ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 3, No. 4, 2008, *pp*. 258-268

Explicit solution of homotopy-perturbation method for some Fifth-order KdV equations and comparing it with other methods

D.D. Ganji [1], G.A. Afrouzi [2,*], R.A. Talarposhti [2]

1. Department of Mechanical Engineering, Mazandaran University, Babol, Iran
2. Department of Mathematics, Facuality of Basic Sciences, Mazandaran University, Babolsar, Iran

(*Received March 12, 2008, accepted September 14, 2008*)

Abstract. One of the newest analytical methods to solve nonlinear dispersive wave equations is using both homotopy and perturbation methods which is called (HPM). Other the reliable methods are variational iteration method (VIM) by He and Adomian's decomposition method (ADM). Here, we compare the exact solution of HPM which are applied to solve a various fifth-order Korteweg-de Vries problems with initial condition with obtained results of (VIM) and (ADM).Comparison of the results with those obtained by (ADM)and (VIM) reveals that (HPM) is very effective, convenient and quite accurate to both linear and nonlinear problems. It is predicted that (HPM) can be widely applied in engineering.

Keywords: Variational iteration method (VIM), Adomian decomposition method (ADM), Homotopy-Perturbation method (HPM), Fifth-order Korteweg-de Vries problems (FKdV), 2dimentional (2D).

AMS subject classification: 35B30, 35B40

1. Introduction

There are few phenomena in different fields of science which occure linearly. Most problems and scientific phenomena such as FKdV equations behave nonlinearly.

There are many standard methods in literature to solve fifth-order Korteweg-de Vries (FKdV) equations. Explicit solutions to the nonlinear equations are of fundamental importance. Various methods for obtaining explicit solutions have been proposed to nonlinear evolution equations. Among them are Hirota's dependent variable transformation, the inverse scattering transform, and the Backlund transformation.

All these methods are described in [1,2] and the references there in.

A common feature of all these methods is that they use different transformation to reduce the equation into a more simple one and then solve it.

The numerical calculation methods have been improving, so as semi-exact analytical methods. Most scientists believe that the combination of the numerical and semi-exact analytical methods can also lead to useful results. One of the most well-known semi-exact methods is the homotopy-perturbation method [2,3-8].

He's variational iteration method (VIM) [9-17] is used to conduct an analytic study on FKdv equation, too. The method gives rapidly convergent successive approximations of the exact solution if such a solution exists, otherwise approximations can be used for numerical purposes. The method is used more effectively in [9-15,18-21] among many others. Another important advantage is that VIM is capable of greatly reducing the size of calculations while still maintaining high accuracy of the numerical solution. In the following part, we will briefly highlight the main points of each method, whose details can be found in [9-17].

Considering Adomian's decomposition method, explicit and numerical solutions are calculated for various Fifth-order Korteweg-de Vries equations with specified initial conditions. The explicit solution of the equation using decomposition series is quickly obtained by existence of the self-canceling "noise" terms where the sum of the components is vanished at infinity.

* Corresponding author: Tel/Fax: +98 11252 42003
E-mail address: afrouzi@umz.ac.ir

We now infer that HPM, VIM and ADM can be used to construct the solution to the initial-value problems for FKdV equation [22,23],

$$u_t - u_{xxxxx} = F(x,t,u,u^2,u_x,u_{xx},u_{xxx}) \tag{1}$$

which occurs, for example, in the theory of magneto-acoustic waves in plasmas [24] and in the theory of shallow water waves with surface tension [25] the FKdV equation has been extensively investigated over the last decade. It has been shown that the traveling-wave solutions of the equation do not vanish at infinity [1,26]. Then the results of these methods have been compared with those of the exact solution.

