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A Fourth Order Dual Method for Iteration Regularization with H^{-1} Fidelity Based Denoising

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Abstract. In this paper, we propose iterative regularization for image denoising problems, based on the total variation minimizing models proposed by Rudin, Osher, and Fatemi(ROF). Besides, considering the staircase occuring in the process of denoising, we combine the higher order derivatives, and use iterative scheme. The fourth order dual method is used to solve the minimization problems. The numerical experiments show the iterative procedure preserves more details and reduces staircasing. Besides, it can be claimed that the fourth order dual method is more faster and stable than time marching algorithms.

Keywords: image denoising; total variation; fourth order dual method; iterative regularization

1. Introduction

Image denoising is a very important problem in image processing that has seen many recent developments. One of the most popular model is proposed by Rudin, Osher, and Fatemi(ROF)[1]：

$$u = \arg \min_{u \in BV(\Omega)} \left\{ |u|_{BV} + \lambda \|f - u\|_{L^2}^2 \right\} \tag{1}$$

Here f is the observed image, u is the restored image, $\Omega \subset R^2$ is an open domain (usually a rectangle in R^2),and $BV(\Omega)$ is the space of functions of bounded variation. $|\cdot|_{BV}$ denotes the seminorm, formally given by

$$|u|_{BV} = \int_\Omega |\nabla u| \, dx$$

The ROF model is well known for its ability to remove noise while preserving sharp edges. However, the ROF model (1) has certain limitations. The ideal result of the minimization procedure (1) would be to decompose f into the true signal u and the additive noise $v := f - u$. In practice, we often find some signals in v. In [2], Meyer did some interesting analysis on the ROF model. According his analysis and the numerical examples, L^2 space is not appropriate for modeling oscillatory patterns. He suggested the use of the dual to the space $BV(\Omega)$ to capture oscillatory functions. Considering the feasibility of numerical implementation, Meyer [2] suggested to approximate $(BV(\Omega))'$ by the following slightly larger space:

Definition. Let G denotes the Banach space consisting of all generalized functionnals $f(x)$ which can be written as

$$f(x) = \partial_1 g_1(x) + \partial_2 g_2(x), g_1, g_2 \in L^\infty(R^2)$$

The norm $\|f\|_*$ of f in G is defined as the following:

$$\|f\|_* = \inf_{\vec{g}=(g_1,g_2)} \left\{ \left\| \sqrt{g_1^2 + g_2^2} \right\|_{L^\infty} \mid f = \partial_1 g_1(x) + \partial_2 g_2(x) \right\}$$

Given a function f defined on Ω, Meyer's decomposition model then follows as:

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$$u = \arg \min_{u \in BV(\Omega)} \left\{ |u|_{BV} + \lambda \|f - u\|_* \right\} \quad (2)$$

However, in practice, model (2) is difficult to implement due to the intrinsic nature of the * norm. Vese and Osher[3] used the space G_p to replace G, which is defined as the following:

$$G_p(\Omega) = \left\{ f = div\, \vec{g}, \vec{g} = (g_1, g_2), g_1, g_2 \in L^P(\Omega) \right\}$$

induced by the norm

$$\|f\|_{G_p(\Omega)} = \inf_{f = div\, \vec{g}, g_1, g_2 \in L^P(\Omega)} \left\| \sqrt{g_1^2 + g_2^2} \right\|_p$$

Then, they proposed the following minimization problem:

$$u = \arg \min_{u \in BV(\Omega), \vec{g} \in L^p(\Omega)^2} \left\{ |u|_{BV} + \lambda \|f - u\|_{G_p(\Omega)}^2 \right\} \quad (3)$$

The model (OSV) by Osher et al.[4], as a continuation of the model in [5],can be seen a modification of the particular case $p = 2$ from (3):

$$u = \arg \min_{u \in BV(\Omega), \vec{g} \in L^2(\Omega)^2} \left\{ |u|_{BV} + \lambda \|f - u\|_{H^{-1}(\Omega)}^2 \right\} \quad (4)$$

Here,

$$\|v\|_{H^{-1}(\Omega)}^2 = \int_\Omega |\nabla \Delta^{-1} v|^2 \, dx$$

It is shown numerically, by the authors in [4,5],that model(4) is indeed a good approximation to Meyer's model(2).

