Intuitionistic Neutrosophic Set

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Abstract. In this paper, we define intuitionistic neutrosophic set (INSs) and truth-value based INSs. In fact, all INSs are neutrosophic set but all neutrosophic sets are not INSs. We define some new operations on INSs with examples for the implementation of the operations in real life problems. Also, we studied some properties of INSs.

Keywords: Neutrosophic sets, truth-value based neutrosophic sets, intuitionistic fuzzy sets.

1. Introduction

In 1965 [12], Zadeh first introduced the concept of fuzzy sets. In many real applications to handle uncertainty, fuzzy set is very much useful and in this one real value $\mu_A(x) \in [0,1]$ is used to represent the grade of membership of a fuzzy set A defined on the universe of discorse X. After two decades Turksen [10] proposed the concept of interval-valued fuzzy set. But for some applications it is not enough to satisfy to consider only the membership-function supported by the evident but also have to consider the non-membership-function against by the evident. Atanassov [1] introduced another type of fuzzy sets that is called intuitionistic fuzzy set (IFS) which is more practical in real life situations. Intuitionistic fuzzy sets handle incomplete information i.e., the grade of membership function and non-membership function but not the indeterminate information and inconsistent information which exists obviously in belief system.

Smarandache [11] introduced another concept of imprecise data called neutrosophic sets. Neutrosophic set is a part of neutrosophy which studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra. Neutrsophic set is a powerful general formal framework that has been recently proposed. In neutrosophic set, indeterminacy by the evident is quantified explicitly and in this concept membership, indeterminacy membership and non-membership functional values are independent. Where membership, indeterminacy membership and non-membership functional values are real standard or non-standard subsets of $]^-0,1^+[$.

In real life problem which is very much useful. For example, when we ask the opinion of an expert about certain statement, he or she may assign that the possibility that the statement true is 0.5 and the statement false is 0.6 and he or she not sure is 0.2. This idea is very much needful in a various problem in real life situation.

The neutrosophic set generalized the concept of classical set, fuzzy set [12], interval-valued-fuzzy set [10], intuitionistic fuzzy set [1], etc.

Definition 1. Let X be a fixed set. A FS A of X is an object having the form $A = \{\langle x, \mu_A(x) \rangle / x \in X\}$. where the function $\mu_A : X \to [0,1]$ define the degree of membership of the element $x \in X$ to the set A, which is a subset of X.

Definition 2. Let X be a fixed set. An IFS A of X is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle / x \in X\}$. where the function $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ define respectively the degree of

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membership and degree of nonmembership of the element $x \in X$ to the set A, which is a subset of X and for every $x \in X$, $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

An element x of X is called significant with respect to a fuzzy subset A of X if the degree of membership $\mu_A(x) > 0.5$, otherwise, it is insignificant. We see that for a fuzzy subset A both the degrees of membership $\mu_A(x)$ and non- membership $\nu_A(x) = 1 - \mu_A(x)$ can not be significant. Further, for an IFS $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle / x \in X\}$ it is observe that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, for all $x \in X$ and hence it is observed that $\min\{\mu_A(x), \nu_A(x)\} \leq 0.5$, for all $x \in X$.

Definition 3. [9] Let X be a fixed set. A generalized intuitionistic fuzzy set (GIFS) A of X is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle | x \in X\}$ where the function $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ define respectively the degree of membership and degree of nonmembership of the element $x \in X$ to the set A, which is a subset of X and for every $x \in X$ satisfy the condition

$$\mu_A(x) \wedge \nu_A(x) \leq 0.5, \text{ for all } x \in X.$$

This condition is called generalized intuitionistic condition (GIC). In fact, all GIFs are IFSs but all IFSs are not GIFSs.

Having motivated from this definition we propose another concept of neutrosophic set.

In this paper, in Section 2 we recall the non-standard analysis by Abraham Robinson and some definitions of neutrosophic sets of Smarandache [11]. In Section 3, we recall some definitions of truth-value based neutrosophic sets [11]. In Section 4, we define a new type of neutrosophic sets called intuitionistic neutrosophic sets (INSs), truth-value based INSs and some results on INSs. Finally, we define two new operations with examples related to of real life situation.

2. Preliminaries

In 1960s Abraham Robinson has developed the non-standard analysis, a formalization and a branch of mathematical logic, that rigorously defines the infinitesimals. Informally, an infinitesimal is an infinitely small number. Formally, x is said to be infinitesimal if and only if for all positive integers n one has $|x| < \frac{1}{n}$. Let $\varepsilon > 0$ be a such infinitesimal number. The hyper-real number set is an extension of the real number set, which includes classes of infinite numbers and classes of infinitesimal numbers. Let's consider the non-standard finite numbers $1^+ = 1 + \varepsilon$, where ``1" is its standard part and ``ε" is non-standard part and $^-0 = 0 - \varepsilon$, where "0" is its standard part and ``ε" is non-standard part. Then we call $]^-0, 1^+[$ is a non-standard unit interval.

Generally, the left and right boundaries of a non-standard interval $]^-a, b^+[$ are vague and imprecise. Combining the lower and upper infinitesimal non-standard variable of an element we can define as $^-c^+ = \{(c - \varepsilon) \cup (c + \varepsilon)\}$.