2. Summery of the methods

2.1. Homotopy-perturbation method

To explain this method, let us consider the following function:

$$A(u) - f(r) = 0\ , \tag{2}$$

with the boundary condition of:

$$B(u\ ,\frac{\partial u}{\partial n}) = 0\,, \tag{3}$$

where $A(u)$ is defined as follows:

$$A(u) = L(u) + N(u)\,. \tag{4}$$

Homotopy-perturbation structure is shown as:

$$\begin{aligned} H(v,p) &= L(v) - L(u_0) + p\,L(u_0) \\ &+ p[N(v) - f(r)] = 0, \end{aligned}\ , \tag{5}$$

or

$$\begin{aligned} H(v,p) &= (1-p)[L(v) - L(u_0)] + p\,L(u_0) \\ &+ p[A(v) - f(r)] = 0, \end{aligned} \tag{6}$$

where,

$$v(r,p) : \Omega \times [0,1] \to R\,. \tag{7}$$

Obviously, considering Eqs. (5) and (6) we have:

$$\begin{aligned} H(v,0) &= L(v) - L(u_0) = 0, \\ H(v,1) &= A(v) - f(r) = 0, \end{aligned} \tag{8}$$

where $p \in [0,1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary condition.

The process of the changes in p from zero to unity is that of $v(r,p)$ changing from u_0 to u_r. We consider v as:

$$v = v_0 + p v_1 + p^2 v_2\,, \tag{9}$$

and the best approximation is:

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \ldots \tag{10}$$

The above convergence is discussed in [27, 28].

2.2. He's variational iteration method

To clarify the basic ideas of VIM, we consider the following differential equation:

$$L\,u + F\,u = g(t)\,, \tag{11}$$

where L is a linear operator, F a nonlinear operator and $g(t)$ a heterogeneous term.

According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(t)=u_n(t) + \int_0^t \lambda\,(L u_n(\tau)+F\,\tilde{u}_n(\tau)-g(\tau))\,d\tau, \tag{12}$$

where λ is a general Lagrangian multiplier [16,17,29] which can be identified optimally via the variational theory. By this method, it is necessary to determine the Lagrangian multiplier λ that will be identified optimally. The successive approximations u_{n+1}, $n \ge 0$, of the solution u will be easily obtained using the determined Lagrangian multiplier and any selective function of u_0. Consequently, the solution is given by $u = \lim_{n\to\infty} u_n$.

Let us consider the standard form of an FKdV equation, (1), in an operator form

$$L_t(u) - L_x(u) = F(x,\, t,\, u,\, u^2,\, u_x,\, u_{xx},\, u_{xxx}), \tag{13}$$

where the notation $\left(\frac{\partial}{\partial t}\right)$ and $\left(\frac{\partial^5}{\partial x^5}\right)$ symbolize the linear differential operators.

According to VIM, we can write down Eq. (13) in the form of:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda[L_t u_n(x,\tau) - L_x u_n(x,\tau) - F(x,\tau,u_n(x,\tau),u_n^2(x,\tau),\frac{\partial}{\partial x}u_n(x,\tau), \frac{\partial^2}{\partial x^2}u_n(x,\tau),\frac{\partial^3}{\partial x^3}u_n(x,\tau)]\,d\tau. \tag{14}$$

The Lagrangian multipliers are therefore identified as $\lambda = -1$.

2.3. Adomian's decomposition method

Considering relation (13), assuming the inverse of the operator L_t is exists and it can conveniently be taken as define integral with respect to t from 0 to t. If we operate the two sides of (13) with the inverse operator of L_t, we have:

$$L_t^{-1}L_t(u) = L_t^{-1}(F(x,t,u,u^2,u_x,u_{xx},u_{xxx})) + L_t^{-1}L_x(u) \tag{15}$$

Substituting the initial condition of (13) in the last formula we have:

$$u(x,t) - u(x,0) = L_t^{-1}(F(x,t,u,u^2,u_x,u_{xx},u_{xxx})) + L_t^{-1}L_x(u). \tag{16}$$

We decompose the unknown function $u(x,t)$ by a series of components defined by:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{17}$$

Substituting the initial condition into (16) and identifying the zeroth component u_0 by terms which arise from the initial condition, we have:

$$u_0 = \begin{cases} f(x), & for \;\; F=\psi(x,t)=0, \\ f(x)+L_t^{-1}(F(x,t)), & for \;\; F=\psi(x,t)\neq 0, \end{cases} \tag{18}$$

and if $F=\phi(x,t,u,u^2,u_x,u_{xx},u_{xxx})$, we obtain the subsequent components as the following recursive relationship:

$$u_{n+1} = L_t^{-1}(A_n) + L_t^{-1}L_x(u_n),\; n \ge 0 \tag{19}$$

and $N\,u=\psi(x,t,u,u^2,u_x,u_{xx},u_{xxx})=\sum_{n=0}^{\infty} A_n$ for $n \ge 0$. See the details of this method in [30].

