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Synchronization and Anti-synchronization of double hump Duffing-Van der Pol Oscillators via Active Control

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Abstract. This paper investigates the synchronization and anti-synchronization behaviour of two identical double-hump Duffing-Van der Pol oscillator (DHDVP) evolving from different initial conditions using the active control technique based on the Lyapunov stability theory (LST) and the Routh-Hurwitz criteria (RHC). The designed controllers, with three different choices of the coefficient matrix of the error dynamics that satisfy the LST and RHC, are found to be effective in the stabilization of the error states at the origin, thereby, achieving synchronization and anti-synchronization between the states variables of two DHDVP oscillators. Interestingly, one of the choices leads to a single control function, thereby, reducing controller complexity for easy implementation. The results are validated by numerical simulations.

Keywords: Synchronization; Active Control; Lyapunov stability theory; Routh-Hurwitz criteria; Double-hump Duffing-Van der Pol Oscillators

1. Introduction

In the last two decades, considerable research has been done in nonlinear systems and their various properties. One of the most important properties of nonlinear dynamical systems is synchronization, which classically, represents the entrainment of frequencies of oscillators due to weak interactions, [1-3]. Studies in this field are partly motivated by experimental realization in lasers, electronic circuits, plasma discharge and chemical reactions [2-4].

For a system of two coupled chaotic oscillators, the master ($\dot{x} = f(x, y)$) and the slave ($\dot{y} = g(x, y)$), where $x(t)$ and $y(t)$ are phase space or state variables, and f and g are the corresponding nonlinear functions, synchronization in a direct sense implies $|x(t) - y(t)| \to 0$ as $t \to \infty$. When this occurs the coupled systems are said to be completely synchronized. Chaos synchronization is related to the observer problem in control theory [5]. The problem may be treated as the design of control laws for full chaotic observer (the slave system) using the known information of the plant (the master system) so as to ensure that the controlled receiver synchronizes with the plant. Hence, the slave chaotic system completely traces the dynamics of the master in the course of time.

On the other hand, anti-synchronization (AS) of two coupled systems $\dot{x} = f(x, y)$ (master system) and $\dot{y} = g(x, y)$ (slave system) means $|x(t) + y(t)| \to 0$ as $t \to \infty$. This phenomenon has been investigated both experimentally and theoretically in many physical systems [6-13]. A recent study of the AS phenomenon in non-equilibrium systems suggests that AS could be used as a technique for particle separation in a mixture of interacting particles [12].

In general, various techniques have been proposed for achieving stable synchronization and ant-synchronization between identical and non-identical systems. Notable among these methods, the active control scheme proposed by Bai and Lonngren [14] has received a considerable attention in the last ten years. Applications to various systems abound, some of which include the electronic circuits, which model a third-order "jerk" equation [15], Lorenz, Chen and Lü system [16], Geophysical systems [17], nonlinear equations of acoustic gravity waves (Lorenz-Stenflo system) [18], Qi *et al* system [18,19], Van-der Pol-Duffing oscillator [20,21], forced damped pendulum [22], nuclear magnetic resonance (NMR) modeled by the Bloch

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equations [23], Parametrically excited oscillators [24,25], permanent magnet reluctance machine [11], inertial ratchets [12,26], and most recently in RCL-shunted Josephson junction [27], Newton-Leipnik system [28] and modified projective synchronization [29].

Recently Lei *et al* [24] applied the Lyapunov stability theory and the Routh-Hurwitz criteria to active control in order to achieve stable synchronization. This version of the active control, which has the advantage of easy implementation has been applied to synchronize a few systems [19,24,25].