Addition of two non-standard finite numbers with themselves or with real numbers defines as:

$$^-a + b = ^-(a+b), \ a + b^+ = (a+b)^+, \ ^-a + b^+ = ^-(a+b)^+,$$

$$^-a + ^-b = ^-(a+b), a^+ + b^+ = (a+b)^+.$$

Similar for subtraction, multiplication, division, root and power of non-standard finite numbers with themselves or real numbers.

Now we recall some definitions of Wang et al.[11].

Let X be a space of points (objects), with a generic elements in X denoted by x. Every element of X is characterized by a truth-membership function T, an indeterminacy function I and a falsity-membership function F, where T, I, F are real standard or non--standard subsets of $]^-0, 1^+[$, that is,

$$T : X \to]^-0, 1^+[,$$

$$I : X \to]^- 0, 1^+[,$$

$$F : X \to]^- 0, 1^+[$$

There is no such restriction on the sum of $T(x), I(x), F(x)$, so $^- 0 \leq T(x) + I(x) + F(x) \leq 3^+$.

Definition 4. A neutrosophic set A on the universe of discourse X is defined as $A = \langle x, T(x), I(x), F(x) \rangle$, for all $x \in X$, where $T(x), I(x), F(x) \to]^- 0, 1^+[$ and $^- 0 \leq T(x) + I(x) + F(x) \leq 3^+$.

Definition 5. The complement of a neutrosophic set A is denoted by A' and is defined as $A' = \langle x, T_{A'}(x), I_{A'}(x), F_{A'}(x) \rangle$, where for all x in X

$$T_{A'}(x) = \{1^+\} - T_A(x),$$

$$I_{A'}(x) = \{1^+\} - I_A(x),$$

$$F_{A'}(x) = \{1^+\} - F_A(x).$$

Definition 6. A neutrosophic set A is contained in another neutrosophic set B i.e., $A \subseteq B$, if for all x in X

$$T_A(x) \leq T_B(x),$$

$$I_A(x) \leq I_B(x),$$

$$F_A(x) \geq F_B(x).$$

Definition 7. The union of two neutrosophic sets A and B is also a neutrosophic set, whose truth-membership, indeterminacy-membership and falsity-membership functions are

$$T_{(A \cup B)}(x) = T_A(x) + T_B(x) - T_A(x)T_B(x),$$

$$I_{(A \cup B)}(x) = I_A(x) + I_B(x) - I_A(x)I_B(x),$$

$$T_{(A \cup B)}(x) = F_A(x) + F_B(x) - F_A(x)F_B(x), \text{ for all } x \in X.$$

Definition 8. The intersection of two neutrosophic sets A and B is also a neutrosophic set, whose truth-membership, indeterminacy-membership and falsity-membership functions are

$$T_{(A \cap B)}(x) = T_A(x)T_B(x),$$

$$I_{(A \cap B)}(x) = I_A(x)I_B(x),$$

$$T_{(A \cap B)}(x) = F_A(x)F_B(x), \text{ for all } x \in X.$$

Definition 9. The difference between two neutrosophic sets A and B is also a neutrosophic set is denoted as $A \setminus B$, whose truth-membership, indeterminacy-membership and falsity-membership functions are

$$T_{(A \setminus B)}(x) = T_A(x) - T_A(x)T_B(x),$$

$$I_{(A \setminus B)}(x) = I_A(x) - I_A(x)I_B(x),$$

$$T_{(A \setminus B)}(x) = F_A(x) - F_A(x)F_B(x), \text{ for all } x \in X.$$

Definition 10. The cartesian product of two neutrosophic sets A and B defined on the universes X and Y respectively is also a neutrosophic set which is denoted by $A \times B$, whose truth-membership, indeterminacy-membership and falsity-membership functions are defined by

$$T_{(A \times B)}(x, y) = T_A(x) + T_B(y) - T_A(x)T_B(y),$$

$$I_{(A \times B)}(x, y) = I_A(x)I_B(y),$$

$$T_{(A \times B)}(x, y) = F_A(x)F_B(y), \text{ for all } x \in X \text{ and } y \in Y.$$

3. Truth-value based neutrosophic sets

Here we recall some definitions of truth-value based neutrosophic sets of Wang et al. [11].

From philosophical point of view the neutrosophic sets takes the value from real standard or non-standard subsets of $]^-0,1^+[$. But in scientific or engineering point of view the neutrosophic sets, which takes the value from real standard or non-standard subsets of $]^-0,1^+[$ will be difficult to apply in the real applications. So, instate of $]^-0,1^+[$ need to take $[0,1]$.

Definition 11. A truth-value based neutrosophic set A on the universe of discourse X is defined as $A = \langle x, T(x), I(x), F(x) \rangle$, for all $x \in X$, where $T(x), I(x), F(x) \to [0,1]$ and $0 \le T(x) + I(x) + F(x) \le 3$.

Definition 12. A truth-value based neutrosophic set A is empty if and only if $T_A(x) = 0$, $F_A(x) = 0$ and $I_A(x) = 1$, i.e., $A = \langle x, 0, 1, 0 \rangle$ for all $x \in X$.

