ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 4, No. 3, 2009, *pp*. 173-182

Physical Parameter Reconstruction of Mass-spring System from its Frequencies and Modes

Zhengsheng Wang [+]

Department of mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, P. R. China

(*Received October 12, 2008, accepted April 2, 2009*)

Abstract. In this paper, an inverse eigenvalue problem of constructing the mass and spring matrices of the mass-spring system from the information of frequency(eigenvalue) and mode(eigenvector) is considered. The solvability of the problem is discussed. Furthermore numerical algorithms and some numerical experiments are given with applications to mass-spring vibrating system.

Keywords: Frequency, Mode, Inverse Eigenvalue problem, Mass-spring system

1. Introduction

Inverse problems in vibration of spring-mass systems aim to determine the physical parameters such as masses and spring constants from natural frequencies and modal data. These problems have important applications in vibration design, structural dynamic design, and structural physical parameter recognition[1-6,10], because n degree of freedom mass-spring system may be thought of as finite-difference or finite-element approximations of continuous systems. Gladwell [10,11], Nylen [1], Ram[3,4], Zhou and Dai [9] had shown that inverse eigenvalue problems arising in classical vibration theory which the reconstruction of mass-spring system may be thought of as basic inverse vibration problem.

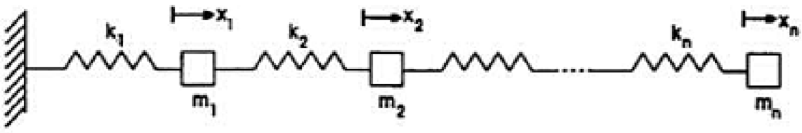

Figure 1: Spring-mass system fixed-free.

A linear n degree of freedom system of n point masses simply interconnected by springs (as Fig1) is characterized by the generalized eigenvalue problem

$$\lambda Mx = Kx \tag{1}$$

Where $\lambda = \omega^2$, ω is the nature frequency, x is the mode, M is the mass matrix, and K is the stiffness matrix as following

+ Corresponding author. Tel.: +86-25-52113705
E-mail address: wangzhengsheng@nuaa.edu.cn

$$M=\begin{bmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_n \end{bmatrix},\quad K=\begin{bmatrix} k_1+k_2 & -k_2 & & & & \\ -k_2 & k_2+k_3 & -k_3 & & & \\ & -k_3 & k_3+k_4 & & \ddots & \\ & & & \ddots & & \\ & & \ddots & & k_{n-1}+k_n & -k_n \\ & & & & -k_n & k_n \end{bmatrix}$$

There are many important results on reconstruction a mass-spring system with given data[1,5,9-11], for example, Ram and Gladwell have given the algorithms for constructing the physical parameters m_i and k_i with all or partial natural frequencies of system by constructing a Jacobi matrix.

In this paper, we focus on the reconstruction of mass-spring system with nature frequency and mode from the modified system. In section 2, we discuss the physical parameter reconstruction of two degree of freedom mass-spring system from the modified system. In section 3, we discuss the physical parameter reconstruction of n degree of freedom system by given two or three nature frequencies and corresponding modes. The solvability of the problem is discussed and some sufficient and necessary conditions for existence of the solution of this problem are proposed. Furthermore, Some numerical examples are given in Section 4.

