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Computational Algorithm to Obtain Multiple Positive Solutions for Sublinear Semipositone Problems

G.A.Afrouzi [+], S.Mahdavi, Z.Naghizadeh

Department of Mathematics, Faculty of Basic Sciences, Mazandaran University, Babolsar, Iran

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Abstract. Using a numerical method based on sub-super solution, we will obtain positive solutions for the problem $\Delta u = g(\lambda, u)$ for $x \in \Omega$ with Dirichlet boundary condition. In particular, we establish multiplicity results for classes of nondecreasing, sublinear functions $g(\lambda, u)$ belongs to $C^1([0,\infty))$.

Keywords: multiple solutions; positive solutions; sub and super-solutions

1. Introduction

In this paper, we consider boundary value problems of the form

$$\begin{aligned} -\Delta u(x) &= g(\lambda, u(x)) \qquad & x \in \Omega \\ u(x) &= 0 & x \in \partial\Omega \end{aligned} \tag{1}$$

Where Δ denotes the laplacian operator, λ is a positive parameter and Ω is a bounded Domain in R^N with smooth boundary.

In order to state the results we recall an anti-maximum principle by Clement and Peletier (see [3]), from which we obtain the existence of a $\delta = \delta(\Omega) > 0$ and a solution z_α, positive in Ω, of

$$\begin{aligned} -\Delta z_\alpha - \alpha z_\alpha &= -1 \qquad & x \in \Omega \\ z_\alpha &= 0 & x \in \partial\Omega \end{aligned} \tag{2}$$

for $\lambda \in (\lambda_1, \lambda_1 + \delta)$, where λ_1 is the first eigenvalue of the $-\Delta$ with Dirichlet boundary condition. Throughout this paper we let α be a fixed number in $(\lambda_1, \lambda_1 + \delta)$ and z_α this corresponding solution.

We use the method of sub-super solutions to obtain positive solutions. By a super solution φ of (1) we mean a $C^1(\overline{\Omega})$ function such that $\varphi = 0$ on $\partial\Omega$ and

$$\int_\Omega \nabla\varphi . \nabla w \geq \int_\Omega g(\lambda, \varphi) w, \qquad \forall w \in W$$

and by a sub solution ψ of (1) we mean a $C^1(\overline{\Omega})$ function such that $\psi = 0$ on $\partial\Omega$ and

$$\int_\Omega \nabla\psi . \nabla w \leq \int_\Omega g(\lambda, \psi) w, \qquad \forall w \in W$$

where $W = \{v \in C_0^\infty \mid v \geq 0 in \Omega\}$. Then by the weak comparison principal (see [5]), if there exist sub and super solutions ψ and φ respectively such that $\psi \leq \varphi$ in Ω then (1) has a $C^1(\overline{\Omega})$ solution u such that $\psi \leq u \leq \varphi$. In the case where $g(\lambda, u) > 0$,clearly $\psi \equiv 0$ in $\overline{\Omega}$ is a sub solution and if $g(\lambda, .)$ is sublinear, $\varphi = Me$ where $e \in C^1(\overline{\Omega})$ is a solution of $-\Delta e = 1$ in Ω, e = 0 0n $\partial\Omega$, is a super

[+] *E-mail address*: afrouzi@umz.ac.ir

solution for M large enough. Thus we have a positive $C^1(\overline{\Omega})$ solution for every $\lambda > 0$.

As noted earlier, since the existence of a positive solution is trivial, we study the question Of multiplicity. In particular, we consider the case $g(\lambda,u) = \lambda f(u)$ and f satisfies:

(i) $f \in C^1([0,\infty))$ is a nondecreasing function such that f (0) = 0, $f(v) > 0 \ \forall v > 0$ and

$$\lim_{v\to\infty} \frac{f(v)}{v} = 0 \text{ (sublinear),}$$

(ii) there exists m > 0 such that $f(v) > v - m$, $\forall v \in [0, m\alpha \| z_\alpha \|_\infty]$, and

(iii) $\alpha < \lambda_1 (\lim_{v\to\infty} \frac{f(v)}{v} = 0)^{-1} = \mu$ (say).

Then we have:

Theorem 1. Consider the boundary value problem

$$\begin{aligned} -\Delta u &= \lambda f(u) \qquad && x \in \Omega \\ u &= 0 && x \in \partial\Omega \end{aligned} \tag{3}$$

Let (i)-(iii) hold. Then problem (3) has at least two positive $C^1(\overline{\Omega})$ solutions u_i; i = 1, 2 for all $\lambda \in [\alpha, \mu)$.

