonometric function solutions of Eq.(3.1.1)

$$u_{12}=\frac{1}{2\alpha}\mp\frac{\sqrt{6}}{2}k\sqrt{\frac{\beta}{\alpha}}\sec[k(x-\frac{2k^2\alpha\beta+1}{4\alpha}t)]\pm\frac{\sqrt{6}}{2}k\sqrt{\frac{\beta}{\alpha}}\tan[k(x-\frac{2k^2\alpha\beta+1}{4\alpha}t)]$$

$$u_{13}=\frac{1}{2\alpha}\mp\frac{\sqrt{6}}{2}k\sqrt{\frac{\beta}{\alpha}}\csc[k(x-\frac{2k^2\alpha\beta+1}{4\alpha}t)]\mp\frac{\sqrt{6}}{2}k\sqrt{\frac{\beta}{\alpha}}\cot[k(x-\frac{2k^2\alpha\beta+1}{4\alpha}t)]$$

(VII) when $A=C=1$, $B=0$; from Appendix, then $F(\xi)=\tan\xi$. By $case2$, we have trigonometric function solutions of Eq.(3.1.1)

$$u_{14}=\frac{1}{2\alpha}\pm\sqrt{6}k\sqrt{\frac{\beta}{\alpha}}\cot[k(x+\frac{16k^2\alpha\beta-1}{4\alpha}t)]\pm\sqrt{6}k\sqrt{\frac{\beta}{\alpha}}\tan[k(x+\frac{16k^2\alpha\beta-1}{4\alpha}t)]$$

$$u_{15} = \frac{1}{2\alpha} \pm \sqrt{6}k\sqrt{\frac{\beta}{\alpha}}\tan[k(x - \frac{8k^2\alpha\beta + 1}{4\alpha}t)]$$

(VIII) when $A = C = -1$, $B = 0$; from Appendix, then $F(\xi) = \cot\xi$. By $case2$, we have trigonometric function solutions of Eq.(3.1.1)

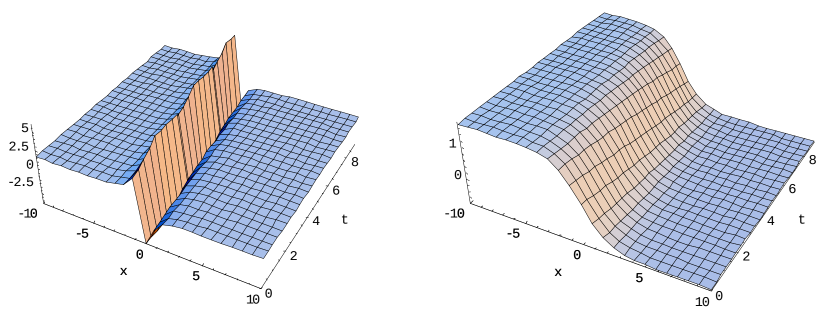
$$u_{16} = \frac{1}{2\alpha} \mp \sqrt{6}k\sqrt{\frac{\beta}{\alpha}}\cot[k(x - \frac{8k^2\alpha\beta + 1}{4\alpha}t)]$$

(IX) when $A = C = 0, C \neq 0$; from Appendix, then $F(\xi) = -\frac{1}{C\xi + \lambda}$. By $case3$, we have rational solutions of Eq.(3.1.1)

$$u_{17} = \frac{1}{2\alpha} \mp \sqrt{6}Ck\sqrt{\frac{\beta}{\alpha}}\frac{1}{Ck(x - \frac{1}{4\alpha}t) + \lambda}$$

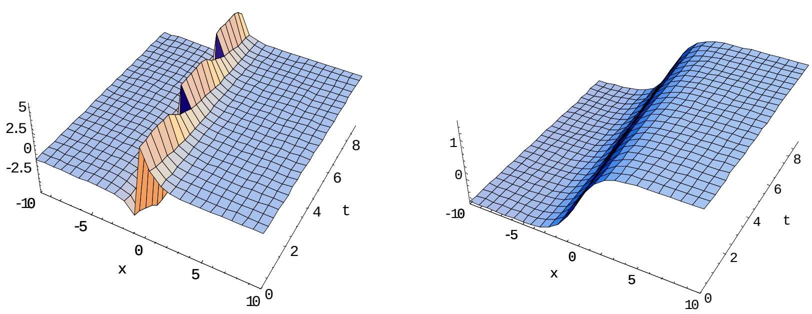
Where C, k, λ are arbitrary constants and $C \neq 0$.

We can see that abundant exact solutions for Gardner equation have been obtained by the modified F-expansion method, and many of them are firstly derived by us. Following, we provide some figures of partial solutions for direct-viewing analysis. We choose $\alpha = \beta = k = 1$.