2. The reconstruction of two degree of freedom system from the modified system

Considering the two degree of freedom mass-spring system

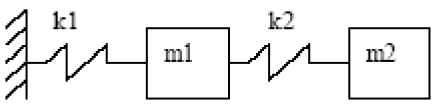

Fig.2 The original systems

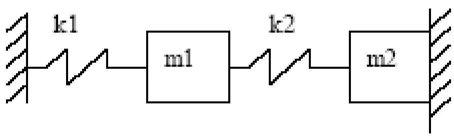

Fig.3 The modified systems with stiffness

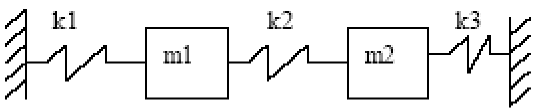

Fig.4 The modified systems with spring

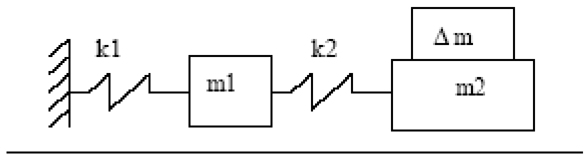

Fig.5 The modified systems with stiffness

The two degree of freedom mass-spring system is characterized $\lambda Mx = Kx$ with

$$M=\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix},\quad K=\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

The eigen-equation of (1) is

$$|K-\lambda M| = m_1 m_2 \lambda^2 - [\,m_2(k_1+k_2)+m_1 k_2\,]\lambda + k_1 k_2 = 0 \tag{2}$$

with roots λ_1, λ_2 satisfing

$$\lambda_1+\lambda_2 = \frac{k_1+k_2}{m_1}+\frac{k_2}{m_2} \tag{3}$$

$$\lambda_1\lambda_2 = \frac{k_1 k_2}{m_1 m_2} \tag{4}$$

If only λ_1, λ_2 are given, it can not reconstruction the four physical parameters m_1, m_2, k_1, k_2 from (3) and (4). It needs the additional condition.

Here three ways are given as the additional condition. (1) The modified system by fixed the m_2 (as Fig.3); (2) The modified system by adding a spring attaching the m_2 (as Fig.4); (3) The modified system by adding a mass Δm attaching to the m_2 (as Fig.5).

(1) The modified system by fixed the last mass

It is easy to get the eigenvalue of the modified system (as fig.3)

$$\lambda^0 = \frac{k_1 + k_2}{m_1} \tag{5}$$

From (3) and (5) ,

$$0 < \lambda^0 < \lambda_1 + \lambda_2$$

So

$$\frac{k_2}{m_2} = \lambda_1 + \lambda_2 - \lambda^0 \tag{6-1}$$

$$\frac{k_1}{m_1} = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2 - \lambda^0} \tag{6-2}$$

$$\frac{k_2}{m_1} = \frac{(\lambda^0 - \lambda_1)(\lambda_2 - \lambda^0)}{\lambda_1 + \lambda_2 - \lambda^0} \tag{6-3}$$

Therefore, (6) gives the relation between m_1, m_2, k_1, k_2. If one of them is given (by experiment), others can by computed by (6).

(2) The modified system by adding spring to the last mass

Considering the modified system by adding spring to the last mass (as fig.4). The mass and stiffness matrices of the modified system are

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

From its generalized eigenvalue problem (1), the eigenvalues of the original system have the relation

$$\lambda_1 + \lambda_2 = \frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \tag{7}$$

$$\lambda_1 \lambda_2 = \frac{k_1 k_2}{m_1 m_2} \tag{8}$$

Also, the eigenvalues $\bar{\lambda}_1$, $\bar{\lambda}_2$ of the modified system have the relation

$$\bar{\lambda}_1 + \bar{\lambda}_2 = \frac{k_1 + k_2}{m_1} + \frac{k_2 + k_3}{m_2} \tag{9}$$

$$\bar{\lambda}_1 \bar{\lambda}_2 = \frac{k_1 k_2 + k_1 k_3 + k_2 k_3}{m_1 m_2} \tag{10}$$

For simplity, denote $\lambda_1 = a$, $\lambda_2 = b$, $\bar{\lambda}_1 = c$, $\bar{\lambda}_2 = d$ and $k_3 = k$.