Definition 13. Let A and B be two truth-value based neutrosophic sets defined on X. Then $A \subseteq B$ if and only if

$$T_A(x) \le T_A(x), I_A(x) \le I_B(x) \text{ and } F_A(x) \ge F_B(x), \text{ for all } x \in X.$$

Two truth-value based neutrosophic sets A and B are equal i.e., $A = B$ if and only if

$$A \subseteq B \text{ and } B \subseteq A, \text{ for all } x \in X.$$

Definition 14. The complement of a truth-value based neutrosophic set A is denoted by A' and defined as

$$T_{(A')}(x) = F_A(x),$$

$$I_{(A')}(x) = 1 - I_A(x),$$

$$F_{(A')}(x) = T_A(x), \text{ for all } x \in X.$$

Definition 15. The intersection of two truth-value based neutrosophic sets A and B is also a neutrosophic set, whose truth-membership, indeterminacy-membership and falsity-membership functions are

$$T_{(A \cap B)}(x) = \min\{T_A(x), T_B(x)\}$$

$$I_{(A \cap B)}(x) = \min\{I_A(x), I_B(x)\}$$

$$F_{(A \cap B)}(x) = \max\{F_A(x), F_B(x)\}, \text{ for all } x \in X.$$

Definition 16. The union of two truth-value based neutrosophic sets A and B is also a neutrosophic set, whose truth-membership, indeterminacy-membership and falsity-membership functions are

$$T_{(A \cup B)}(x) = \max\{T_A(x), T_B(x)\}$$

$$I_{(A \cup B)}(x) = \max\{I_A(x), I_B(x)\}$$

$$F_{(A \cup B)}(x) = \min\{F_A(x), F_B(x)\}, \text{ for all } x \in X.$$

The following two definitions are based on two new operators i.e., truth-favorite (ΔA) and falsity-favorite (∇A) of truth-value based neutrosophic sets. These operators remove the indeterminacy-membership value in the neutrosophic set and transform it into IFS.

Definition 17. The truth-favorite of a truth-value based neutrosophic set A is a neutrosophic set and denote as $\Delta A = \langle x, T_{(\Delta A)}(x), I_{(\Delta A)}(x), F_{(\Delta A)}(x) \rangle$, where

$$T_{(\Delta A)}(x) = \min\{T_A(x) + I_A(x), 1\},$$

$$I_{(\Delta A)}(x) = 0,$$

$$F_{(\Delta A)}(x) = F_A(x), \text{ for all } x \in X.$$

Definition 18. The falsity-favorite of a truth-value based neutrosophic set A is a neutrosophic set and denote as $\nabla A = \langle x, T_{(\nabla A)}(x), I_{(\nabla A)}(x), F_{(\nabla A)}(x)\rangle$, where

$$T_{(\nabla A)}(x) = T_A(x),$$

$$I_{(\nabla A)}(x) = 0,$$

$$F_{(\nabla A)}(x) = \min\{F_A(x) + I_A(x), 1\}, \text{ for all } x \in X.$$

4. Intuitionistic neutrosophic sets and some results

Having motivated from the observation, we define an intuitionistic neutrosophic set (INS) as follows:

Definition 19. An element x of X is called significant with respect to neutrsophic set A of X if the degree of truth-membership or falsity-membership or indeterminancy-membership value, i.e., $T_A(x) \text{ or } F_A(x) \text{ or } I_A(x) \geq 0.5$. Otherwise, we call it insignificant. Also, for neutrosophic set the truth-membership, indeterminacy-membership and falsity-membership all can not be significant.

We define an intuitionistic neutrosophic set by $A^* = \langle x, T_{A^*}(x), I_{A^*}(x), F_{A^*}(x)\rangle$, where

$$\min\{T_{A^*}(x), F_{A^*}(x)\} \leq 0.5,$$

$$\min\{T_{A^*}(x), I_{A^*}(x)\} \leq 0.5,$$

and $\min\{F_{A^*}(x), I_{A^*}(x)\} \leq 0.5$, for all $x \in X$,

with the condition $\quad 0 \leq \{T_{A^*}(x) + I_{A^*}(x) + F_{A^*}(x)\} \leq 2$.

Definition 20. The intersection of two INSs A^* and B^* is also an INS, whose truth-membership, indeterminacy-membership and falsity-membership functions are

$$T_{(A^* \cap B^*)}(x) = \min\{T_{A^*}(x), T_{B^*}(x)\}$$

$$I_{(A^* \cap B^*)}(x) = \min\{I_{A^*}(x), I_{B^*}(x)\}$$

$$F_{(A^* \cap B^*)}(x) = \max\{F_{A^*}(x), F_{B^*}(x)\}, \text{ for all } x \in X.$$

Definition 21. The union of two two INSs A and B is also an INS, whose truth-membership, indeterminacy-membership and falsity-membership functions are

$$T_{(A^* \cup B^*)}(x) = \max\{T_{A^*}(x), T_{B^*}(x)\}$$

$$I_{(A^* \cup B^*)}(x) = \min\{I_{A^*}(x), I_{B^*}(x)\}$$

$$F_{(A^* \cup B^*)}(x) = \min\{F_{A^*}(x), F_{B^*}(x)\}, \text{ for all } x \in X.$$

Now we define truth-favorite and falsity-favorite of the truth value based of a INS.