The proof of Theorem 1 discusses in [7].

We shall construct supersolution φ_1 and φ_2 and subsolutions ψ_1 and ψ_2 as follow:

Clearly, $\psi_1 \equiv 0$ is a sub solution since f (0) =0 and $\psi_2 = m\alpha z_\alpha$ is a strict sub solution for $\lambda \geq \alpha$. Also $\psi_2 > \psi_1$. Now consider $\varphi_1 = \varepsilon v$ where $\varepsilon > 0$ is to be chosen sufficiently small so that $\psi_2 \neq\leq \varphi_1$ and $v \in C^1(\overline{\Omega})$ is a positive solution of the eigenvalue problem

$$\begin{aligned} -\Delta v &= \lambda_1 v \qquad && x \in \Omega \\ v &= 0 && x \in \partial\Omega \end{aligned} \tag{4}$$

Finally let $\varphi_2 = Me$ where $M = M(\lambda)$ is to be choosen. Applying lemma 1.1 in [2], we obtain three positive $C^1(\overline{\Omega})$ solutions for $\lambda \in [\alpha, \mu)$ that one of them is trivial solution.

Here we give a simple example that satisfies assumption of Theorem 1. Consider

$$\begin{aligned} f(u) &= m^{\frac{3}{2}} u^2 \qquad && u \leq 1 \\ &= 4m^{\frac{3}{2}} u^{\frac{1}{2}} - 3m^{\frac{3}{2}} && u > 1 \end{aligned} \tag{5}$$

Clearly $f(0) = 0$ and $\lim_{v\to\infty} \frac{f(v)}{v} = 0$, that is, (i) is satisfied . Also $\lim_{v\to\infty} \frac{f(v)}{v} = 0$ satisfying condition (iii). Now let v_0 be the unique solution of $f(v) = S(v)$ where $S(v) = v - m$. Then for m large enough $v_0 > 16m^{\frac{3}{2}}$ hence for m sufficiently large $v_0 > m\alpha \| z_\alpha \|_\infty$ and $f(v)$ satisfies (ii). Thus for f in this class of nonlinearities, the equation $-\Delta u(x) = \lambda f(u)$ has at least two positive solutions for $\lambda \in [\alpha, \infty)$.

We investigate numerically positive solutions. Our numerical method is based on monotone iteration.

2. Numerical Results

It is well-known that there must always exists a solution for problems such as (1) between a sub-solution $\underline{v}$ and a super-solution $\overline{u}$ such that $\underline{v} \leq \overline{u}$ for all $x \in \Omega$ (see [1]).

Consider the boundary value problem

$$\begin{cases} \Delta u(x) + f(x, u(x)) = 0 & on\ \Omega \\ u(x) = 0 & on\ \partial\Omega. \end{cases}$$

Let $\overline{u}$, $\underline{v} \in C^2(\overline{\Omega})$ satisfy $\overline{u} \geq \underline{v}$ as well as

$$\Delta\overline{u}(x) + f(x, \overline{u}(x)) \leq 0 \quad on \Omega \quad \overline{u} \geq 0$$

$$\Delta\underline{v}(x) + f(x, \underline{v}(x)) \geq 0 on \Omega \quad \underline{v} \leq 0.$$

Choose a number $c > 0$ such that $c + \frac{\partial f(x,u)}{\partial u} > 0 \quad \forall (x,u) \in \overline{\Omega} \times [\underline{v}, \overline{u}]$ and such that the operator $(\Delta - c)$ with Dirichlet boundary condition has its spectrum strictly contained in the open left-half complex plane. Then the mapping

$$T : \phi \to w, \quad w = T\phi, \quad \phi \in C^2(\overline{\Omega}), \quad \phi(x) \in [\underline{v}, \overline{u}], \quad \forall x \in \overline{\Omega} \tag{5.1}$$

where $w(x)$ is the unique solution of the BVP

$$\begin{cases} \Delta w(x) - cw(x) = -[c\phi(x) + f(x, \phi(x))] & on\ \Omega \\ w(x) = 0 & on\ \partial\Omega, \end{cases}$$

is monotone, i.e. for any ϕ_1, ϕ_2 satisfying (3.1) and $\phi_1 \leq \phi_2$, we have $T\phi_1, T\phi_2$ satisfies (5.1), and $T\phi_1 \leq T\phi_2$ on Ω.