From （7）-（10）,

$$m_1 = \frac{kP^2}{(ad^2 + ac^2 + a^3 + bcd + 3adc + a^2 b - 2a^2 d - 2a^2 c - c^2 d - cd^2 - abc - abd)Q} \tag{11-1}$$

$$m_2 = \frac{k}{c + d - a - b} \tag{11-2}$$

$$k_1 = \frac{kabP}{(cd + a^2 - ac - ad)Q} \quad (11\text{-}3)$$

$$k_2 = \frac{kP}{2ac + 2ad + 2bc + 2bd - a^2 - 2ab - b^2 - c^2 - 2cd - d^2} \quad (11\text{-}4)$$

where $P = a^2 + b^2 + ab + cd - ac - ad - bd - bc$， $Q = b^2 + cd - bc - bd$.

Therefore, if λ_1、λ_2 and $\bar{\lambda}_1$， $\bar{\lambda}_2$ are given, all the physical parameters of the original system from (11).

(3) The modified system by adding a mass Δm attaching to the last mass

Considering the modified system by adding mass Δm to the last mass (as fig.4). The mass and stiffness matrices of the modified system are

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 + \Delta m \end{bmatrix},\ K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

Then the eigenvalues of modified system have the relation

$$\bar{\lambda}_1 + \bar{\lambda}_2 = \frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2 + \Delta m} \quad (12)$$

$$\bar{\lambda}_1 \bar{\lambda}_2 = \frac{k_1 k_2}{m_1(m_2 + \Delta m)} \quad (13)$$

denote $\lambda_1 = a$, $\lambda_2 = b$, $\bar{\lambda}_1 = c$, $\bar{\lambda}_2 = d$ and $\Delta m = m$.

Connecting (12) (13) with (3) （4）, we have

$$m_2 = \frac{mcd}{ab - cd} \quad (14\text{-}1)$$

$$k_2 = \frac{mabcd(a + b - c - d)}{(ab - cd)^2} \quad (14\text{-}2)$$

$$m_1 = \frac{m_2 k_2^2}{m_2 k_2 (a + b) - k_2^2 - abm_2^2} \quad (14\text{-}3)$$

$$k_1 = \frac{m_1 m_2 ab}{k_2} \quad (14\text{-}4)$$

Therefore, if λ_1, λ_2, $\bar{\lambda}_1, \bar{\lambda}_2$ and Δm are given, all the physical parameters of the original system from (14).

The three models can be summarized as the inverse eigenvalue problem: The reconstruction of the physical parameters of two degree of freedom system from its nature frequencies of the original and modified system.

(4) The reconstruction form the nature frequencies and corresponding modes

The generalized eigenvalue problem (1) can be reweitten as the following standard eigenvalue problem

$$Dx = \lambda x \quad (15)$$

where

$$D = M^{-1} K \quad (16)$$

For two degree of freedom system,

$$D = \begin{bmatrix} a_1 & -b \\ -c & a_2 \end{bmatrix}$$

where

$$a_1 = \frac{k_1 + k_2}{m_1}, b = \frac{k_2}{m_1}, c = \frac{k_2}{m_2}, a_2 = \frac{k_2}{m_2} \quad (17)$$

If the eigenpairs $\lambda_1,\varphi_1,\phi_2,\lambda_2$ are given which is satisfied with (15), rewriting into matric form

$$D\phi = \phi\Lambda$$

where

$$\varphi = [\varphi_1 \quad \varphi_2], \Lambda = diag[\lambda_1 \quad \lambda_2]$$

then

$$D = \varphi\Lambda\varphi^{-1} \tag{18}$$

Also, from（17）,we get

$$\frac{k_1}{m_1}, \frac{k_2}{m_1}, \frac{k_2}{m_2}$$

Similar as（6）, we can get the physical parameters m_1,m_2,k_1,k_2 if one of them is given.