Definition 22. For the truth value based INS we define the truth-favorite as follows:

If $\{I_{A^*}(x) \geq 0.5, T_{A^*}(x) \leq 0.5 \text{ and } F_{A^*}(x) \leq 0.5\}$

or $\{T_{A^*}(x) \geq 0.5, I_{A^*}(x) \leq 0.5 \text{ and } F_{A^*}(x) \leq 0.5\}$, then

$$T_{\Delta A^*}(x) = \min\{T_{A^*}(x) + I_{A^*}(x), 1\}$$

$$I_{\Delta A^*}(x) = 0$$

$$F_{\Delta A^*}(x) = F_{A^*}(x).$$

and if $T_{A^*}(x) \leq 0.5$, $I_{A^*}(x) \leq 0.5$ and $F_{A^*}(x) \geq 0.5$, then

$$T_{\Delta A^*}(x) = \min\{T_{A^*}(x) + I_{A^*}(x), 0.5\}$$

$$I_{\Delta A^*}(x) = 0$$

$$F_{\Delta A^*}(x) = F_{A^*}(x), \text{ for all } x \in X.$$

So, the truth favorite of a truth value based INS becomes a generalized intuitionistic fuzzy set and it is denoted as $\Delta A^* = \langle T_{\Delta A^*}(x), F_{\Delta A^*}(x) \rangle$.

Note 1. For the INS A^* on X when $T_{A^*}(x) \geq 0.5$ then the truth-membership i.e., $T_{(\Delta A^*)}(x)$ can take maximum value 1.

For the INS A^* on X when $T_{A^*}(x) \leq 0.5$ then the truth-favorite i.e., $T_{(\Delta A^*)}(x)$ can take maximum value 0.5.

Definition 23. For the truth value based INS A^* we define the false favorite as follows:

If $\{I_{A^*}(x) \geq 0.5, T_{A^*}(x) \leq 0.5 \text{ and } F_{A^*}(x) \leq 0.5\}$

or $\{T_{A^*}(x) \leq 0.5, I_{A^*}(x) \leq 0.5 \text{ and } F_{A^*}(x) \geq 0.5\}$, then

$$T_{\nabla A^*}(x) = T_{A^*}(x)$$

$$I_{\nabla A^*}(x) = 0$$

$$F_{\nabla A^*}(x) = \min\{F_{A^*}(x) + I_{A^*}(x), 1\}$$

and if $F_{A^*}(x) \leq 0.5$, $I_{A^*}(x) \leq 0.5$ and $T_{A^*}(x) \geq 0.5$ then

$$T_{\nabla A^*}(x) = T_{A^*}(x)$$

$$I_{\nabla A^*}(x) = 0$$

$$F_{\nabla A^*}(x) = \min\{F_{A^*}(x) + I_{A^*}(x), 0.5\}, \text{for all } x \in X.$$

So, the false favorite of a truth value based INS becomes a generalized intuitionistic fuzzy setand it is denoted as $\nabla A^* = \langle T_{\nabla A^*}(x), F_{\nabla A^*}(x) \rangle$.

Note 2. For INS when $F \geq 0.5$ then the false-favorite i.e., $F_{(\nabla A^*)}(x)$ can take maximum value 1.

For INS when $F \leq 0.5$ then the false-favorite i.e., $F_{\nabla A^*}(x)$ can take maximum value 0.5.

Here we recall the Definition 5, 7, 14, 16, 17, and 18 and we will show by means of examples that they are not valid for INSs.

Example 1. Let $A^* = \{\langle x_1, 0.8, 0.2, 0.4 \rangle, \langle x_2, 0.4, 0.5, 0.8 \rangle\}$ and

$B^* = \{\langle x_1, 0.4, 0.3, 0.8 \rangle, \langle x_2, 0.4, 0.2, 0.8 \rangle\}$ be two INSs of X .

By definition 5, $(A^*)' = \{\langle x_1, 0.2, 0.8, 0.6 \rangle, \langle x_2, 0.6, 0.5, 0.2 \rangle\}$.

Here we see that for x_1 both I and F are ≥ 0.5 . So $(A^*)'$ is not a INS.

By definition 7, we get $A^* \cup B^* = \{\langle x_1, 0.88, 0.44, 0.88 \rangle, \langle x_2, 0.64, 0.6, 0.96 \rangle\}$.

Here all T, I, F for x_1 and also for x_2 are ≥ 0.5 . So $A^* \cup B^*$ is not a INS.

By definition 14, $(A^*)' = \{\langle x_1, 0.4, 0.8, 0.8 \rangle, \langle x_2, 0.8, 0.5, 0.4 \rangle\}$.

Here we see that I and F both are ≥ 0.5 . So $(A^*)'$ is not a INS.

Now by definition 17 and 18, we get $\Delta A^* = \{\langle x_1, 1.0, 0.4 \rangle, \langle x_2, 0.9, 0.8 \rangle\}$,

$\Delta B^* = \{\langle x_1, 0.7, 0.8 \rangle, \langle x_2, 0.7, 0.8 \rangle\}$ and $\nabla A^* = \{\langle x_1, 0.8, 0.6 \rangle, \langle x_2, 0.4, 0.1 \rangle\}$.