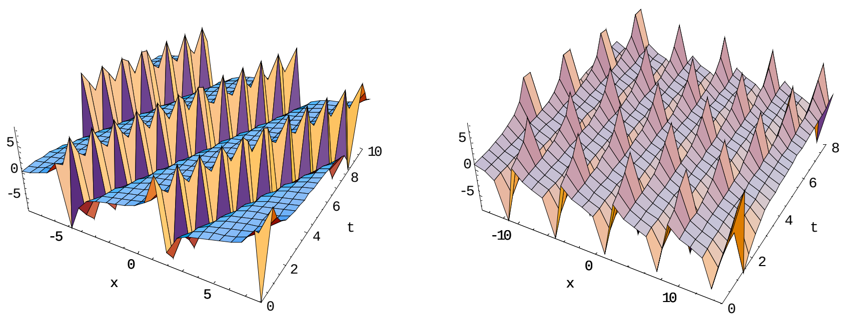
(a) (b)

Fig. 1.1: The soliton-like solution u_5 is shown at “-” and “+”((a)) and “-” and “-”((b))



(c) (d)

Fig. 1.2: The soliton-like solution u_5 is shown at “+” and “+” ((c)) and “+” and “-”((d))



(a) (b)

Fig. 2: The periodic solution u_{10} is shown at "+" and "+" ((a)) and "-" and "-"((b))

3.2. Breaking Soliton Equation

$$u_{xt} - 4u_x u_{xy} - 2u_y u_{xx} + u_{xxxy} = 0 \tag{3.2.1}$$

Eq.(3.2.1) could describe the (2+ 1)-dimensional interaction of Riemann wave propagation along the y-axis with long-wave propagation along the x-axis[16]. The Breaking Soliton Equation has been researched by many scientists[17-20], and many exact solutions for Eq.(3.2.1) have been derived. Here we use our method to solve it and get a series of its soliton-like solutions, trigonometric function solutions and rational solutions.

(i) we assume that Eq.(3.2.1) has traveling wave solution in the form

$$u(x,y,t) = u(\xi),\ \xi = k(x + ly + \omega t)\ \left(k \neq 0\right) \tag{3.2.2}$$

Substituting (3.2.2) into（3.2.1）,we have

$$\omega u'' - 6klu'u'' + k^2 l u^{(4)} = 0 \tag{3.2.3}$$

(ii) Considering the homogeneous balance between $u'u''$ and $u^{(4)}$ in (3.2.3), we suppose that the solution of ODE(3.2.3) can be expressed as

$$u(\xi) = a_0 + a_{-1} F(\xi)^{-1} + a_1 F(\xi) \tag{3.2.4}$$

where a_0, a_{-1}, a_1 are constants to be determined. Substituting (3.2.4) into Eq.(3.2.3), and using (2.5), the left-hand side of Eq.(3.2.3) can be converted into a finite series in $F^p(\xi)$ （p=-5,…,-1,0,1,…,5）, equating each coefficient of $F^p(\xi)$ to zero yields a system of algebraic equations for $a_{-1}, a_0, a_1, l, \omega$.

$$F^5:\ 24C^4k^4la_1 - 12C^3k^3la_1^2 = 0 \tag{3.2.5.1}$$

$$F^4:\ 60BC^3k^4la_1 - 30BC^2k^3la_1^2 = 0 \tag{3.2.5.2}$$

$$F^3:\ \begin{aligned}&50B^2C^2k^4la_1 + 40AC^3k^4la_1 + 2C^2k^2\omega a_1 + 12C^3k^3la_{-1}a_1 - \\ &24B^2Ck^3la_1^2 - 24AC^2k^3la_1^2 = 0\end{aligned} \tag{3.2.5.3}$$

$$F^2:\ \begin{aligned}&15B^3Ck^4la_1 + 60ABC^2k^4la_1 + 3BCk^2\omega a_1 + 24BC^2k^3la_{-1}a_1 - \\ &6B^3k^3la_1^2 - 36ABCk^3la_1^2 = 0\end{aligned} \tag{3.2.5.4}$$

$$F^1:\ \begin{aligned}&B^4k^4la_1 + 22AB^2Ck^4la_1 + 16A^2C^2k^4la_1 + B^2k^2\omega a_1 + 2ACk^2\omega a_1 + 12B^2Ck^3la_{-1}a_1 + \\ &12AC^2k^3la_{-1}a_1 - 12AB^2k^3la_1^2 - 12A^2Ck^3la_1^2 = 0\end{aligned} \tag{3.2.5.5}$$

$$F^0:\ \begin{aligned}&B^3Ck^4la_{-1} + 8ABC^2k^4la_{-1} + BCk^2\omega a_{-1} + 6BC^2k^3la_{-1}^2 + AB^3k^4la_1 + 8A^2BCk^4la_1 + \\ &ABk^2\omega a_1 - 6A^2Bk^3la_1^2 = 0\end{aligned} \tag{3.2.5.6}$$