3. The reconstruction of n degree of freedom system from its a few nature frequencies and corresponding modes

The generalized eigenvalue problem (1) of n degree of freedom system can be rewritten as the following standard eigenvalue problem (15) with

$$D = M^{-1}K = \begin{bmatrix} \frac{k_1+k_2}{m_1} & \frac{-k_2}{m_1} & & & \\ \frac{-k_2}{m_2} & \frac{k_2+k_3}{m_2} & \frac{-k_3}{m_2} & & \\ & \frac{-k_3}{m_3} & \frac{k_3+k_4}{m_3} & \ddots & \\ & & \ddots & \frac{k_{n-1}+k_n}{m_{n-1}} & \frac{-k_n}{m_{n-1}} \\ & & & \frac{-k_n}{m_n} & \frac{k_n}{m_n} \end{bmatrix} \tag{19}$$

$$= \begin{bmatrix} a_1 & -b_1 & & & \\ -c_1 & a_2 & -b_2 & & \\ & -c_2 & \ddots & \ddots & \\ & & \ddots & a_{n-1} & -b_{n-1} \\ & & & -c_{n-1} & a_n \end{bmatrix} \tag{20}$$

where

$$a_i = \frac{k_i + k_{i+1}}{m_i}; b_i = \frac{k_{i+1}}{m_i}; c_i = \frac{k_{i+1}}{m_{i+1}}; k_{n+1} = 0, b_n = c_n = 0 \qquad (i = 1,2,\cdots,n) \tag{21}$$

If matrix D is known，we can get all the physical parameters of the system by (21).

Algorithm 1

(1) If m_1 given, from a_1, b_1 ,

$$k_2 = m_1 b_1, k_1 = m_1 a_1 - k_2 = m_1(a_1 - b_1)$$

(2) From c_1, b_2, we get m_2, k_3

$$m_2 = \frac{k_2}{c_1} = m_1\frac{b_1}{c_1}, \qquad k_3 = m_2 b_2 = m_1\frac{b_1}{c_1} b_2$$

(3) From c_{i-1}, b_i, we get m_i, k_{i+1}

$$m_i = \frac{k_i}{c_{i-1}} = m_1 \prod_{j=1}^{i-1} \frac{b_j}{c_j}, \qquad k_{i+1} = m_i b_i = m_1 b_i \prod_{j=1}^{i-1} \frac{b_j}{c_j} \qquad (i = 3,4,\cdots,n-2)$$

(4) From c_{n-2}, b_{n-1}, we get m_{n-1}, k_n

$$m_{n-1} = \frac{k_{n-1}}{c_{n-2}} = m_1 \prod_{j=1}^{n-2} \frac{b_j}{c_j}, \qquad k_n = m_{n-1} b_{n-1} = m_1 b_{n-1} \prod_{j=1}^{n-2} \frac{b_j}{c_j}$$

(5) From a_n, we get m_n

$$m_n = \frac{k_n}{a_n} = m_1 \frac{b_{n-1}}{a_n} \prod_{j=1}^{n-2} \frac{b_j}{c_j}$$

Note that the tri-diagonal matrix D in (20) can be rewritten into Jacobi matrix J by similar transform as folloeing

Choosing the nonsingular matrix

$$P = \begin{bmatrix} 1 & & & \\ & p_1 & & \\ & & \ddots & \\ & & & p_{n-1} \end{bmatrix}$$

where

$$p_i = \sqrt{\frac{c_1 c_2 \cdots c_i}{b_1 b_2 \cdots b_i}} \qquad (i = 1,2,\cdots,n-1)$$

Do similar transform $x = Py$ to （15），

$$Jy = \lambda y$$

Where $J = P^{-1}DP$ is symmetric tri-diagonal matrix， which has the same eigenvalues of matrix D. Therefore, the problem can be summarized as the inverse eigenvalue problem of symmetric tri-diagonal matrix.