To overcome the staircasing in reconstruction from TV regularization, Chambolle and Lions(CL)[6] introduced higher derivatives into the energy. A variant of the CL model, combined the H^{-1} norm, is introduced by Chan, Esedoglu, and Park (CEP)[7,8] for fast staicasing reduction. Their model, which is called $CEP2 - H^{-1}$, has the following formulation:

$$u = \arg \min_{u = u_1 + u_2} \left\{ |u_1|_{BV} + \alpha \|\Delta u_2\|_{L^1} + \lambda \|f - u_1 - u_2\|_{H^{-1}(\Omega)}^2 \right\} \quad (5)$$

Numerical examples have shown $CEP2 - H^{-1}$ model (5) can effectively reduces staircasing.

2. Iterative Regularization on ROF Model, OSV Model and $CEP2 - H^{-1}$ Model

In ROF model, in practice, v not only consists of noise, but also contains many details, e.g. "texture". However, in the process of removing noise, the details of the image are also lost. Naturally, we proposed the following iterative ROF model:

$$\begin{aligned} & v_0 = 0 \\ & u_{k+1} = \arg \min_{u \in BV(\Omega)} \left\{ |u|_{BV} + \lambda \|f + v_k - u\|_{L^2}^2 \right\} \\ & v_{k+1} = f + v_k - u_{k+1} \end{aligned} \quad (6)$$

which is also derived from the Bregman distance(see[9,10]) and the multiscale representation(see[11]).

As the same reason as discussed in[9], the parameter λ should decrease in order to access small scale features that have been eliminated in the previous steps. Then we derived the following modified iterative process:

$$
\begin{aligned}
& v_0 = 0 \\
& u_{k+1} = \arg\min_{u \in BV(\Omega)} \left\{ |u|_{BV} + \frac{1}{2^k \lambda} \| f + v_k - u \|_{L^2}^2 \right\} \\
& v_{k+1} = f + v_k - u_{k+1}
\end{aligned}
\tag{7}
$$

Likewise, we proposed the iterative OSV model:

$$
\begin{aligned}
& v_0 = 0 \\
& u_{k+1} = \arg\min_{u \in BV(\Omega)} \left\{ |u|_{BV} + \frac{1}{2^k \lambda} \| f + v_k - u \|_{H^{-1}(\Omega)}^2 \right\} \\
& v_{k+1} = f + v_k - u_{k+1}
\end{aligned}
\tag{8}
$$

The iterative $CEP2 - H^{-1}$ model has the following formulation:

$$
\begin{aligned}
& v_0 = 0 \\
& u_{k+1} = \arg\min_{u = u_1 + u_2} \left\{ |u_1|_{BV} + \alpha \|\Delta u_2\|_{L^1} + \frac{1}{2^k \lambda} \| f + v_k - u_1 - u_2 \|_{H^{-1}(\Omega)}^2 \right\} \\
& v_{k+1} = f + v_k - u_{k+1}
\end{aligned}
\tag{9}
$$

3. The Fourth Order Method for the Iterative $CEP2 - H^{-1}$ Model

In this section, we only present the fourth order method (which is introduced by Tony F.Chan et al. in [7,8]) for the iterative $CEP2 - H^{-1}$ model, which is similar as used on other models.

To derive u_{k+1} from (9) at step k, we minimize the coupled problems:

for $u_{k,2}$ fixed, solve for $u_{k+1,1}$:

$$u_{k+1,1} = \arg\min_{u_1} \left\{ \int_\Omega |\nabla u_1| + \frac{1}{2^k \lambda} \int_\Omega |\nabla \Delta^{-1}(f + v_k - u_1 - u_{k,2})|^2 dx \right\} \tag{10}$$

for $u_{k+1,1}$ fixed, solve for $u_{k+1,2}$:

$$u_{k+1,2} = \arg\min_{u_2} \left\{ \alpha \int_\Omega |\Delta u_2| + \frac{1}{2^k \lambda} \int_\Omega |\nabla \Delta^{-1}(f + v_k - u_{k+1,1} - u_2)|^2 dx \right\} \tag{11}$$

For simplicity, we will drop the $k+1$ from $u_{k+1,1}$, $u_{k+1,2}$ in the following.