3. Applications

Example 3.1. We consider an equation with the initial condition given by:

$$u_t + u u_x - u u_{xxx} - u_{xxxxx} = 0,$$

$$u(x,0) = e^x. \tag{20}$$

A variational iteration can be constructed as follows:

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) + \int_0^t \lambda [\frac{\partial}{\partial t} u_n(x,\tau) \\ & u_n(x,\tau)(\frac{\partial}{\partial x} u_n(x,\tau)) - u_n(x,\tau)(\frac{\partial^3}{\partial x^3} u_n(x,\tau)) \\ & - \frac{\partial^5}{\partial x^5} u_n(x,\tau)] d\tau. \end{aligned} \tag{21}$$

The Lagrangian multipliers are therefore identified as $\lambda = -1$, so we have:

$$\begin{aligned} & u_0(x,t) = e^x, \\ & u_1(x,t) = e^x + t e^x, u_2(x,t) = e^x + t e^x + \frac{1}{2} t^2 e^x, \\ & u_3(x,t) = e^x + t e^x + \frac{1}{2} t^2 e^x + \frac{1}{6} t^3 e^x, \end{aligned} \tag{22}$$

and so on. The solution $u(x,t)$ is given as :

$$u(x,t) = e^x \left(1 + t + \frac{1}{2!} t^2 + \frac{1}{3!} t^3 + \dots \right). \tag{23}$$

$u(x,t)$ in a closed form is found to be:

$$u(x,t) = e^x e^t = e^{x+t}. \tag{24}$$

Now we solve Eq. (20) with ADM.

Following the outlined scheme, Eq. (20) is rewritten in an operator form of:

$$L_t u = -u u_x + u u_{xxx} + L_x u. \tag{25}$$

thus,

$$\begin{aligned} u(x,t) = u(x,0) - L_t^{-1}(\sum_{n=0}^{\infty} A_n) + L_t^{-1}(\sum_{n=0}^{\infty} B_n) \\ + L_t^{-1} L_x(u), \end{aligned} \tag{26}$$

where $\sum_{n=0}^{\infty} A_n = u u_x$ and $\sum_{n=0}^{\infty} B_n = u u_{xxx}$.

Therefore we find:

$$\begin{aligned} u_1(x,t) &= -L_t^{-1}(A_0) + L_t^{-1}(B_0) + L_t^{-1} L_x(u_0) \\ &= t e^x, \\ u_2(x,t) &= -L_t^{-1}(A_1) + L_t^{-1}(B_1) + L_t^{-1} L_x(u_1) \\ &= \frac{1}{2!} t^2 e^x, \end{aligned} \tag{27}$$

$$u_3(x,t) = -L_t^{-1}(A_2) + L_t^{-1}(B_2) + L_t^{-1}L_x(u_2)$$
$$= \frac{1}{3!}t^3 e^x,$$

and so on. The solution $u(x,t)$ in the series form is given as:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = e^x\left(1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \ldots\right). \tag{28}$$

The solution $u(x,t)$ in a closed form is:

$$u(x,t) = e^x e^t = e^{x+t}. \tag{29}$$

A homotopy-perturbation method can be constructed as follows:

$$H(v,p) = (1-p)\left(\frac{\partial v(x,t)}{\partial t} - \frac{\partial v_0(x,t)}{\partial t}\right)$$
$$+ p\left(\frac{\partial v(x,t)}{\partial t} + v(x,t)\frac{\partial v(x,t)}{\partial x}\right. \tag{30}$$
$$\left. - v(x,t)\frac{\partial^3 v(x,t)}{\partial x^3} - \frac{\partial^5 v(x,t)}{\partial x^5}\right) = 0.$$