The Duffing-Van der Pol oscillator (DVP) models describe periodically self-excited oscillators in Physics, engineering, electronics, biology, neurology and many other disciplines [30] and references therein . The dynamics of the system include the exhibition of strange attractors, Hobf and Neimark-Sacker bifurcations, Smale horseshoe chaos, multi-stability, etc [30-37] and references therein. While the DVP model has been extensively studied in the double-well and single-well configurations, it has not been studied in the double-hump configuration. The double hump is the unbounded case which can lead to chaotic escape. However, Tchoukuegno *et al*. [38] showed that chaotic escape can be inhibited by introducing a parametric control. Moreover if the external forcing is not above a certain threshold value there will be no chaotic escape.

The aim of this article is to use the active control technique based on the Lyapunov stability theory and the Routh-Hurwitz criteria to synchronize and anti-synchronize two identical double-hump Duffing-Van der Pol oscillators evolving from different initial conditions. To the best of my knowledge this has not been done. The rest of the paper is organized as follows: section 2 describes the DVP model, sections 3 and 4 deal with the synchronization and anti-synchronization of the DVP respectively while section 5 concludes the paper.

2. Description of the Duffing-Van der Pol oscillator model

The Duffing-Van der Pol oscillator (DVP) with external excitation is described by the following differential equation:

$$\ddot{x} - \mu\left(1 - x^2\right)\dot{x} + \frac{dV(x)}{dx} = g(f, \omega, t) \tag{1}$$

where μ>0 is the damping parameter, the dots over the state variable $x(t)$ denote derivative with respect to time t, $g(f, \omega, t)$ is the periodic external excitation of amplitude f and angular frequency ω, and $V(x)$, the potential, is approximated by a finite Taylor series as follows.

$$V(x) = \frac{1}{2}\alpha x^2 + \frac{1}{4}\beta x^4 \tag{2}$$

where α and β are constant parameters. System (1) is a generalization of the classic Duffing-Van der Pol oscillator equation and can be considered in at least three physically relevant situations wherein the potential (2) is a single-well (α >0, β >0), double-well (α <0, β >0) and double-hump (α >0, β <0). In this article the double-hump configuration of the potential (2) as shown in Fig. 1 is considered. By substituting the potential (2) into (1) and letting the external excitation $g(f, \omega, t) = f \cos \omega t$, (1) becomes

$$\ddot{x} - \mu\left(1 - x^2\right)\dot{x} + \alpha x + \beta x^3 = f \cos \omega t \tag{3}$$

Using the definitions $\dot{x} = \dot{x}_1 = x_2$ and $\ddot{x} = \dot{x}_2$ (3) becomes

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \mu\left(1 - x_1^2\right)x_2 - \alpha x_1 - \beta x_1^3 + f \cos \omega t \end{aligned} \tag{4}$$

System (4) exhibit chaotic dynamics in the double-hump configuration for the parameter values α = 4.5, β = −0.79, μ = 0.1, f = 0.079 and ω = 0.675, as shown by the chaotic attractor in phase space, Fig. 2.

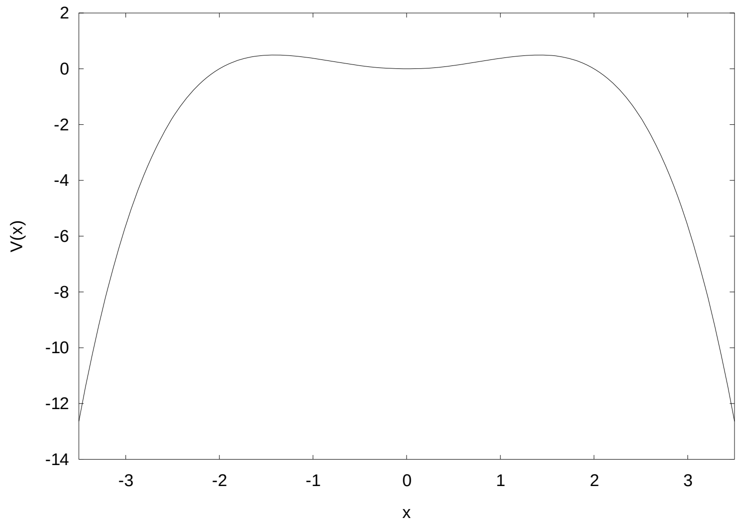

Fig. 1: The double-hump configuration of the potential (2) with parameter values $\alpha = 1,\ \beta = -0.5$