Also, the generalized eigenvalue problem (1) can be rewritten into standard eigenvalue probelm

$$Ay = \lambda y \tag{22}$$

where

$$A = M^{-\frac{1}{2}} K M^{-\frac{1}{2}} = \begin{bmatrix} \frac{k_1+k_2}{m_1} & \frac{-k_2}{\sqrt{m_1 m_2}} & & & \\ \frac{-k_2}{\sqrt{m_1 m_2}} & \frac{k_2+k_3}{m_2} & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \frac{k_{n-1}+k_n}{m_{n-1}} & \frac{-k_n}{\sqrt{m_{n-1} m_n}} \\ & & & \frac{-k_n}{\sqrt{m_{n-1} m_n}} & \frac{k_n}{m_n} \end{bmatrix} \tag{23}$$

is real symmetric tri-diagonal matrix

$$y = M^{\frac{1}{2}} x = \begin{bmatrix} \sqrt{m_1} & & \\ & \ddots & \\ & & \sqrt{m_n} \end{bmatrix} x$$

If matrix A is known，by Algorithm 1，all the physical parameters (mass matrix M and stiffness matrix K)can be found。

Lemma 1[9]: Suppose two different real numbers λ, μ ($\lambda > \mu$) and two nonzero vectors x, y are given. There exists an unique Jacobi matrix J with (λ, x) , (μ, y) as its two eigenpairs if and only if the following conditions are satisfied:

(i) $d_n = 0$;

(ii) $d_k / D_k > 0$ $(k = 1,2,\cdots,n-1$;

where

$$\begin{cases} d_k = \sum_{i=1}^{k} x_i y_i, (k = 1,2,...,n); \\ D_k = \begin{vmatrix} x_k & x_{k+1} \\ y_k & y_{k+1} \end{vmatrix} \neq 0, (k = 1,2,...,n-1); \end{cases} \tag{24}$$

And the elements of matrix J are

$$\begin{cases} b_k = (\lambda - \mu) d_k / D_k, (k = 1,2,...,n-1); \\ a_1 = \lambda - b_1 x_2 / x_1, a_n = \lambda - b_{n-1} x_{n-1} / x_n; \\ a_k = \begin{cases} \lambda - (b_{k-1} x_{k-1} + b_k x_{k+1}) / x_k, x_k \neq 0; \\ \mu - (b_{k-1} y_{k-1} + y_{k+1}) / y_k, x_k = 0; \end{cases} (k = 2,3,...,n-1); \end{cases} \tag{25}$$

Lemma 2[8] Suppose three different real numbers λ, μ, ν and three nonzero vectors x, y, z are given. There exists an unique Jacobi matrix J with (λ, x)，(μ, y)，(ν, z) as its three eigenpairs if and only if the following conditions are satisfied:

(i) $d_n^{(1)} = d_n^{(2)} = d_n^{(3)} = 0$

(ii) $(\lambda - \mu) d_k^{(1)} / D_k^{(1)} = (\lambda - \nu) d_k^{(2)} / D_k^{(2)} > 0$;

(iii) If $x_k = 0$s， $(\lambda - \mu) d_j^{(1)} / D_j^{(1)} = (\mu - \nu) d_j^{(3)} / D_j^{(3)}$, $j = k, k-1$

where

$$\begin{cases} d_k^{(1)} = \sum_{i=1}^{k} x_i y_i, d_k^{(2)} = \sum_{i=1}^{k} x_i z_i, d_k^{(3)} = \sum_{i=1}^{k} y_i z_i, k = (1,2,..,n); \\ D_k^{(1)} = \begin{vmatrix} y_k & y_{k+1} \\ x_k & x_{k+1} \end{vmatrix}, D_k^{(2)} = \begin{vmatrix} z_k & z_{k+1} \\ x_k & x_{k+1} \end{vmatrix}, D_k^{(3)} = \begin{vmatrix} z_k & z_{k+1} \\ y_k & y_{k+1} \end{vmatrix}, (k = 1,2,...,n-1); \end{cases} \tag{26}$$

And the elements of matrix J are

$$\begin{cases} b_k = (\lambda - \mu) d_k / D_k, (k = 1,2,...,n-1); \\ a_1 = \lambda - b_1 x_2 / x_1, a_n = \lambda - b_{n-1} x_{n-1} / x_n; \\ a_k = \begin{cases} \lambda - (b_{k-1} x_{k-1} + b_k x_{k+1}) / x_k, x_k \neq 0; \\ \mu - (b_{k-1} y_{k-1} + y_{k+1}) / y_k, x_k = 0; \end{cases} (k = 2,3,...,n-1); \end{cases} \quad (27)$$

Acoording to Lemma 1 , it is shown that under some conditions the physical parameters of the mass-spring system from its two nature frequencies and corresponding modes.