One can now try to obtain a solution of Eq. (30) in the form of:

$$v(x,t) = v_0(x,t) + p v_1(x,t) + \ldots, \tag{31}$$

where $v_i(x,y),\ i = 0,1,2,\ldots$ are functions yet to be determined. According to Eq. (30) the initial approximation is:

$$v_0(x,t) = u_0(x,t) = e^x. \tag{32}$$

Substituting Eqs. (31) and (32) into Eq. (30) yields:

$$\frac{\partial}{\partial t}v_1(x,t) - e^x = 0, \tag{33}$$

$$-\frac{\partial^5}{\partial x^5}v_1(x,t) + e^x\frac{\partial}{\partial x}v_1(x,t) + \frac{\partial}{\partial t}v_2(x,t)$$
$$-e^x\frac{\partial^3}{\partial x^3}v_1(x,t) = 0, \tag{34}$$

$$-\frac{\partial^5}{\partial x^5}v_2(x,t) + e^x\left(\frac{\partial}{\partial x}v_2(x,t) - \frac{\partial^3}{\partial x^3}v_2(x,t)\right)$$
$$+\frac{\partial}{\partial t}v_3(x,t) + v_1(x,t)\frac{\partial}{\partial x}v_1(x,t) \tag{35}$$
$$-v_1(x,t)\frac{\partial^3}{\partial x^3}v_2(x,t) = 0,$$

with the following conditions:

$$v_i(x,0) = 0, \qquad i = 0,1,2,3. \tag{36}$$

The solutions of Eqs. (33)-(35) may be written as follows:

$$v_1(x,t) = e^x t, \tag{37}$$

$$v_2(x,t) = \frac{1}{2!} t^2 e^x , \tag{38}$$

$$v_3(x,t) = \frac{1}{3!} t^3 e^x , \tag{39}$$

In the same manner, the rest of components were obtained by using the Maple Package.

According to the HPM, we can conclude:

$$u(x,t) = \lim_{p \to 1} v(x,t) = v_0(x,t) + v_1(x,t) + \dots \tag{40}$$

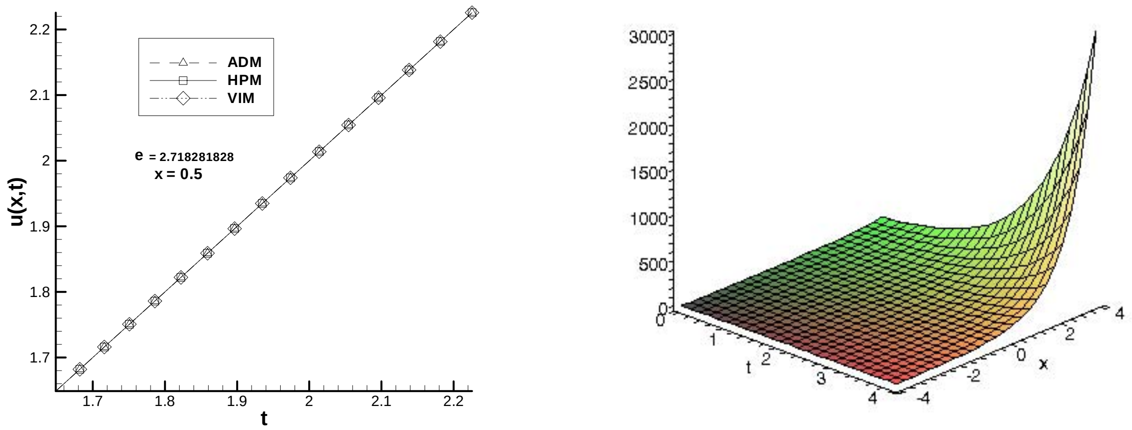
therefore,

$$u(x,t) = e^x \left(1 + t + \frac{1}{2!} t^2 + \frac{1}{3!} t^3 + \dots \right). \tag{41}$$

The solution $u(x,t)$ in a closed form is found to be:

$$u(x,t) = e^x e^t = e^{x+t} . \tag{42}$$

The exact solution of Eq. (20) with HPM is $u(x,t) = e^{x+t}$, that is equal with the obtained results of ADM, VIM.