Consequently, by letting $f_c(x,u) = cu + f(x,u)$, the iterations

$$\begin{cases} u_0(x) = \overline{u}(x) & \\ (\Delta - c)u_{n+1}(x) = -f_c(x, u_n(x)) & on\ \Omega, \\ & n = 0,1,2,\ldots \\ u_{n+1}(x) = 0 & on\ \partial\Omega, \end{cases}$$

and

$$\begin{cases} v_0(x) = \underline{v}(x) & \\ (\Delta - c)v_{n+1}(x) = -f_c(x, v_n(x)) & on\ \Omega, \\ & n = 0,1,2,\ldots \\ v_{n+1}(x) = 0 & on\ \partial\Omega, \end{cases}$$

yield iteration u_n and v_n satisfying

$$\underline{v} = v_0 \leq v_1 \leq \cdots \leq v_n \leq \cdots \leq u_n \leq \cdots \leq u_1 \leq u_0 = \overline{u},$$

so that the limits

$$u_\infty(x) = \lim_{n\to\infty} u_n(x), \qquad v_\infty(x) = \lim_{n\to\infty} v_n(x)$$

exists in $C^2(\overline{\Omega})$. We have

(i) $v_\infty(x) \leq u_\infty(x)$ on $\overline{\Omega}$

(ii) u_∞ and v_∞ are, respectively, stable from above and below;

(iii) if $u_\infty \not\equiv v_\infty$ and both u_∞ and v_∞ are asymptotically stable, then there exists an unstable solution $\phi \in C^2(\overline{\Omega})$ such that $v_\infty \leq \phi \leq u_\infty$.

We use following algorithm.

sub- and super-solution algorithm:

1. Find a subsolution v_0 and a supersolution u_0. Choose a number $c > 0$;

2. Solve the boundary value problem

$$-\Delta w_{n+1}(x) - c w_{n+1}(x) = -f_c(x, w_n(x)) \qquad x \in \Omega$$
$$w_n(x) = 0 \qquad x \in \partial\Omega$$

for $w_n = v_n$ and $w_n = u_n$, respectively;

If $\| w_{n+1} - w_n \| < \varepsilon$, output and stop. Else go to step 2.

We will use the notation **u** to represent an array of real numbers agreeing with u on a grid $\Omega \subset \overline{\Omega}$. We will take the grid to be regular.

We consider the problem $-\Delta u(x) = \lambda f(x, u(x))$ with $\Omega = [0,1]\times[0,1]$ and $f(u)$ is defined by (5).

For doing step 1, we solve problem (2) to obtain subsolution.

We know from Introduction that problem (2) has a positive solution for $(\lambda_1, \lambda_1 + \delta)$.

The obtained results shows there is an array of positive solution for $\lambda \in (17,35)$ so λ_1 is around 17.

Let $\psi_1 \equiv 0$ and $\psi_2 = m\alpha z_\alpha$ where z_α , α and m obtained from section 1 as subsolutions in our algorithm for $\lambda \ge \alpha$ and to obtain supersolution for $\lambda < \mu$ we solve

$$-\Delta e(x) = 1 \quad x \in \Omega$$
$$e(x) = 0 \quad x \in \partial\Omega \tag{6}$$

by finite difference (see [4,6]). We execute algorithm for $\lambda \in [17.1, \infty)$ for ψ_1 and φ_1 as sub and super solutions to obtain u_1 and for ψ_2 and φ_2 as sub and super solutions to obtain second solution u_2. For brevity we express just some of those numerical results.

Approximation of u_2 and u_1 for $\lambda = 17.1$ respectively

x/y	0.2	0.4	0.6	0.8
0.2	1.539×10^3	2.304×10^3	2.304×10^3	1.539×10^3
0.4	2.304×10^3	3.475×10^3	3.475×10^3	2.304×10^3
0.6	2.304×10^3	3.475×10^3	3.475×10^3	2.304×10^3
0.8	1.539×10^3	2.304×10^3	2.304×10^3	1.539×10^3

x/y	0.2	0.4	0.6	0.8
0.2	0.018×10^{-4}	0.093×10^{-4}	0.093×10^{-4}	0.018×10^{-4}
0.4	0.093×10^{-4}	0.211×10^{-4}	0.211×10^{-4}	0.093×10^{-4}
0.6	0.093×10^{-4}	0.211×10^{-4}	0.211×10^{-4}	0.093×10^{-4}
0.8	0.018×10^{-4}	0.093×10^{-4}	0.093×10^{-4}	0.018×10^{-4}