$$F^{-1}:\begin{array}{l} B^4k^4la_{-1}+22AB^2Ck^4la_{-1}+16A^2C^2k^4la_{-1}+B^2k^2\omega a_{-1}+2ACk^2\omega a_{-1}+ \\ 12B^2Ck^3la_{-1}^2+12AC^2k^3la_{-1}^2-12AB^2k^3la_{-1}a_1-12A^2Ck^3la_{-1}a_1=0 \end{array} \quad (3.2.5.7)$$

$$F^{-2}:\begin{array}{l} 15AB^3k^4la_{-1}+60A^2BCk^4la_{-1}+3ABk^2\omega a_{-1}+6B^3k^3la_{-1}^2+ \\ 36ABCk^3la_{-1}^2-24A^2Bk^3la_{-1}a_1=0 \end{array} \quad (3.2.5.8)$$

$$F^{-3}:\begin{array}{l} 50A^2B^2k^4la_{-1}+40A^3Ck^4la_{-1}+2A^2k^2\omega a_{-1}+24AB^2k^3la_{-1}^2+ \\ 24A^2Ck^3la_{-1}^2-12A^3k^3la_{-1}a_1=0 \end{array} \quad (3.2.5.9)$$

$$F^{-4}:\ 60A^3Bk^4la_{-1}+30A^2Bk^3la_{-1}^2=0 \quad (3.2.5.10)$$

$$F^{-5}:\ 24A^4k^4la_{-1}+12A^3k^3la_{-1}^2=0 \quad (3.2.5.11)$$

(iii) Solving the algebraic equations(3.2.5), we have the following solutions of $a_{-1}, a_0, a_1, l, \omega$

*Case*1: when $A=0$, we have

$$a_0=a_0,\ a_{-1}=0,\ a_1=2Ck,\ l=l,\ \omega=-B^2k^2l. \quad (3.2.6)$$

*Case*2: when $B=0$, we have

$$a_0=a_0,\ a_{-1}=0,\ a_1=2Ck,\ l=l,\ \omega=4ACk^2l; \quad (3.2.7)$$

$$a_0=a_0,\ a_{-1}=-2Ak,\ a_1=2Ck,\ l=l,\ \omega=16ACk^2l. \quad (3.2.8)$$

*Case*3: when A=B=0, we have

$$a_0=a_0,\ a_{-1}=0,\ a_1=2Ck,\ l=l,\ \omega=0; \quad (3.2.9)$$

$$a_0=a_0,\ a_1=-\frac{\omega}{6Ckl},\ a_1=2Ck,\ l=l,\ \omega=\omega. \quad (3.2.10)$$

Substituting these solutions into (3.2.4),from Appendix,we can obtain many soliton-like solutions, trigonometric function solutions and rational solutions of Eq.(3.2.1)(where we left the same type solutions out):

(I) when $A=0$, $B=1$, $C=-1$; from Appendix, then $F(\xi)=\frac{1}{2}+\frac{1}{2}\tanh(\frac{1}{2}\xi)$. By *case*1, we have soliton-like solutions of Eq.(3.2.1)

$$u_1=a_0-k-k\tanh[\frac{1}{2}k(x+ly-k^2lt)]$$

(II) when $A=0$, $B=-1$, $C=1$; from Appendix, then $F(\xi)=\frac{1}{2}-\frac{1}{2}\coth(\frac{1}{2}\xi)$. By *case*1, we have soliton-like solutions of Eq.(3.2.1)

$$u_2=a_0+k-k\coth[\frac{1}{2}k(x+ly-k^2lt)]$$

(III) when $A=\frac{1}{2}$, $B=0$, $C=-\frac{1}{2}$; from Appendix, then $F(\xi)=\coth\xi\pm\text{csch}\xi$ or $\tanh\xi\pm i\,\text{sech}\xi$. By *case*2, we have soliton-like solutions of Eq.(3.2.1)

$$u_3=a_0-k\coth[k(x+ly-k^2lt)]\mp k\,\text{csch}[k(x+ly-k^2lt)]$$

$$u_4=a_0-k\tanh[k(x+ly-k^2lt)]\mp \mathrm{i}\,k\,\text{sech}[k(x+ly-k^2lt)]$$

(IV) when $A=1$, $B=0$, $C=-1$; from Appendix, then $F(\xi)=\tanh\xi$ or $\coth\xi$. By *case*2,we have combined soliton-like solutions of Eq.(3.2.1)

$$u_5=a_0-2k\tanh[k(x+ly-16k^2lt)]-2k\coth[k(x+ly-16k^2lt)]$$

(V) when $A=C=\frac{1}{2}$, $B=0$; from Appendix, then $F(\xi)=\sec\xi+\tan\xi$ or $\csc\xi-\cot\xi$. By *case*2, we have trigonometric function solutions of Eq. (3.2.1)

$$u_6 = a_0 + k\sec[k(x+ly+k^2lt)] + k\tan[k(x+ly+k^2lt)]$$
$$u_7 = a_0 + k\csc[k(x+ly+k^2lt)] - k\cot[k(x+ly+k^2lt)]$$
$$u_8 = a_0 + 2k\tan[k(x+ly+4k^2lt)]$$
$$u_9 = a_0 - 2k\cot[k(x+ly+4k^2lt)]$$