(a) (b)

Fig1. The numerical result of the exact solution u(x,t) of Eq. (20) with HPM when $e = 2.718281828$, $x = 0.5$ which is equal to the obtained results of the VIM and ADM. (a) is the figure of 2D for exact solution and (b) is the figure of 3D for exact solution.

Example 3.2. Let's seek the explicit solution of the inhomogeneous FKdV equation, as follows:

$$u_t + u u_x - u_{xxx} + u_{xxxxx} = \sin(x) + 2t\cos(x) + \frac{1}{2} t^2 \sin(2x),$$

$$u(x,0) = 0 . \tag{43}$$

A variational iteration can be constructed as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda [\frac{\partial}{\partial t} u_n(x,\tau) + u_n(x,\tau)(\frac{\partial}{\partial x} u_n(x,\tau)) - \frac{\partial^3}{\partial x^3} u_n(x,\tau) + \frac{\partial^5}{\partial x^5} u_n(x,\tau) - \sin(x) + 2\tau\cos(x) - \frac{1}{2}\tau^2 \sin(2x)] d\tau \ . \tag{44}$$

The Lagrangian multiplier is therefore identified as $\lambda = -1$. So we have:

$$u_0(x,t) = t\sin(x), \tag{45}$$

and $u_i(x,t) = t\sin(x)$, $i = 0,1,2,3,\ldots$

hence,

$$u(x,t) = t\sin(x). \tag{46}$$

Now we solve Eq. (43) with ADM. We first used (43) in an operator form in the same manner as in (16) and then we used (19) to determine the individual terms of the decomposition series, we have:

$$u_0(x,t) = t\sin(x), \tag{47}$$

$$u_1(x,t) = -L_t^{-1}(A_0) + L_t^{-1}((u_0)_{xxx}) - L_t^{-1}L_x(u_0) + t^2\cos(x) + \frac{t^3}{6}\sin(2x) = 0, \tag{48}$$

$$u_{n+1}(x,t) = 0, \quad n \geq 1. \tag{49}$$

In case of right choice of these functions, the modified technique accelerates the convergence of the decomposition series solution by computing just u_0 and u_1 terms of the series. So,

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = t\sin(x). \tag{50}$$

A homotopy-perturbation method can be constructed as follows:

$$H(v,p) = (1-p)\left(\frac{\partial v(x,t)}{\partial t} - \frac{\partial v_0(x,t)}{\partial t}\right) + p\left(\frac{\partial v(x,t)}{\partial t} + v(x,t)\frac{\partial v(x,t)}{\partial x} - \frac{\partial^3 v(x,t)}{\partial x^3} + \frac{\partial^5 v(x,t)}{\partial x^5} - \sin(x) - \frac{t^2}{2}\sin(2x) - 2t\cos(x)\right) = 0. \tag{51}$$

One can now try to obtain a solution of Eq. (51), in the form of:

$$v(x,t) = v_0(x,t) + p\,v_1(x,t) + \ldots, \tag{52}$$

where $v_i(x,y)$, $i = 0,1,2,\ldots$ are functions yet to be determined. According to Eq. (51) the initial approximation is:

$$v_0(x,t) = u_0(x,t) = t\sin(x). \tag{53}$$

Substituting Eqs. (52) and (53) into Eq. (51) yields:

$$\frac{\partial}{\partial t} v_1(x,t) = 0, \tag{54}$$

$$\begin{aligned} &\frac{\partial^5}{\partial x^5} v_1(x,t) + \frac{\partial}{\partial t} v_2(x,t) + v_1(x,t)\, t\cos(x) \\ &t\sin(x)\frac{\partial}{\partial x} v_1(x,t) - \frac{\partial^3}{\partial x^3} v_1(x,t) = 0, \end{aligned} \tag{55}$$

$$\begin{aligned} &\frac{\partial^5}{\partial x^5} v_2(x,t) + \frac{\partial}{\partial t} v_3(x,t) + v_2(x,t)\, t\cos(x) \\ &t\sin(x)\frac{\partial}{\partial x} v_2(x,t) - \frac{\partial^3}{\partial x^3} v_2(x,t) \\ &+ v_1(x,t)\frac{\partial}{\partial x} v_1(x,t) = 0, \end{aligned} \tag{56}$$