Here we seen that for $\Delta A^*(x_1), \Delta A^*(x_2), \Delta B^*(x_1), \Delta B^*(x_1)$ $and \nabla A^*(x_1)$ the values of T and F are ≥ 0.5, so they are not INSs.

Example 2. Let $A^* = \{\langle x_1, 0.8, 0.2, 0.4 \rangle, \langle x_2, 0.4, 0.5, 0.8 \rangle\}$ and
$B^* = \{\langle x_1, 0.4, 0.6, 0.3 \rangle, \langle x_2, 0.3, 0.2, 0.8 \rangle\}$ be two INSs of X.

Then by definition 16, we get $A^* \cup B^* = \{\langle x_1, 0.8, 0.6, 0.3 \rangle, \langle x_2, 0.4, 0.5, 0.8 \rangle\}$. Here T, I for x_1 are ≥ 0.5. So, $A^* \cup B^*$ is not a INS.

Theorem 1. Let A^*, B^* and C^* be three INSs, then the following results hold good.

(i) Commutative $A^* \cup B^* = B^* \cup A^*$.

(ii) Associative $A^* \cup (B^* \cup C^*) = (A^* \cup B^*) \cup C^*$, $A^* \cap (B^* \cap C) = (A^* \cap B^*) \cap C^*$.

(iii) Distributive $A^* \cup (B^* \cap C^*) = (A^* \cup B^*) \cap (A^* \cup C^*)$,

$$A^* \cap (B^* \cup C^*) = (A^* \cap B^*) \cup (A^* \cap C^*).$$

(iv) Idempotency $A^* \cup A^* = A^*$, $A^* \cap A^* = A^*$.

(v) If ϕ be absolutely false truth-value based INS and $\phi = \langle 0, 0, 1 \rangle$, then

$A^* \cap \phi = \phi$ *and* $A^* \cup \phi$ *becomes a generalized intuitionistic fuzzy set*.

(vi) If $\bar{\phi}$ be absolutely true truth-value based INS and $\bar{\phi} = \langle 1, 0, 0 \rangle$, then

$A^* \cup \bar{\phi} = \bar{\phi}$ *and* $A^* \cap \bar{\phi}$ *becomes a generalized intuitionistic fuzzy set*.

Example 3. Let $A^* = \{\langle 0.6, 0.3, 0.1 \rangle, \langle 0.4, 0.7, 0.5 \rangle, \langle 0.4, 0.1, 0.8 \rangle\}$, $B^* = \{\langle 0.2, 0.2, 0.6 \rangle, \langle 0.7, 0.2, 0.4 \rangle, \langle 0.1, 0.6, 0.7 \rangle\}$ and $C^* = \{\langle 0.3, 0.8, 0.2 \rangle, \langle 0.4, 0.1, 0.6 \rangle, \langle 0.9, 0.1, 0.2 \rangle\}$ be three INSs of X. Then

$$A^* \cup B^* = \{\langle 0.6, 0.2, 0.1 \rangle, \langle 0.7, 0.2, 0.4 \rangle, \langle 0.4, 0.1, 0.7 \rangle\}\}$$

$$A^* \cup C^* = \{\langle 0.6, 0.3, 0.1 \rangle, \langle 0.4, 0.1, 0.5 \rangle, \langle 0.9, 0.1, 0.2 \rangle\}\}$$

$$B^* \cap C^* = \{\langle 0.2, 0.2, 0.6 \rangle, \langle 0.4, 0.1, 0.6 \rangle, \langle 0.1, 0.1, 0.7 \rangle\}\}$$

$$A^* \cup (B^* \cap C^*) = \{\langle 0.6, 0.2, 0.1 \rangle, \langle 0.4, 0.1, 0.5 \rangle, \langle 0.4, 0.1, 0.7 \rangle\}\}$$

$$(A^* \cup B^*) \cap (A^* \cup C^*) = \{\langle 0.6, 0.2, 0.1 \rangle, \langle 0.4, 0.1, 0.5 \rangle, \langle 0.4, 0.1, 0.7 \rangle\}\}.$$

Hence distributive property verified.

Theorem 2. For any two truth-value based INSs A^* and B^*

$(i) \Delta(A^* \cup B^*) \subseteq \Delta A^* \cup \Delta B^*, \Delta(A^* \cap B^*) \subseteq \Delta A^* \cap \Delta B^*$.

$(ii) \nabla A^* \cup \nabla B^* \subseteq \nabla(A^* \cap B^*), \nabla A^* \cap \nabla B^* \subseteq \nabla(A^* \cup B^*)$.