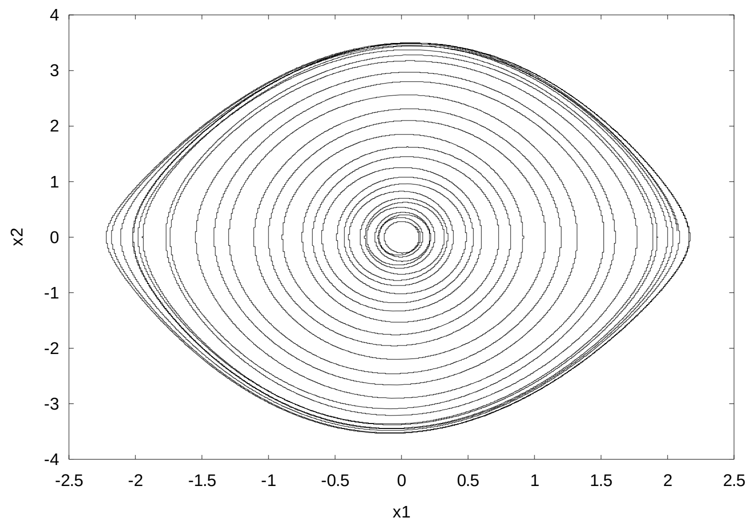

Fig. 2: Phase portrait showing the chaotic attractor of the double-hump Duffing-Van der Pol oscillator for parameter values $\alpha = 0.5,\ \beta = -0.5,\ \lambda = 0.1$

3. Synchronization of two double-hump Duffing-Van der Pol oscillators

3.1. Formulation of the active controllers

The method employed here was used in references [24,25]. Let system (4) be the master system and the following system be the slave system.

$$\dot{y}_1 = y_2 + u_1(t)$$
$$\dot{y}_2 = \mu(1 - y_1^2)y_2 - \alpha y_1 - \beta y_1^3 + f\cos\omega t + u_2(t) \quad (5)$$

where $u_1(t)$ and $u_2(t)$ are control functions to be determined. Subtracting (4) from (5) and using the notation $e_1 = y_1 - x_1$ and $e_2 = y_2 - x_2$ we have

$$\dot{e}_1 = e_2 + u_1(t)$$
$$\dot{e}_2 = \mu e_2 - \mu(y_1^2 y_2 - x_1^2 x_2) - \alpha e_1 - \beta(y_1^3 - x_1^3) + u_2(t) \quad (6)$$

We now re-define the control functions such as to eliminate terms in (6) which cannot be expressed as linear terms in e_1 and e_2 as follows:

$$u_1(t) = v_1(t)$$
$$u_2(t) = \mu(y_1^2 y_2 - x_1^2 x_2) + \beta(y_1^3 - x_1^3) + v_2(t) \tag{7}$$

Substituting (7) into (6) we have

$$\dot{e}_1 = e_2 + v_1(t)$$
$$\dot{e}_2 = \mu e_2 - \alpha e_1 + v_2(t) \tag{8}$$

Eq.(8) is the error dynamics, which can be interpreted as a control problem where the system, to be controlled is a linear system with control inputs $v_1(t) = v_1\ (e_1(t), e_2(t))$ and $v_2(t) = v_2\ (e_1(t), e_2(t))$. As long as these feedbacks stabilize the system, $\left|e_i(t)_{,i=1,2}\right| \to 0$ as $t \to \infty$. This simply implies that the two systems (4) and (5) evolving from different initial conditions are synchronized. As functions of e_1 and e_2 we choose $v_1(t)$ and $v_2(t)$ as follows:

$$\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = \mathbf{D} \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} \tag{9}$$

where $\mathbf{D} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, is a 2×2 constant feedback matrix to be determined. Hence the error system (8) can be written as

$$\begin{pmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \end{pmatrix} = \mathbf{C} \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} \tag{10}$$

where $\mathbf{C} = \begin{pmatrix} a & 1+b \\ c-\alpha & \mu+d \end{pmatrix}$, is the coefficient matrix.