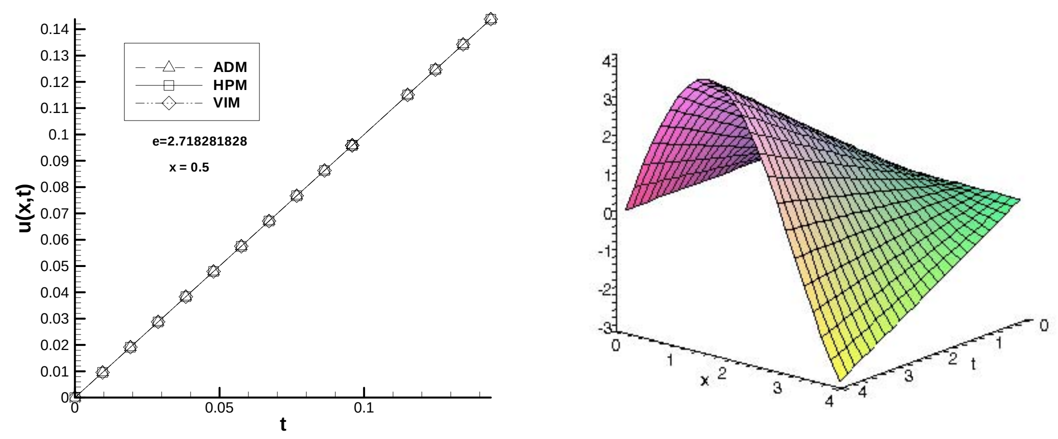
with the following conditions:

$$v_i(x,0) = 0, \qquad i = 0,1,2,3. \tag{57}$$

The solutions of Eqs. (54)-(56) may be written as follows:

$$v_i(x,0) = 0, \quad i = 1,2,3. \tag{58}$$

In the same manner, the rest of components were obtained by using the Maple Package.



(c) (d)

Fig2. The numerical result of the exact solution u(x,t) of Eq. (43) with HPM when $e = 2.718281828$, $x = 0.5$ which is equal to the obtained results of VIM and ADM. (c) is the figure of 2D for exact solution and (d) is the figure of 3D for exact solution.

According to the HPM, we can conclude:

$$u(x,t) = \lim_{p\to 1} v(x,t) = v_0(x,t) + v_1(x,t) + \ldots \tag{59}$$

therefore,

$$u(x,t) = t\sin(x)\,, \tag{60}$$

which is exactly the same as obtained by ADM and VIM.

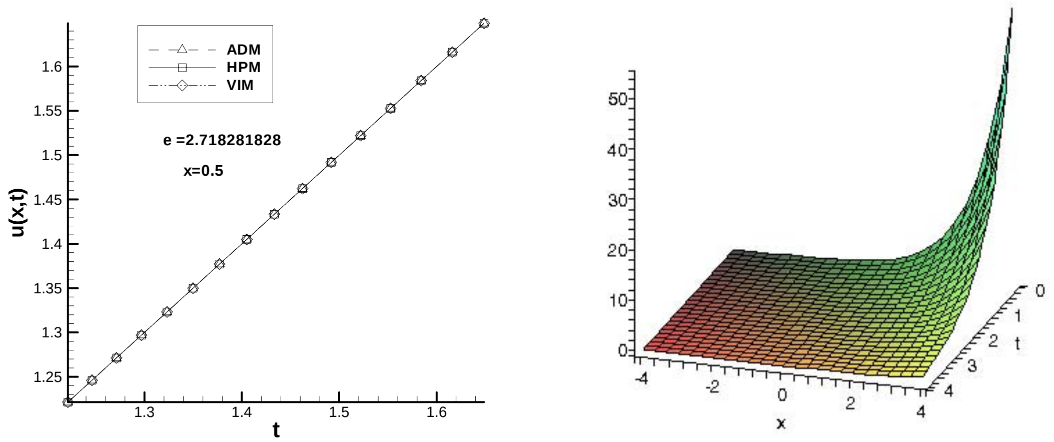
Example 3.3. We consider an equation with the initial condition given by:

$$u_t + u u_x - u u_{xxx} + u_{xxxxx} = 0\,,$$

$$u(x,0) = e^{x}\,. \tag{61}$$

A variational iteration can be constructed as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda[\frac{\partial}{\partial t} u_n(x,\tau) + u_n(x,\tau)(\frac{\partial}{\partial x} u_n(x,\tau)) - u_n(x,\tau)(\frac{\partial^3}{\partial x^3} u_n(x,\tau)) + \frac{\partial^5}{\partial x^5} u_n(x,\tau)]\,d\tau\,. \tag{62}$$