Algorithm 2

(1) using (26) computing the D_i and d_i;

(2) If $d_n \neq 0$, there is no solution to the problem;

(3) For $k = 1,2,\cdots,n,$

If $D_k \neq 0, (\mu - \lambda) d_k / D_k \leq 0$, or $D_k = 0, d_k \neq 0$, there is no solution to the problem;

If $D_k = 0, d_k = 0, x_k = 0$, but $x_{k-1} x_{k+1} \geq 0$ or $\begin{vmatrix} x_{k-1} & x_{k+1} \\ y_{k-1} & y_{k+1} \end{vmatrix} \neq 0$, there is no solution to the problem;

If $D_k \neq 0$, Computing

$$\begin{cases} b_k = (\lambda - \mu) d_k / D_k, (k = 1,2,...,n-1); \\ a_1 = \lambda - b_1 x_2 / x_1, a_n = \lambda - b_{n-1} x_{n-1} / x_n; \\ a_k = \begin{cases} \lambda - (b_{k-1} x_{k-1} + b_k x_{k+1}) / x_k, x_k \neq 0; \\ \mu - (b_{k-1} y_{k-1} + y_{k+1}) / y_k, x_k = 0; \end{cases} (k = 2,3,...,n-1); \end{cases}$$

If $D_k = 0, d_k = 0, x_k \neq 0$, solving the following linear equations with $b_k > 0$

$$\begin{cases} x_k a_k + x_{k+1} b_k = \lambda x_k - x_{k-1} b_{k-1} \\ y_k a_k + y_{k+1} b_k = \mu y_k - y_{k-1} b_{k-1} \end{cases}$$

If $D_k = 0, d_k = 0, x_k = 0$,computing $b_k = -x_{k-1} b_{k-1} / x_{k+1}$, a_k is any real number

(4) Using Algoritm1, the physical parameters of mass-spring system can be reconstructed from the computed Jacobi matrix.

Also, according to Lemma 2, we can set up the Algorithm 3 to reconstruct the physical parameters of mass-spring system from its three nature frequencies and corresponding modes. . Here we can not describe any more.

4. Numerical Examples

Using the above theory and algorithms, we give some numerical examples here to illustrate the results obtained in this paper are correct.

Example 1 : Considering the mass-spring system of two degree of freedom shown in Fig.2-Fig.5.

(a) Suppose $\lambda_1 = 1, \lambda_2 = 3, \lambda^0 = 2$ and $m_1 = 2$, from (6) we get

$$M=\begin{bmatrix}2 & 0\\ 0 & \frac{1}{2}\end{bmatrix},\ K=\begin{bmatrix}4 & -1\\ -1 & 1\end{bmatrix}$$

So the other three physical parameters of the original mass-spring system are $m_2=0.5, k_1=3, k_2=1$.

(b) Suppose $\lambda_1=2, \lambda_2=9, \bar{\lambda}_1=3, \bar{\lambda}_2=12$ and $k_3=4$, from （11）we get

$$M=\begin{bmatrix}\frac{169}{45} & 0\\ 0 & 1\end{bmatrix},\ K=\begin{bmatrix}16.9 & -6.5\\ -6.5 & 6.5\end{bmatrix}$$

So all the physical parameters of the original mass-spring system are $m_1=3.7556$ $m_2=1$, $k_1=10.4$, $k_2=6.5$.