According to the Lyapunov stability theory and the Routh-Hurwitz criteria, if

$$a + d + \mu < 0$$
$$(c-\alpha)(1+b) - a(\mu+d) < 0 \tag{11}$$

then the eigenvalues of the coefficient matrix of error system (8) must be real or complex with negative real parts and, hence, stable synchronized dynamics between systems (4) and (5) is guaranteed. Let

$$a + d + \mu = -E$$
$$(c-\alpha)(1+b) - a(\mu+d) = -E \tag{12}$$

where $E > 0$ is a real number which is usually set equal to 1. There are several ways of choosing the constant elements a, b, c, d of matrix D in order to satisfy the Lyapunov stability theory and the Routh-Hurwitz criteria (11). The following common choices are easily obtained from (12).

$$\mathbf{D_1} = \begin{pmatrix} -E & 0 \\ \alpha - E & -\mu \end{pmatrix}, \mathbf{D_2} = \begin{pmatrix} 0 & 0 \\ \alpha - E & -\mu - E \end{pmatrix}, \mathbf{D_3} = \begin{pmatrix} -\mu - E & 0 \\ \alpha - (\mu+E)\mu - E & 0 \end{pmatrix} \tag{13}$$

3.2. Numerical simulation

Using the fourth-order Runge-Kutta algorithm with initial conditions (x_1 , x_2)=(0.1,0.2), (y_1, y_2)=(1.3,1.4), a time step of 0.001, E=1, and parameter values as in Fig. 2 to ensure chaotic dynamics of the state variables, the systems (4) and (5) with $u_1(t)$ and $u_2(t)$ as defined in (7) were numerically solved for each of the cases $\mathbf{D_1}$, $\mathbf{D_2}$, $\mathbf{D_3}$ of (13). The results obtained show that the error states oscillate chaotically when the controllers are switch off, Fig. 3, and when the controllers are switched on at $t = 0$, Fig. 4, the error states converge to zero for each of the matrices $\mathbf{D_1}$, $\mathbf{D_2}$, $\mathbf{D_3}$, thereby, achieving synchronization between systems (4) and (5). Therefore, any 2×2 constant matrix that satisfies (11) can bring about synchronization between systems (4) and (5). This is also confirmed by the convergence of the synchronization quality defined by

$$e = \sqrt{e_1^2 + e_2^2} \tag{14}$$

Even though the synchronization time t_s, defined as the time taken for e to decrease to 10^{-6}, is slightly longer for $\mathbf{D_2}$ (t_s=29.01) than for $\mathbf{D_1}$ (t_s=27.83) and $\mathbf{D_3}$ (t_s=27.58), the fact that $\mathbf{D_2}$ gives only a single control function $u_2(t)$, thereby, reducing controller complexity makes this result interesting. The problem of controller complexity is a very crucial issue in the practical implementation of control techniques [39], since the cost implication, the density requirement for designing controllers and the need to make the complexity of the controller to be, at least comparable to or less than the device being controlled are fundamental, if the controlling technique is desired to serve a useful purpose.