(e) (f)

Fig3. The numerical result of the exact solution u(x,t) of Eq. (61) with HPM when $e = 2.718281828$, $x = 0.5$ which is equal to the obtained results of the VIM and ADM. (e) is the figure of 2D for exact solution and (f) is the figure of 3D for exact solution.

The Lagrangian multiplier is therefore identified as $\lambda = -1$. So we have:

$$u_0(x,t) = e^{x}\,,$$

$$u_1(x,t) = e^{x} - t e^{x}\,,$$

$$u_2(x,t) = e^{x} - t e^{x} + \frac{1}{2} t^2 e^{x}\,,$$

$$u_3(x,t) = e^{x} - t e^{x} + \frac{1}{2} t^2 e^{x} - \frac{1}{6} t^3 e^{x}\,, \tag{63}$$

and so on. The solution $u(x,t)$ is given as :

$$u(x,t) = e^{x}\left(1 - t + \frac{1}{2!} t^2 - \frac{1}{3!} t^3 + \dots\right). \tag{64}$$

The solution $u(x,t)$ in a closed form is:

$$u(x,t) = e^{x} e^{-t} = e^{x-t}. \qquad (65)$$

which is exactly the same as obtained by ADM [30] and the exact solution with HPM [31].

4. Conclusions

In this paper, the exact solution with HPM is equal with the obtained results with ADM and VIM. HPM has been successfully applied to finding the exact solutions of some nonlinear fifth-order Korteweg-de Vries FKdV partial differential equations with specified initial conditions. The obtained solutions are compared with the ADM and VIM. All the examples show that the results of HPM are in excellent agreement with those obtained by ADM and VIM. So that it can be introduced to overcome the difficulties arising in calculation of Adomian's polynomials. VIM is to construct correction functional using general Lagrange multipliers identified optimally via the variational theory. But HPM does not require small parameters in the equation. The results show that HPM is a powerful mathematical tool for solving linear and nonlinear partial differential equations, and therefore can be applied in engineering.

5. References

[1] J.P. Body. Weak non-local solitons for capillary-gravity waves: fifth-order Korteweg-de Vries equation. *Physica D*. 1991,**48**: 129-146.

[2] J.H. He. Some asymptotic methods for strongly nonlinear equations. *International Journal of Modern Physics B* . 2006, **20** (10): 1141-1199.

[3] D.D. Ganji and A. Sadighi. Application of homotopy-perturbation and variational iteration methods to nonlinear heat transfer and porous media equations. *Journal of Computational and Applied Mathematics.* In Press, Corrected Proof, Available online 8 September 2006.

[4] D.D. Ganji and M. Rafei. Solitary wave solutions for a generalized Hirota–Satsuma coupled KdV equation by homotopy perturbation method. *Short Communication Physics Letters A.* 2006, **356**(2): 131-137.

[5] D.D. Ganji. The application of He's homotopy perturbation method to nonlinear equations arising in heat transfer. *Short Communication Physics Letters A.* 2006, **355**(4-5): 337-341.

[6] D.D. Ganji and A. Rajabi. Assessment of homotopy–perturbation and perturbation methods in heat radiation equations. *International Communications in Heat and Mass Transfer.* 2006, **33**(3): 391-400.

[7] E.M. Abulwafa, M.A. Abdou, A.A. Mahmoud. The solution of nonlinear coagulation problem with mass loss. *Chaos Solitons and Fractals.* 2006, **29**: 313-330.

[8] J.H. He. New interpretation of homotopy perturbation method. *International Journal of Modern Physics B*. 2006, **20** (18): 2561-2568.