Proof. (i) Let, $A^* = \langle T_{A*}(x_1), I_{A*}(x_1), F_{A*}(x_1) \rangle$ and $B^* = \langle T_{B*}(x_1), I_{B*}(x_1), F_{B*}(x_1) \rangle$ be two truth-value based INSs for $x_1 \in X$, where

$T_{A*}(x_1) \leq 0.5$, $I_{A*}(x_1) \geq 0.5$, $F_{A*}(x_1) \leq 0.5$ and

$T_{B*}(x_1) \geq 0.5$, $I_{B*}(x_1) \leq 0.5$, $F_{B*}(x_1) \leq 0.5$. Then

$$\Delta(A^* \cup B^*) = \Delta \langle \max\{T_{A*}(x_1), T_{B*}(x_1)\}, \min\{I_{A*}(x_1), I_{B*}(x_1)\}, \min\{F_{A*}(x_1), F_{B*}(x_1)\} \rangle$$

$$= \Delta \langle T_{B^*}(x_1), I_{B^*}(x_1), \min\{F_{A^*}(x_1), F_{B^*}(x_1)\} \rangle$$

$$= \langle \max\{T_{A^*}(x_1) + I_{A^*}(x_1), 0.5\}, \min\{F_{A^*}(x_1), F_{B^*}(x_1)\} \rangle$$

$$= \langle \{0 \leq T_{\Delta(A^* \cup B^*)} \leq 0.5\}, F_{\Delta(A^* \cup B^*)} = \min\{F_{A^*}(x_1), F_{B^*}(x_1)\} \rangle \quad (1)$$

Now,

$$\Delta A^* \cup \Delta B^* = \{\langle \max\{T_{A^*}(x_1) + I_{A^*}(x_1), 1^+\}, F_{\Delta A^*}(x_1) \rangle\}$$

$$\cup \{\langle \max\{T_{B^*}(x_1) + I_{B^*}(x_1), 0.5\}, F_{\Delta B^*}(x_1) \rangle\}$$

$$= \{\langle 0.5 \leq T_{\Delta A^*}(x_1) \leq 1^+, F_{\Delta A^*}(x_1) \rangle\} \cup \{\langle 0 \leq T_{\Delta B^*}(x_1) \leq 1, F_{\Delta B^*}(x_1) \rangle\}$$

$$= \langle 0.5 \leq T_{(\Delta A^* \cup \Delta B^*)}(x_1) \leq 1^+,$$

$$F_{(\Delta A^* \cup \Delta B^*)}(x_1) = \min\{F_{A^*}(x_1), F_{B^*}(x_1)\} \rangle. \quad (2)$$

So, by equations (1) and (2) we get $\Delta(A^* \cup B^*) \subseteq \Delta A^* \cup \Delta B^*$. Similarly for any combinations of two or more INSs we can prove the results.

Proof. (ii) Proof is similar to (i).

Example 4. *Let* $A^* = \{\langle 0.6,0.3,0.1 \rangle, \langle 0.4,0.7,0.5 \rangle, \langle 0.4,0.1,0.8 \rangle\}$, $B^* = \{\langle 0.2,0.2,0.6 \rangle,$ $\langle 0.7,0.2,0.4 \rangle, \langle 0.1,0.6,0.7 \rangle\}$ *and* $C^* = \{\langle 0.3,0.8,0.2 \rangle, \langle 0.4,0.1,0.6 \rangle, \langle 0.9,0.1,0.2 \rangle\}$ *be three INSs of* X. *Then*

$$A^* \cup B^* \cup C^* = \{\langle 0.6,0.1,0.2 \rangle, \langle 0.7,0.1,0.4 \rangle, \langle 0.9,0.1,0.2 \rangle\}\}$$

$$A^* \cap B^* \cap C^* = \{\langle 0.2,0.1,0.6 \rangle, \langle 0.4,0.1,0.6 \rangle, \langle 0.1,0.1,0.8 \rangle\}\}$$

$$\Delta(A^* \cup B^* \cup C^*) = \{\langle 0.7,0.2 \rangle, \langle 0.8,0.4 \rangle, \langle 0.1,0.2 \rangle\}\}$$

$$\Delta(A^* \cap B^* \cap C^*) = \{\langle 0.3,0.6 \rangle, \langle 0.5,0.6 \rangle, \langle 0.2,0.8 \rangle\}\}$$

$$\Delta A^* = \{\langle 0.7,0.3 \rangle, \langle 0.1,0.5 \rangle, \langle 0.5,0.8 \rangle\}$$

$$\Delta B^* = \{\langle 0.4,0.6 \rangle, \langle 0.9,0.4 \rangle, \langle 0.5,0.7 \rangle\}$$

$$\Delta C^* = \{\langle 1.0,0.2 \rangle, \langle 0.5,0.6 \rangle, \langle 1.0,0.2 \rangle\}$$

$$\Delta A^* \cup \Delta B^* \cup \Delta C^* = \{\langle 0.7,0.2 \rangle, \langle 0.1,0.4 \rangle, \langle 1.0,0.2 \rangle\}$$

$$\Delta A^* \cap \Delta B^* \cap \Delta C^* = \{\langle 0.4,0.6 \rangle, \langle 0.5,0.6 \rangle, \langle 0.5,0.8 \rangle\}$$

Hence $\Delta(A^* \cup B^* \cup C^*) \subseteq \Delta A^* \cup \Delta B^* \cup \Delta C^*$ and $\Delta(A^* \cap B^* \cap C^*) \subseteq \Delta A^* \cap \Delta B^* \cap \Delta C^*$.