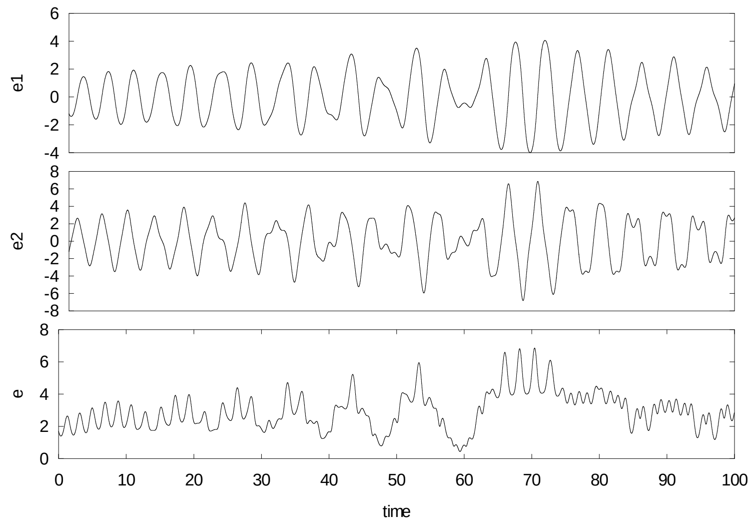

Fig. 3: Time series of the error states e_1, e_2 and the error propagation e with control switched off.

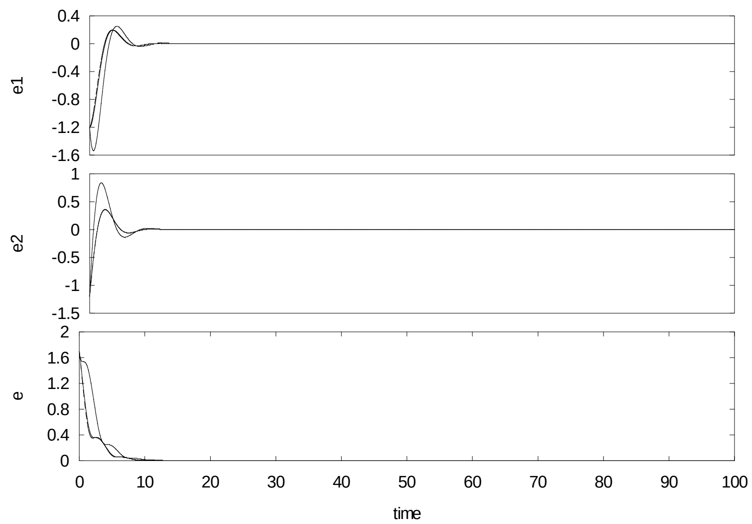

Fig. 4: Time series of the error states e_1, e_2 and the error propagation e with control switched on at $t = 0$ for the cases $\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}$

4. Anti-synchronization of two double-hump Duffing-Van Pol der oscillators

4.1. Formulation of the active controllers

Again we consider systems (4) and (5) as the master system and slave system respectively. Adding (4) and (5) and using the notation $e_1 = x_1 + y_1$ and $e_2 = x_2 + y_2$ we obtain

$$\dot{e}_1 = e_2 + u_1(t)$$
$$\dot{e}_2 = \mu e_2 - \mu\left(y_1^2 y_2 + x_1^2 x_2\right) - \alpha e_1 - \beta\left(y_1^3 + x_1^3\right) + u_2(t) \tag{15}$$

We again re-define the control functions such as to eliminate terms in (15) which cannot be expressed as linear terms in e_1 and e_2 as follows

$$u_1(t) = v_1(t)$$
$$u_2(t) = \mu\left(y_1^2 y_2 + x_1^2 x_2\right) + \beta\left(y_1^3 + x_1^3\right) + v_2(t) \tag{16}$$

Substituting (16) into (15) we obtain (8) exactly. The rest of the steps taken from (8) to (13) also follow exactly.

4.2. Numerical simulation

Using the fourth-order Runge-Kutta algorithm with the same initial conditions, time step, and parameter values as in subsection 3.2, systems (4) and (5) with $u_1(t)$ and $u_2(t)$ as defined in (16) were computed for the case $\mathbf{D_2}$ of (13) since $\mathbf{D_2}$ leads to a single control function, thereby, reducing controller complexity for easy implementation. The results obtained show that there is not any form of synchronization between the state variables (x_1, x_2) of the master system and (y_1, y_2) of the slave system when the controller is deactivated, Fig. 5, and when the controller is activated at $t = 0$, there is anti-synchronization between x_1 and y_1 as well as between and y_2, Fig. 6. The dynamics of y_1 and y_2 are controlled to anti-synchronize with those of x_1 and x_2 respectively.