(VI) when $A = C = -\frac{1}{2}$, $B = 0$; from Appendix, then $F(\xi) = \sec\xi - \tan\xi$ or $\csc\xi + \cot\xi$. By *case*2, we have trigonometric function solutions of Eq. (3.2.1)

$$u_{10} = a_0 - k\sec[k(x+ly+k^2lt)] + k\tan[k(x+ly+k^2lt)]$$
$$u_{11} = a_0 - k\csc[k(x+ly+k^2lt)] - k\cot[k(x+ly+k^2lt)]$$

(VII) when $A = C = 1$, B=0; from Appendix, then $F(\xi) = \tan\xi$. By *case*2, we have combined trigonometric function solutions of Eq. (3.2.1)

$$u_{12} = a_0 + 2k\tan[k(x+ly+16k^2lt)] - 2k\cot[k(x+ly+16k^2lt)]$$

where a_0, k, l are arbitrary constants in Ⅰ-Ⅶ.

(VIII) when $A = B = 0$, $C \neq 0$; from Appendix, then $F(\xi) = -\frac{1}{C\xi + \lambda}$. By *case*3, we have rational solutions of Eq.(3.2.1)

$$u_{13} = a_0 - \frac{2Ck}{Ck(x+ly)+\lambda}$$

$$u_{14} = a_0 + \frac{\omega}{6Ckl}[Ck(x+ly+\omega t)+\lambda] - \frac{2Ck}{Ck(x+ly+\omega t)+\lambda}$$

where a_0,*C*, *k*, l, ω are arbitrary constants and *C*≠0.

Following, we provide some figures of partial solutions for direct-viewing analysis. We choose $a_0 = 0, k = l = 1, t = 0$.

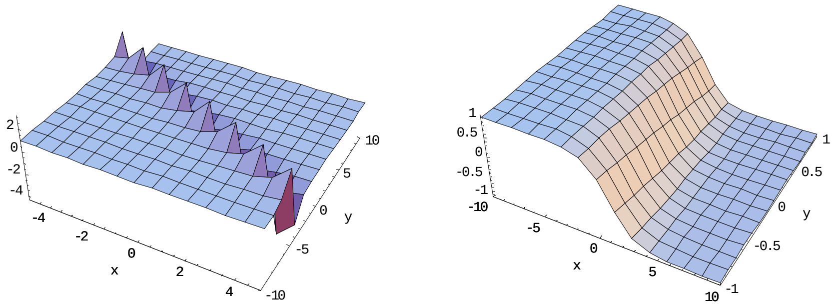
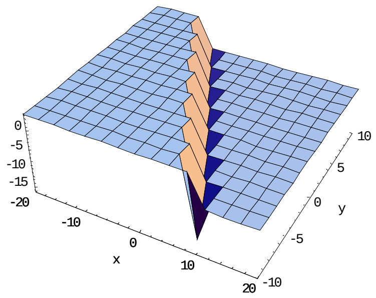


Fig. 3: The soliton-like solution u_3 is shown at “-” and “+”, respectively



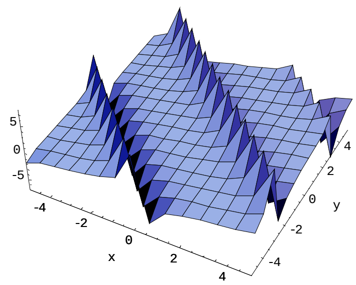
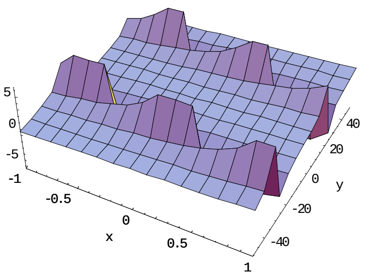


Fig. 4: The soliton-like solution u_5

Fig. 5: The periodic solution u_6



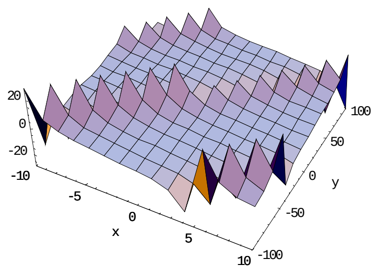


Fig. 6: The periodic solution u_7

Fig. 7: The periodic solution u_{12}

3.3. Variant Boussinesq equations

$$\begin{cases} u_t + (uv)_x + v_{xxx} = 0 & (3.3.1.1) \\ v_t + u_x + vv_x = 0 & (3.3.1.2) \end{cases}$$

As models for water waves, v is the velocity and u is the total depth. Eq.(3.3.1) has been researched by Wang[4], Fan[21] and Lv[22], and some exact solutions have been derived. Here we will obtain more exact solutions, including soliton-like solutions, trigonometric function solutions and rational solutions, and some of them are new.