To solve the u_1 component, we consider the following primal-dual formulation for (10):

$$\min_{u_1} \max_{\xi} \left\{ \int_\Omega u_1 div\xi dx + \frac{1}{2^k \lambda} \int_\Omega |\nabla \Delta^{-1}(f + v_k - u_1 - u_2)|^2 dx \mid \xi \in C_c^1(\Omega; R^2), |\xi(x)| \le 1, \forall x \in \Omega \right\} \tag{12}$$

The above equation (12) is convex in u_1 and concave in ξ, so swapping the min and max yields:

$$\max_{w = div(\xi)} \min_{u_1} \left\{ \int_\Omega u_1 w dx + \frac{1}{2^k \lambda} \int_\Omega |\nabla \Delta^{-1}(f + v_k - u_1 - u_2)|^2 dx \mid \xi \in C_c^1(\Omega; R^2), |\xi(x)| \le 1, \forall x \in \Omega \right\} \tag{13}$$

Then, for each fixed w,

$$\min_{u_1} \left\{ \int_\Omega u_1 w dx + \frac{1}{2^k \lambda} \int_\Omega |\nabla \Delta^{-1}(f + v_k - u_1 - u_2)|^2 dx \mid \xi \in C_c^1(\Omega; R^2), |\xi(x)| \le 1, \forall x \in \Omega \right\} \tag{14}$$

yieds a minimizer $u_1 = f + v_k - u_2 + 2^{k+1} \lambda \Delta w$. Substituting this expression for u_1 back into (13), yields the following full dual minimization problem with respect to w:

$$-\min_{w,|w|\le 1}\left\{2^{k-2}\lambda\int_\Omega|\nabla w|^2\,dx-\int_\Omega(f+v_k-u_2)w\,dx\right\} \tag{15}$$

Since we only concern with $v=div(\xi)$, minus "-" can be removed. Besides, in the discrete setting, equation (15) can be set as a constrained minimization problem with inequality constrains:

$$\min_{p,|p|\le 1}\left\{2^{k-2}\lambda\int_\Omega|\nabla div(p)|^2\,dx-\int_\Omega(f+v_k-u_2)div(p)\,dx\right\} \tag{16}$$

Optimality conditions of (16) read:

$$\nabla(f+v_k-u_2)_{i,j}+2^{k-1}\lambda\nabla\Delta div(p)_{i,j}+\alpha_{i,j}p_{i,j}=0 \tag{17}$$

where $\alpha_{i,j}$ are the Lagrange Multipiers. Using the observation and essential contribution in [12],

$$\alpha_{i,j}=|\nabla(f+v_k-u_2)_{i,j}+2^{k-1}\lambda\nabla\Delta div(p)_{i,j}|$$

Setting $A(p)_{i,j}=\nabla(f+v_k-u_2)_{i,j}+2^{k-1}\lambda\nabla\Delta div(p)_{i,j}$, then (17) reduces to:

$$A(p)_{i,j}+|A(p)_{i,j}|\,p_{i,j}=0$$

This equation can be solved by the semi-implicit gradient decent (fixed point) iteration introduced in [12,13]:

$$p^{n+1}=p^n-\tau(A^n_{i,j}+|A^n_{i,j}|\,p^{n+1}_{i,j})$$

The above scheme can simplify to the explicit iteration schemes:

$$p^0=0,\ p^{n+1}_{i,j}=\frac{p^n_{i,j}-\tau A^n_{i,j}}{1+\tau|A^n_{i,j}|} \tag{18}$$

where $A^n_{i,j}=\nabla(f+v_k-u_2)_{i,j}+2^{k-1}\lambda\nabla\Delta div(p^n)_{i,j}$,where $u_1^n=f+v_k-u_2+2^{k-1}\lambda\Delta p^n\to u_1$ as $n\to\infty$ for τ small enough, with u_1 a solution to (10) (see [8] for the proof).