$$\nabla(A^* \cup B^* \cup C^*) = \{\langle 0.6,0.3 \rangle, \langle 0.7,0.5 \rangle, \langle 0.9,0.3 \rangle\}\}$$

$$\nabla(A^* \cap B^* \cap C^*) = \{\langle 0.2,0.7 \rangle, \langle 0.4,0.7 \rangle, \langle 0.1,0.9 \rangle\}\}$$

$$\nabla A^* = \{\langle 0.6,0.4 \rangle, \langle 0.4,1.0 \rangle, \langle 0.4,0.9 \rangle\}$$

$$\nabla B^* = \{\langle 0.2,0.8 \rangle, \langle 0.7,0.5 \rangle, \langle 0.1,1.0 \rangle\}$$

$$\nabla C^* = \{\langle 0.3,1.0 \rangle, \langle 0.4,0.7 \rangle, \langle 0.9,0.3 \rangle\}$$

$$\nabla A^* \cup \nabla B^* \cup \nabla C^* = \{\langle 0.6,0.4 \rangle, \langle 0.7,0.5 \rangle, \langle 0.9,0.3 \rangle\}$$

$$\nabla A^* \cap \nabla B^* \cap \nabla C^* = \{\langle 0.2,1.0 \rangle, \langle 0.4,1.0 \rangle, \langle 0.4,1.0 \rangle\}$$

Hence $\nabla A^* \cup \nabla B^* \cup \nabla C^* \subseteq \nabla(A^* \cup B^* \cup C^*)$ and $\nabla A^* \cap \nabla B^* \cap \nabla C^* \subseteq \nabla(A^* \cap B^* \cap C^*)$.

Definition 24. For any two truth-value based INSs ΔA^* and ΔB^*, we define

$$\Delta A^* \oplus \Delta B^* = \langle \frac{T_{(\Delta A^*)} + T_{(\Delta B^*)}}{2(T_{(\Delta A^*)}.T_{(\Delta B^*)} + 1)}, \frac{T_{(\Delta A^*)} + T_{(\Delta B^*)}}{2(T_{(\Delta A^*)}.T_{(\Delta B^*)} + 1)} \rangle \text{ and}$$

$$\Delta A^* \odot \Delta B^* = \langle (T_{(\Delta A^*)}.T_{(\Delta B^*)}), (F_{(\Delta A^*)}.F_{(\Delta B^*)}) \rangle.$$

Note 3. As for $a, b \in [0,1]$, $\frac{(a+b)}{2(ab+1)} \le 0.5$. Therefore $\Delta A \oplus \Delta B$ is a truth-value based INSs.

Note 4. $\Delta A \odot \Delta B$ is a valid operation for truth value based INSs, as for $T_{(\Delta A^*)}$ or $T_{(\Delta B^*)} \le 0.5$ and $F_{(\Delta A^*)}$ or $F_{(\Delta B^*)} \le 0.5$. Hence one of $T_{(\Delta A^*)}.T_{(\Delta B^*)}$ or $F_{(\Delta A^*)}.T_{(\Delta B^*)}$ is less than or equal to 0.5.

Definition 25. Let ΔA_i^* for $i = 1,2,\ldots,n$ be a set of truth value based INSs. Then the product $\bigotimes_{i=1}^{n} \Delta A_i^* = \Delta A^*$ (say), whose truth-membership function and false-membership function are respectively defined as follows:

$$T_{(\Delta A^*)} = \begin{cases} \frac{1}{2}, & \text{if } T_{\Delta A_i^*} = 1 \text{ for } i = 1,2,\ldots,n. \\ \dfrac{\sum_{k=1}^{n-1}[(-1)^{k+1} \sum_{i_1 < i_2 < \cdots < i_k} (T_{(\Delta A_i^*)_1}.T_{(\Delta A_i^*)_2} \cdots T_{(\Delta A_i^*)_k})]}{2[1+(-1)^n(T_{A_1^*}.T_{A_2^*} \cdots T_{A_n^*})]}, & \text{otherwise,} \end{cases}$$

where $i_k = 1,2,\ldots,n$ and

$$F_{\Delta A^*} = \begin{cases} \frac{1}{2}, & \text{if } F_{\Delta A_i^*} = 1 \text{ for } i = 1,2,\ldots,n. \\ \dfrac{\sum_{k=1}^{n-1}[(-1)^{k+1} \sum_{i_1 < i_2 < \cdots < i_k} (F_{(\Delta A_i^*)_1}.F_{(\Delta A_i^*)_2} \cdots F_{(\Delta A_i^*)_k})]}{2[1+(-1)^n(F_{A_1^*}.F_{A_2^*} \cdots F_{A_n^*})]}, & \text{otherwise,} \end{cases}$$

where $i_k = 1,2,\ldots,n$.