(i) We assume that Eq.(3.3.1) has traveling wave solution in the form

$$u(x,t) = u(\xi), \quad v(x,t) = v(\xi), \qquad \xi = k(x+\omega t) \quad (k \neq 0) \qquad (3.3.2)$$

Substituting (3.3.2) into（3.3.1）,we have

$$\begin{cases} \omega u' + u'v + uv' + k^2 u^{(3)} = 0 & (3.3.3.1) \\ \omega v' + u' + vv' = 0 & (3.3.3.2) \end{cases}$$

(ii) Balancing between the governing nonlinear term(s) and highest order derivatives in

Eq.(3.3.3).Therefore we may choose

$$u(\xi) = a_0 + a_{-1}F^{-1}(\xi) + a_1F(\xi) + a_{-2}F^{-2}(\xi) + a_2F^2(\xi) \quad (3.3.4)$$

$$v(\xi) = b_0 + b_{-1}F^{-1}(\xi) + b_1F(\xi) \quad (3.3.5)$$

Substituting (3.3.4)and(3.3.5)into Eq.(3.3.3), and using (5), the left-hand side of Eq.(3.3.3.1) can be converted into a finite series in $F^p(\xi)\left(p=-4,...,-1,0,1,...,4\right)$ and the left-hand side of Eq.(3.3.3.2) can be converted into a finite series in $F^q(\xi)\left(q=-3,...,-1,0,1,...,3\right)$, equating each coefficient of $F^p(\xi)$ ($F^q(\xi)$)to zero yields a system of algebraic equations for a_0, a_{-1}, a_{-2}, a_1, a_2, b_0, b_{-1}, b_1, ω.

$$6C^3k^3b_1 + 3Cka_2b_1 = 0 \quad (3.3.6.1)$$

$$2Ck\omega a_2 + 2Cka_2b_0 + 12BC^2k^3b_1 + 2Cka_1b_1 + 3Bka_2b_1 = 0 \quad (3.3.6.2)$$

$$\begin{aligned} &Ck\omega a_1 + 2Bk\omega a_2 + Cka_2b_{-1} + Cka_1b_0 + 2Bka_2b_0 + 7B^2Ck^3b_1 + \\ &8AC^2k^3b_1 + Cka_0b_1 + 2Bka_1b_1 + 3Aka_2b_1 = 0 \end{aligned} \quad (3.3.6.3)$$

$$\begin{aligned} &Bk\omega a_1 + 2Ak\omega a_2 + Bka_2b_{-1} + Bka_1b_0 + 2Aka_2b_0 + \\ &B^3k^3b_1 + 8ABCk^3b_1 + Bka_0b_1 + 2Aka_1b_1 = 0 \end{aligned} \quad (3.3.6.4)$$

$$\begin{aligned} &-Ck\omega a_{-1} + Ak\omega a_1 - B^2Ck^3b_{-1} - 2AC^2k^3b_{-1} - Cka_0b_{-1} + Aka_2b_{-1} - Cka_{-1}b_0 + \\ &Aka_1b_0 + AB^2k^3b_1 + 2A^2Ck^3b_1 - Cka_{-2}b_1 + Aka_0b_1 = 0 \end{aligned} \quad (3.3.6.5)$$

(iii) Solving the algebraic equation(3.3.6), we have the following solutions of a_0, a_{-1}, a_{-2}, a_1, a_2, b_0, b_{-1}, b_1, ω.

*Case*1: when $A=0$, we have

$$a_0 = 0, a_{-1} = 0, a_1 = -2BCk^2, a_{-2} = 0, a_2 = -2C^2k^2,$$
$$b_0 = b_0, b_{-1} = 0, b_1 = \pm 2Ck, \omega = \pm Bk - b_0 \quad (3.3.7)$$

*Case*2: when $B=0$, we have

$$a_0 = 0, a_{-1} = 0, a_1 = 0, a_{-2} = -2A^2k^2, a_2 = -2C^2k^2, b_0 = b_0, b_{-1} = \pm Ak, b_1 = \pm 2Ck, \omega = -b_0; \quad (3.3.8)$$

$$a_0 = -2ACk^2, a_{-1} = 0, a_1 = 0, a_{-2} = -2A^2k^2, a_2 = -2C^2k^2, b_0 = b_0, b_{-1} = \mp Ak, b_1 = \pm 2Ck, \omega = -b_0; \quad (3.3.9)$$

$$a_0 = -2ACk^2, a_{-1} = 0, a_1 = 0, a_{-2} = 0, a_2 = -2C^2k^2, b_0 = b_0, b_{-1} = 0, b_1 = \pm 2Ck, \omega = -b_0; \quad (3.3.10)$$