[9] J.H. He. A new approach to nonlinear partial differential equations, Comm. *Nonlinear Sci. Numer. Simul*. 1997, **2** (4): 203-205.

[10] J.H. He. A variational iteration approach to nonlinear problems and its applications. *Mech. Appl*. 1998, **20** (1): 30-31.

[11] J.H. He. Variational iteration method for autonomous ordinary differential systems. *Appl. Math. Comput.* 2000, **114** (2/3): 115-123.

[12] J.H. He. Homotopy perturbation method: a new nonlinear technique. *Appl. Math. Comput*. 2003, **135**: 73-79.

[13] J.H. He. The homotopy perturbation method for nonlinear oscillators with discontinuities. *Appl. Math. Comput*. 2004, **151**: 287-292.

[14] J.H. He. A generalized variational principle in micromorphic thermoelasticity. *Mech. Res. Comm*. 2005, **32** (1): 93-98.

[15] J.H. He. Some asymptotic methods for strongly nonlinearly equations. *Iterant. J. Modern Math.* 2006, **20** (10) : 1141-1199.

[16] J. H. He. Approximate analytical solution for seepage flow with fractional derivatives in porous media. *Comput. Methods Appl. Mech. Eng.* 1998, **167**: 57-68.

[17] J. H. He. Variational iteration method: a kind of nonlinear analytical technique: some examples. *Int. Non-Linear Mech*. 1999, **34**: 699-708.

[18] D.D. Ganji. M. Jannatabadi and E. Mohseni, Application of He's variational iteration method to nonlinear Jaulent–Miodek equations and comparing it with ADM. *Journal of Computational and Applied Mathematics.* In Press, Corrected Proof, Available online, 8 September 2006.

[19] M.A. Abdou, A.A. Soliman. Variational iteration method for solving Burgers' and coupled Burgers' equation. *J. Comput. Appl. Math*. 2005, **181**: 245-251.

[20] S. Momani, S. Abusaad. Application of he's variational-iteration method to Helmholtz equation. *Chaos Solitons and Fractals*. 2005, **27** (5): 1119-1123.

[21] S. Momani, Z. Odibat. Analytical approach to linear fractional partial differential equations arising in fluid mechanics. *Phys. Lett.* 2006, **A 1** (53): 1-9.

[22] L.Xiqiang, B. Chenglin. Exact solutions of some fifth-order nonlinear equations. *Appl. Math. Scr.B (Engl.Ed.),* 2000, **15**: 28-32.

[23] P.G. Drazin, R.S. Johnson. Solutions: An Introduction, Cambridge University Press, 29. S. Momani, S. Abusaad, Application of he's variational-iteration method to Helmholtz equation, Chaos, Solitons and Fractals. 2005, **27** (5): 1119-1123.

[24] T.R. Akylas, T.-S. Yang. On short-scale oscillatory tails of long-wave disturbances. *Stud. Appl.Math.* 1995, **94** : 1-20.

[25] J.K. Hunter, J. Scheurle. Existence of perturbed solitary wave solutions to a model equation for water waves. Physica . 1988, D 32: 253-268.

[26] J.T. Beale. Exact solitary waves with capillary ripples at infinity, Commun. *Pure Appl. Math*. 1991, **44**: 211-247.

[27] J.H. He. The homotopy perturbation method for nonlinear oscillators with discontinuities. *Applied Mathematics and Computation.* 2004, **151**:287-292.

[28] J.H. He. Comparison of homotopy perturbation method and homotopy analysis method. *Applied Mathematics and Computation*. 2004, **156**:527-539.

[29] J.H. He. Approximate solution for nonlinear differential equations with convolution product nonlinearities. *Comput. Methods Appl. Mech. Eng.* 1998, **167**: 67-73.

[30] Kaya D. An explicit and numerical solutions of some fifth-order KdV equation by decomposition method. *Applied mathematics and Computation*. 2003, **144**: 353-363.

[31] M.Rafei, D. D. Ganji. Explicit Solutions of Helmholtz Equation and Fifth-order KdV Equation using Homotopy Perturbation Method. *International Journal of Nonlinear Sciences and Numerical Simulation*. 2006, **7**(3): 321-328.