A solution u_2 of (11) can be obtained by solving a dual formulation in the same way as (12). Indeed, the primal-dual formulation of (11) follows as:

$$\min_{u_2}\max_{\xi}\left\{\alpha\int_\Omega u_2\Delta\xi dx+\frac{1}{2^k\lambda}\int_\Omega|\nabla\Delta^{-1}(f+v_k-u_1-u_2)|^2dx\,|\,\xi\in C_c^2(\Omega;R),-1\le\xi\le 1\right\} \tag{19}$$

By using the same method as above,(19) is equivalent to solving the following full dual problem:

$$\min_{p,|p|\le 1}\left\{2^{k-2}\alpha\lambda\int_\Omega|\nabla(\Delta p)|^2\,dx-\int_\Omega(f+v_k-u_1)\Delta p\,dx\right\} \tag{20}$$

whose solution can be obtained from the semi-implicit gradient descent (fixed point iteration):

$$p^0=0,\ p^{n+1}_{i,j}=\frac{p^n_{i,j}+\tau A^n_{i,j}}{1+\tau|A^n_{i,j}|} \tag{21}.$$

where $A^n_{i,j}=\Delta(2^{k-1}\alpha\lambda\Delta^2p^n+(f+v_k-u_1))_{i,j}$. It can be shown that $u_2^n=f+v_k-u_1+2^{k-1}\alpha\lambda\Delta p^n\to u_2$ as $n\to\infty$ for τ small enough, with u_2 a solution to (11).

Based the above analysis, we present the algorithm:

Initialize $v_0=0$;

For $k=0,1,2\cdots$:

$u_{k+1,1}=f+v_k;u_{k+1,2}=0;$

compute $u_{k+1,1}$ and $u_{k+1,2}$ respectively as minimizers of (10) and (11), then

$$u_{k+1} = u_{k+1,1} + u_{k+1,2}, v_{k+1} = f + v_k - u_{k+1}$$

4. Numerical Results and Comparisons

In this section, we present numerical results of image denoising, obtained with the proposed iterative model. SNR (noise-to-signal ratio) is formulated as following:

$$SNR = \frac{\sum_{i,j}\left[f_{i,j} - \frac{\sum_{i,j} u_{i,j}}{MN}\right]^2}{\sum_{i,j}\left[f_{i,j} - u_{i,j} - \frac{\sum_{i,j}(f_{i,j} - u_{i,j})}{MN}\right]^2}$$