*case*3: when $A = B = 0$, we have

$$a_0 = 0, a_{-1} = 0, a_1 = 0, a_{-2} = 0, a_2 = -2C^2k^2, b_0 = b_0, b_{-1} = 0, b_1 = \pm 2Ck, \omega = -b_0 \quad (3.3.11)$$

Substituting these solutions into (3.3.4) and (3.3.5),from Appendix,we can obtain many soliton-like solutions, trigonometric function solutions and rational solutions of Eq.(3.3.1)(where we left the same type solutions out):

(I) when $A=0$, $B=1$, $C=-1$; from Appendix, then $F(\xi)$ $F(\xi) = \frac{1}{2} + \frac{1}{2}\tanh(\frac{1}{2}\xi)$. By *case*1, we have soliton-like solutions of Eq.(3.3.1)

$$u_1 = \frac{1}{2}k^2\,\mathrm{sech}^2[\frac{1}{2}k(x+(k-b_0)t)],\ v_1 = b_0 - k - k\tanh[\frac{1}{2}k(x+(k-b_0)t)];$$

$$u_2 = \frac{1}{2}k^2\,\mathrm{sech}^2[\frac{1}{2}k(x-(k+b_0)t)],\ v_2 = b_0 + k + k\tanh[\frac{1}{2}k(x-(k+b_0)t)].$$

(II) when $A=0$, $B=-1$, $C=1$; from Appendix, then $F(\xi) = \frac{1}{2} - \frac{1}{2}\coth(\frac{1}{2}\xi)$. By *case*1, we have soliton-like solutions of Eq.(3.3.1)

$$u_3 = -\frac{1}{2}k^2 \operatorname{csch}^2[\frac{1}{2}k(x-(k+b_0)t)],\ v_3 = b_0 + k - k\coth[\frac{1}{2}k(x-(k+b_0)t)];$$

$$u_4 = -\frac{1}{2}k^2 \operatorname{csch}^2[\frac{1}{2}k(x+(k-b_0)t)],\ v_4 = b_0 - k + k\coth[\frac{1}{2}k(x+(k-b_0)t)].$$

(III) when $A=\frac{1}{2}$, $B=0$, $C=-\frac{1}{2}$; from Appendix, then $F(\xi)=\coth\xi \pm \operatorname{csch}\xi$ or $\tanh\xi \pm i\operatorname{sech}\xi$. By $case2$,we have soliton-like solutions of Eq.(3.3.1)

$$u_5 = -k^2(1+2\operatorname{csch}^2[k(x-b_0t)]),\ v_5 = b_0 \pm 2k\operatorname{csch}[k(x-b_0t)];$$

$$u_6 = -\frac{(e^{4kx}+e^{4kb_0t}-6e^{2k(x+b_0t)})k^2}{(e^{2kx}+e^{2kb_0t})^2},\ v_6 = b_0 \mp 2ik\operatorname{sech}[k(x-b_0t)];$$

$$u_7 = -2k^2\operatorname{csch}^2[k(x-b_0t)],\ v_7 = b_0 \pm k(\coth[\frac{1}{2}k(x-b_0t)]+\tanh[\frac{1}{2}k(x-b_0t)]);$$

$$u_8 = -2k^2\operatorname{csch}^2[k(x-b_0t)],\ v_8 = b_0 \pm 2k\coth[k(x-b_0t)];$$

$$u_9 = \frac{8k^2e^{2k(x+b_0t)}}{(e^{2kx}+e^{2kb_0t})^2},\ v_9 = b_0 \pm 2k\tanh[k(x-b_0t)];$$

$$u_{10} = \frac{-k^2}{\cosh[k(x-b_0t)]-1},\ v_{10} = b_0 \pm k\coth[\frac{1}{2}k(x-b_0t)];$$

$$u_{11} = \frac{k^2}{\cosh[k(x-b_0t)]+1},\ v_{11} = b_0 \pm k\tanh[\frac{1}{2}k(x-b_0t)];$$

$$u_{12} = \frac{-ik^2}{\sinh[k(x-b_0t)]-i},\ v_{12} = b_0 \pm k(i\operatorname{sech}[k(x-b_0t)]+\tanh[k(x-b_0t)]);$$

$$u_{13} = \frac{ik^2}{\sinh[k(x-b_0t)]+i},\ v_{13} = b_0 \pm k(\tanh[k(x-b_0t)]-i\operatorname{sech}[k(x-b_0t)]).$$