Theorem 3. If $a_i \in [0,1]$, for $i = 1,2,\ldots,n$ and $a_i \ne 1$ for at least one i, then following inequality holds good.

$$\frac{1}{2} \ge \frac{\sum_{k=1}^{n-1}[(-1)^{k+1} \sum_{i_1 < i_2 < \cdots < i_k} (a_{i_1}.a_{i_2} \cdots a_{i_k})]}{2[1+(-1)^n(a_1.a_2 \cdots a_n)]}, \text{ for } i_k = 1,2,\ldots n.$$

Proof. We have

$$\prod_{i=1}^{k}(1-a_i) \ge 0 \Rightarrow 1 + \sum_{k=1}^{n-1}[(-1)^{k+1} \sum_{i_1 < i_2 < \cdots < i_k} (a_{i_1}.a_{i_2} \cdots a_{i_k})] \ge 0$$

$$\Rightarrow 1 + (-1)^n(a_1.a_2 \cdots a_n) \ge \sum_{k=1}^{n-1}[(-1)^{k+1} \sum_{i_1 < i_2 < \cdots < i_k} (a_{i_1}.a_{i_2} \cdots a_{i_k})]$$

$$\Rightarrow 1 \ge \frac{\sum_{k=1}^{n-1}[(-1)^{k+1} \sum_{i_1 < i_2 < \cdots < i_k} (a_{i_1}.a_{i_2} \cdots a_{i_k})]}{2[1+(-1)^n(a_1.a_2 \cdots a_n)]} \quad (\text{as } a_1.a_2 \cdots a_n \ne 1)$$

$$\Rightarrow \frac{1}{2} \geq \frac{\sum_{k=1}^{n-1}[(-1)^{k+1}\sum_{i_1<i_2<\cdots<i_k}(a_{i_1}.a_{i_2}\cdots a_{i_k})]}{2[1+(-1)^n(a_1.a_2\cdots a_n)]}.$$

Theorem 4. Let $\bigotimes_{i=1}^{n}\Delta A_i^* = \Delta A^*$. Then ΔA^* is a truth value based INS. In fact $T_{A^*} \leq 0.5$ and $F_{A^*} \leq 0.5$.

Proof. The proof follows from Theorem 3.

Note that $\bigotimes_{i=1}^{n}\Delta A_i^* = \Delta A^*$ is not only a truth value based INS but also an intuitionistic fuzzy set.

We take an example in practical field to illustrate this operation.

Example 5. Suppose in a small company there are only three employees. An appraisal report has to made by three experts regarding the work culture of that employees. Each expert gave his opinion as for and against of the employees in the form of a truth value based INSs. Let the set of employees be $X = \{x_1, x_2, x_3\}$. Let A_i^* $(i = 1,2,3)$ be the experts report are given as follows:

$$A_1^* = \{\langle x_1, 0.7, 0.2, 0.4\rangle, \langle x_2, 0.2, 0.6, 0.3\rangle, \langle x_3, 0.5, 0.1, 0.7\rangle\},$$
$$A_2^* = \{\langle x_1, 0.5, 0.6, 0.3\rangle, \langle x_2, 0.5, 0.2, 0.4\rangle, \langle x_3, 0.4, 0.3, 0.6\rangle\},$$
$$A_3^* = \{\langle x_1, 0.4, 0.1, 0.8\rangle, \langle x_2, 0.5, 0.3, 0.6\rangle, \langle x_3, 0.9, 0.1, 0.3\rangle\}.$$

Then the truth-favorite of each expert is

$$\Delta A_1^* = \{\langle x_1, 0.9, 0.4\rangle, \langle x_2, 0.8, 0.3\rangle, \langle x_3, 0.5, 0.7\rangle\},$$
$$\Delta A_2^* = \{\langle x_1, 1.0, 0.3\rangle, \langle x_2, 0.7, 0.4\rangle, \langle x_3, 0.5, 0.6\rangle\},$$
$$\Delta A_3^* = \{\langle x_1, 0.5, 0.8\rangle, \langle x_2, 0.5, 0.6\rangle, \langle x_3, 1.0, 0.3\rangle\},$$

Then the final conclusion of the experts be given by $\bigotimes_{i=1}^{3}\Delta A_i^* = \Delta A^*$ (say), where

$$T_{\Delta A^*}(x_1) = \frac{(0.9+1.0+0.5)^+ - (0.9+0.5+0.45)}{2.(1-0.45)} = 0.5 \text{ (roundoff)},$$

$$F_{\Delta A^*}(x_1) = \frac{(0.4+0.3+0.8) - (0.12+0.32+0.24)}{2.(1-0.096)} = 0.45 \text{ (roundoff)},$$

$$T_{\Delta A^*}(x_2) = \frac{(0.8+0.7+0.5) - (0.56+0.4+0.35)}{2.(1-0.28)} = 0.48 \text{ (roundoff)},$$

$$F_{\Delta A^*}(x_2) = \frac{(0.3+0.4+0.6) - (0.12+0.18+0.24)}{2.(1-0.072)} = 0.41 \text{ (roundoff)},$$

$$T_{\Delta A^*}(x_3) = \frac{(0.5+0.5+1.0) - (0.25+0.5+0.5)}{2.(1-0.25)} = 0.5 \text{ (roundoff)},$$

$$F_{\Delta A^*}(x_3) = \frac{(0.7+0.6+0.3) - (0.42+0.18+0.21)}{2.(1-0.126)} = 0.45 \text{ (roundoff)}.$$

Thus the final conclusions of x_1, x_2 and x_3 given by three experts regarding for and against is $A^* = \{\langle x_1, 0.5, 0.45\rangle, \langle x_2, 0.48, 0.41\rangle\}$ and $\langle x_3, 0.5, 0.45\rangle\}$.

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