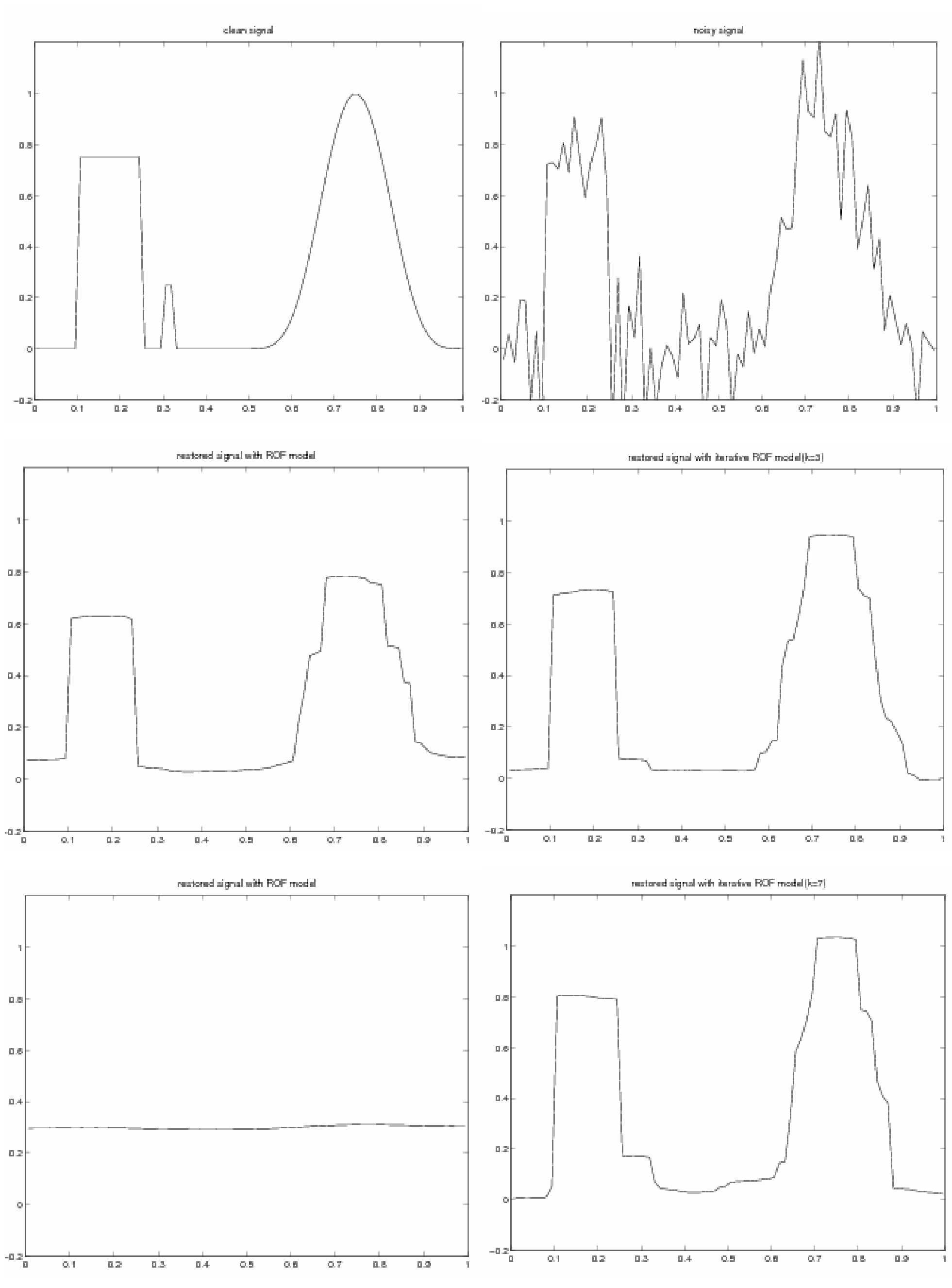

Figure 1. *1-d signal*. Top left and right: clean and noisy signals, SNR=5.2623. Middle left and right: recovered signals from ROF model and iterative ROF model at step 3. Here, λ =0.009, SNR=9.5569, 29.6689. Bottom left and right, recovered signals from ROF model and iterative ROF model at step 7,λ = 0.09,SNR=0.0003,28.1183.

In the first experiment, we deal with a 1-d signal. In Figure 1, top left and right, clean and noisy signals are respectively observed. Middle left and right, recovered signals obtained from the ROF model and the iterative ROF model respectively. Staircasing appears comparable in the two signals, but parameters are less sensitive in the iterative ROF model. Signals bottom left and right better illustrate the flexility of parameters selection with the iterative regulation. In table 1, we tabulate the different SNR as k increases when λ = 0.009.From the table, we can see that selecting a proper " k " is critical.

In the following example, we process an intensity image with 256 × 256 pixels and range [0,255].The noisy image is obtained by adding a gaussian white noise with variance $\sigma = 0.01$ to the original image. In Figure 3, left and right, the original and observed image are presented, SNR=2.5662. Figure 4 shows restored images and removed noises at different steps k obtained with iterative $CEP2 - H^{-1}$ model. At earlier stage, u_k is over-smoothed, and v_k contains many "texture". As k increases, u_k is getting closer to the exact image. But as k continues to increase, noises again "come back". In table 2, we tabulate the different values of SNR in the iterative process.

Figure 3. *Clean and noisy image.* Left: clean image; right: noisy image. SNR=2.5662.

Figure 4. *Images obtained with iterative* $CEP2 - H^{-1}$ *model.* Top, from left to right: recovered images obtained at step 1,5,10 respectively, and its corresponding SNR = 16.5485,20.6155,17.6302. Bottom, from left to right, removed noises $v = f - u_1 - u_2$ corresponding to the top one respectively. Apparently, images are clearer as k increases at the early steps, but noises return again after five steps.

Table1 Different SNR as k increases in the first experiment

k	1	2	3	4	5	6
SNR	9.5569	15.2664	29.6689	22.8270	19.1732	13.6038

Table2 Different SNR as k increases in the second experiment

k	1	2	5	7	8	10
SNR	16.5485	17.9523	20.6155	19.2053	18.8985	17.6302

5. Conclusion and future works

In this paper, we introduced iterative regularization in ROF, OSV and $CEP2-H^{-1}$ model, and used the fourth dual method for numerical solution. Numerical examples using iterative procedures represent better results than previous models. Some future works include some appropriate stop criteria and some discussions about selection of parameters.

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