(IV) when $A=1$, $B=0$, $C=-1$; from Appendix, then $F(\xi)$ $F(\xi)=\tanh\xi$ or $\coth\xi$. By $case2$,we have soliton-like solutions of Eq.(3.3.1)

$$u_{14} = -2k^2(\coth^2[k(x-b_0t)]+\tanh^2[k(x-b_0t)]),\ v_{14} = b_0 \pm 4k\operatorname{csch}[2k(x-b_0t)];$$

$$u_{15} = -8k^2\operatorname{csch}^2[2k(x-b_0t)],\ v_{15} = b_0 \pm 2k(\coth[k(x-b_0t)]+\tanh[k(x-b_0t)]);$$

$$u_{16} = 2k^2\operatorname{sech}^2[k(x-b_0t)],\ v_{16} = b_0 \pm 2k\tanh[k(x-b_0t)];$$

$$u_{17} = -2k^2\operatorname{csch}^2[k(x-b_0t)],\ v_{17} = b_0 \pm 2k\coth[k(x-b_0t)].$$

(V) when $A=C=\frac{1}{2}$, $B=0$, from Appendix, then $F(\xi)=\sec\xi+\tan\xi$ or $\csc\xi-\cot\xi$. By $case2$, we have trigonometric function solutions of Eq.(3.3.1)

$$u_{18} = \frac{k^2(-1-(\sec[k(x-b_0t)]+\tan[k(x-b_0t)])^4)}{2(\sec[k(x-b_0t)]+\tan[k(x-b_0t)])^2},\ v_{18} = b_0 \pm 2k\sec[k(x-b_0t)];$$

$$u_{19} = k^2(1-2\csc^2[k(x-b_0t)]),\ v_{19} = b_0 \pm 2k\csc[k(x-b_0t)];$$

$$u_{20} = \frac{-8k^2e^{2ik(x+b_0t)}}{(e^{2ikx}+e^{2ikb_0t})^2},\ v_{20} = b_0 \pm 2k\tan[k(x-b_0t)];$$

$$u_{21} = -2k^2\csc^2[k(x-b_0t)],\ v_{21} = b_0 \pm 2k\cot[k(x-b_0t)];$$

$$u_{22} = \frac{k^2}{\sin[k(x-b_0t)]-1},\ v_{22} = b_0 \pm k(\sec[k(x-b_0t)]+\tan[k(x-b_0t)]);$$

$$u_{23}=\frac{-k^2}{\cos[k(x-b_0t)]+1},\quad v_{23}=b_0\pm k\tan[\frac{1}{2}k(x-b_0t)].$$

(VI) when $A=C=-\frac{1}{2},\ B=0$; from Appendix, then $F(\xi)=\sec\xi-\tan\xi$ or $\csc\xi+\cot\xi$. By $case2$, we have trigonometric function solutions of Eq.(3.3.1)

$$u_{24}=\frac{(e^{4ikx}+e^{4ikb_0t}-6e^{2ik(x+b_0t)})k^2}{(e^{2ikx}+e^{2ikb_0t})^2},\quad v_{24}=b_0\pm 2k\sec[k(x-b_0t)];$$

$$u_{25}=-2k^2\sec^2[k(x-b_0t)],\quad v_{25}=b_0\pm 2k\tan[k(x-b_0t)];$$

$$u_{26}=-2k^2\csc^2[k(x-b_0t)],\quad v_{26}=b_0\pm k(\tan[\frac{1}{2}k(x-b_0t)]-\cot[\frac{1}{2}k(x-b_0t)]);$$

$$u_{27}=\frac{-k^2}{\sin[k(x-b_0t)]+1},\quad v_{27}=b_0\pm k(\tan[k(x-b_0t)]-\sec[k(x-b_0t)]);$$

$$u_{28}=\frac{k^2}{\cos[k(x-b_0t)]-1},\quad v_{28}=b_0\pm k\cot[\frac{1}{2}k(x-b_0t)].$$

(VII) when $A=C=1,\ B=0$; from Appendix, then $F(\xi)=\tan\xi$. By $case2$, we have trigonometric function solutions of Eq.(3.3.1)

$$u_{29}=-2k^2(\cot^2[k(x-b_0t)]+\tan^2[k(x-b_0t)]),\qquad v_{29}=b_0\pm 4k\csc[2k(x-b_0t)];$$

$$u_{30}=-8k^2\csc^2[2k(x-b_0t)],\qquad v_{30}=b_0\pm 2k(\tan[k(x-b_0t)]-\cot[k(x-b_0t)]).$$

(Ⅷ) when $A=C=1,\ B=0$; from Appendix, then $F(\xi)=\cot\xi$. By $case2$, we have trigonometric function solutions of Eq.(3.3.1)

$$u_{31}=-2k^2\csc^2[k(x-b_0t)],\qquad v_{31}=b_0\pm 2k\cot[k(x-b_0t)].$$

(IX) when $A=B=0, C\neq 0$; from Appendix, then $F(\xi)=-\frac{1}{C\xi+\lambda}$. By $case3$, we have rational solutions of Eq.(3.3.1)

$$u_{32}=\frac{-2C^2k^2}{(Ck(x-b_0t)+\lambda)^2},\qquad v_{32}=b_0\pm\frac{2Ck}{Ck(x-b_0t)+\lambda}.$$

Where C,k,b_0, λ are arbitrary constants and $C\neq 0$.