(c) Suppose $\lambda_1=100$, $\lambda_2=10$, $\bar{\lambda}_1=90$, $\bar{\lambda}_2=5$ and $\Delta m=4$, from（14）we get

$$M=\begin{bmatrix}\frac{405}{209} & 0\\ 0 & \frac{36}{11}\end{bmatrix},\ K=\begin{bmatrix}\frac{10800}{121}+\frac{1350}{19} & -\frac{1350}{19}\\ -\frac{1350}{19} & \frac{1350}{19}\end{bmatrix}$$

So all the physical parameters of the original mass-spring system are $m_1=1.9378$ $m_2=3.2727$, $k_1=71.0526$, $k_2=89.2562$.

Example 2 : Considering the mass-spring of five degree of freedom shown in Fig.1.

(1) Suppose the three nature frequencies and corresponding modes of the mass-spring are given as following 432.7400，242.2059 and mode matrix φ with $m_1=4$.

$$\varphi=\begin{bmatrix}-0.1638 & -0.6925\\ 0.5224 & 0.3897\\ -0.8232 & 0.5082\\ 0.1428 & -0.3106\\ -0.0260 & 0.1179\end{bmatrix}$$

Using algorithm 2, we can reconstruct the physical parameters of the mass-spring system.

Table1 The reconstructed physical parameters

m_1	m_2	m_3	m_4	m_5
4	3	2	5	6
k_1	k_2	k_3	k_4	k_5
500	300	350	250	400

5. Conclusion

As a summary, we have presented some methods for reconstruction of the physical parameters of mass-spring system from its nature frequencies and corresponding modes. Numerical examples have been given to illustrate the effectiveness of our results and the proposed methods. Also, the idea in this paper may provide some insights for the physical parameters reconstruction of other vibration models by the inverse eigenvalue process , such as rods and beams and so on.

6. Acknowledgements

The author would like to thank Professor Dai Hua for his useful discussions. Meanwhile, the author thanks the references for their valuable comments that led to improvements of the paper.

7. References

[1] P. Nylen, F. Uhlig. Inverse eigenvalue problems associated with spring-mass systems. *Linear Algebra and its Application*. 1997, **254**: 409-425.

[2] K. Farahani, H. Bahai. An inverse strategy for relocation of eigenfrequencies in structural design. Part I: first order approximate solutions. *Journal of Sound and Vibration.* 2004, **274**(3-5): 481-505.

[3] S. G. Braun, Y. M. Ram. Modal modification of vibrating systems: some problems and their solutions. *Mechanical Systems and Signal Processing.* 2001, **15**(1): 101-119.

[4] Y. M. Ram. S. Elhay. The construction of band symmetric models for vibratory sysrems from modal analysis data. *Journal of Sound and Vibration.* 1995, **181**(4): 583 - 591.

[5] Peter Nylen. Inverse eigenvalue problem: existence of special mass-damper-spring systems. *Linear Algebra and its Applications.* 1999, **297**: 107-132 .

[6] Y. Halevi, P. Kenigsbuch. Model updating of the complex modeshapes and the damping matrix. *Inverse Problem in Engineering.* 2000, **8**: 143-162.

[7] Z. S. Wang, H Dai. On the construction of a real symmetric five-diagonal matrix from its three eigenpairs. *Applied Mathematics and Computation.* 2006, **175**(1): 597-608.

[8] Z. S. Wang. Inverse eigenvalue problems for real symmetric banded matrix. *Appl. Math. J. Chinese Univ..* 2004, **19** (Ser.A): 451-459(in Chinese).

[9] S. Q. Zhou, H Dai. *Inverse problems of Algebra eigenvalue.* Henan Science and Technology Press, 1991(in Chinese).

[10] G. M. L. Gladwell. *Inverse Problems in Vibration. 2nd.* Kluwer: Academic Publishers, 2004.

[11] G. M. L. Gladwell. *Inverse finite element vibration problems, Sound and Vibration*. 1999, **221**(2): 309-324.