Approximation of u_2 and u_1 for $\lambda = 30$ respectively

x/y	0.2	0.4	0.6	0.8
0.2	0.480×10^4	0.718×10^4	0.718×10^4	0.480×10^4
0.4	0.718×10^4	1.083×10^4	1.083×10^4	0.718×10^4
0.6	0.718×10^4	1.083×10^4	1.083×10^4	0.718×10^4
0.8	0.480×10^4	0.718×10^4	0.718×10^4	0.480×10^4

x/y	0.2	0.4	0.6	0.8
0.2	0.030×10^{-4}	0.068×10^{-4}	0.068×10^{-4}	0.030×10^{-4}
0.4	0.068×10^{-4}	0.106×10^{-4}	0.106×10^{-4}	0.068×10^{-4}
0.6	0.068×10^{-4}	0.106×10^{-4}	0.106×10^{-4}	0.068×10^{-4}
0.8	0.030×10^{-4}	0.068×10^{-4}	0.068×10^{-4}	0.030×10^{-4}

Approximation of u_2 and u_1 for $\lambda = 100$ respectively

x/y	0.2	0.4	0.6	0.8
0.2	0.540×10^{5}	0.808×10^{5}	0.808×10^{5}	0.540×10^{5}
0.4	0.808×10^{5}	1.217×10^{5}	1.217×10^{5}	0.808×10^{5}
0.6	0.808×10^{5}	1.217×10^{5}	1.217×10^{5}	0.808×10^{5}
0.8	0.540×10^{5}	0.808×10^{5}	0.808×10^{5}	0.540×10^{5}

x/y	0.2	0.4	0.6	0.8
0.2	0.105×10^{-3}	0.196×10^{-3}	0.196×10^{-3}	0.105×10^{-3}
0.4	0.196×10^{-3}	0.362×10^{-3}	0.362×10^{-3}	0.196×10^{-3}
0.6	0.196×10^{-3}	0.362×10^{-3}	0.362×10^{-3}	0.196×10^{-3}
0.8	0.105×10^{-3}	0.196×10^{-3}	0.196×10^{-3}	0.105×10^{-3}

Approximation of and u_1 for $\lambda = 1000$ respectively

x/y	0.2	0.4	0.6	0.8
0.2	0.542×10^{7}	0.811×10^{7}	0.811×10^{7}	0.542×10^{7}
0.4	0.811×10^{7}	1.223×10^{7}	1.223×10^{7}	0.811×10^{7}
0.6	0.811×10^{7}	1.223×10^{7}	1.223×10^{7}	0.811×10^{7}
0.8	0.542×10^{7}	0.811×10^{7}	0.811×10^{7}	0.542×10^{7}

x/y	0.2	0.4	0.6	0.8
0.2	0.340×10^{3}	0.631×10^{3}	0.631×10^{3}	0.340×10^{3}
0.4	0.631×10^{3}	1.192×10^{3}	1.192×10^{3}	0.631×10^{3}
0.6	0.631×10^{3}	1.192×10^{3}	1.192×10^{3}	0.631×10^{3}
0.8	0.340×10^{3}	0.631×10^{3}	0.631×10^{3}	0.340×10^{3}

Approximation of u_2 and u_1 for $\lambda = 10000$ respectively

x/y	0.2	0.4	0.6	0.8
0.2	0.543×10^{9}	0.812×10^{9}	0.812×10^{9}	0.543×10^{9}
0.4	0.812×10^{9}	1.223×10^{9}	1.223×10^{9}	0.812×10^{9}
0.6	0.812×10^{9}	1.223×10^{9}	1.223×10^{9}	0.812×10^{9}
0.8	0.543×10^{9}	0.812×10^{9}	0.812×10^{9}	0.543×10^{9}

x/y	0.2	0.4	0.6	0.8
0.2	0.497×10^6	0.769×10^6	0.769×10^6	0.497×10^6
0.4	0.769×10^6	1.196×10^6	1.196×10^6	0.769×10^6
0.6	0.769×10^6	1.196×10^6	1.196×10^6	0.769×10^6
0.8	0.497×10^6	0.769×10^6	0.769×10^6	0.497×10^6

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