Following, we provide some figures of partial solutions for direct-viewing analysis. we choose k=1.

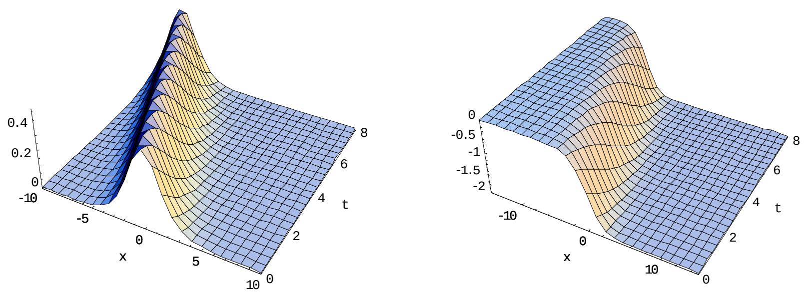


Fig. 8: The solitary solutions u_1, v_1 at b_0=0

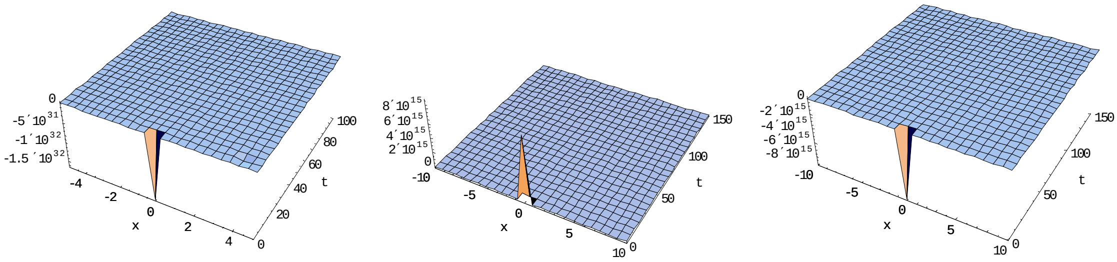


Fig. 9: The soliton-like solutions u_{14} , v_{14} at b_0=1

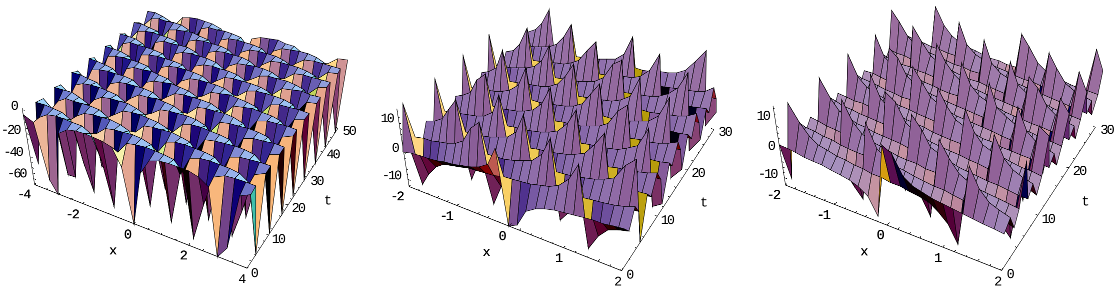


Fig. 10: The periodic solutions u_{26} , v_{26} at b_0=1

4. Conclusion

In this paper, using our modified F-expansion method, we have considered some illustrative equations and derived abundant solutions for them, including soliton-like solutions, trigonometric function solutions and rational solutions, most of which have not appeared in those known literatures. They should be meaningful to explain some physics phenomena. We can also see that the method overcomes some disadvantages of F-expansion method and can be applied to more nonlinear PDEs. Moreover, with the aid of computer symbolic systems like *Maple* or *Mathematica*, the method can be conveniently operated.

5. Acknowledgments

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6. Appendix

Relations between values of (A, B, C) and corresponding $F(\xi)$ in Riccati equation

$$F'(\xi) = A + BF(\xi) + CF^2(\xi)$$

A	B	C	F
0	1	-1	$\frac{1}{2} + \frac{1}{2}\tanh(\frac{1}{2}\xi)$
0	-1	1	$\frac{1}{2} - \frac{1}{2}\coth(\frac{1}{2}\xi)$
$\frac{1}{2}$	0	$-\frac{1}{2}$	$\coth\xi \pm \operatorname{csch}\xi$, $\tanh\xi \pm i\operatorname{sech}\xi$
1	0	-1	$\tanh\xi$, $\coth\xi$
$\frac{1}{2}$	0	$\frac{1}{2}$	$\sec\xi + \tan\xi$, $\csc\xi - \cot\xi$

$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\sec\xi - \tan\xi$, $\csc\xi + \cot\xi$
1(-1)	0	1(-1)	$\tan\xi$ ($\cot\xi$)
0	0	$\neq 0$	$-\frac{1}{C\xi+\lambda}$ (λ is arbitrary constant)

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