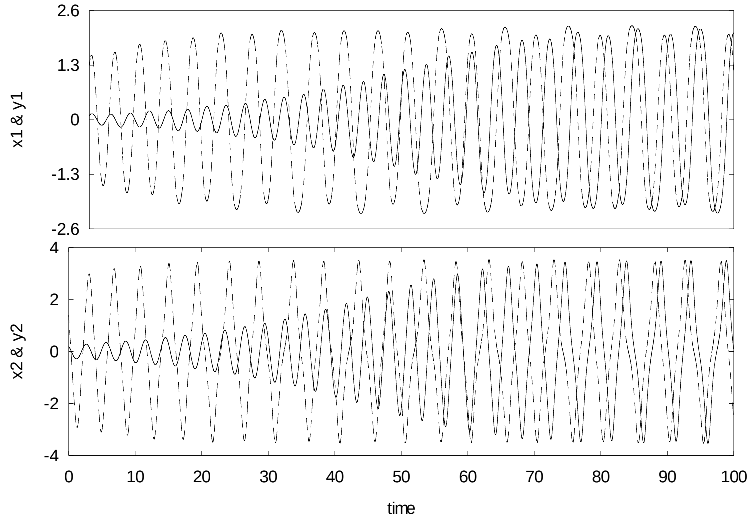

Fig. 5: Time series showing non-synchronous dynamics between the state variables x_1 (solid) & y_1 (dashed) and x_2 (solid) & y_2 (dashed) when the control is switched off.

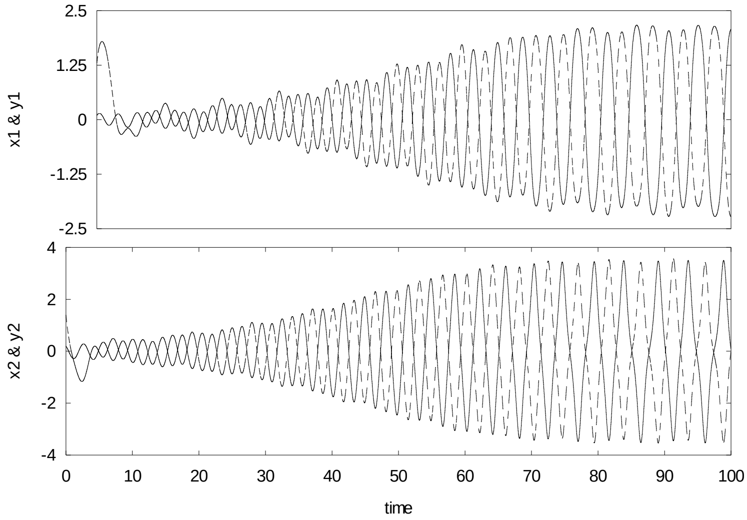

Fig. 6: Time series showing anti-synchronous dynamics of the state variables x_1 (solid) & y_1 (dashed) and x_2 (solid) & y_2 (dashed) when the control is switched on at t =0.

5. Concluding Remarks

This paper has investigated the synchronization and anti-synchronization behaviour of two identical double-hump Duffing oscillators (DHDO) evolving from different initial conditions via the active control technique based on the Lyapunov stability theory (LST) and the Routh-Hurwitz criteria (RHC). Three different choices of the coefficient matrix of the error dynamics that satisfy the LST and RHC were made. All the three choices were effective in the stabilization of the error states at the origin, thereby, achieving synchronization and anti-synchronization between the states variables of the two DHDO's. Interestingly one of the choices led to a single control function, thereby, reducing controller complexity for easy implementation. The results were validated by numerical simulations.

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