

# On Stability of Positive Solutions to Reaction-Diffusion Systems with Unequal Diffusion Coefficients

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**Abstract.** We study stability of positive stationary solutions to the system of boundary value problem of the form

$$\begin{cases} -\Delta u(x) = \lambda f(u, v) & x \in \Omega, \\ -\Delta v(x) = \mu g(u, v) & x \in \Omega, \\ Bu = 0 = Bv & x \in \partial\Omega, \end{cases}$$

where  $\lambda, \mu > 0$  are parameters,  $\Omega \subset \mathbb{R}^N (N \geq 1)$  is a bounded domain with a smooth boundary  $Bz(x) = \alpha h(x)z + (1 - \alpha) \frac{\partial z}{\partial n}$  where  $\alpha \in [0, 1], h: \partial\Omega \rightarrow \mathbb{R}^+$  with  $h = 1$  when  $\alpha = 1$ , and  $f, g: [0, +\infty) \times [0, +\infty) \rightarrow \mathbb{R}$  are  $C^1$  functions. We establish stability/instability of positive stationary solutions, under certain conditions.

**Keywords:** Linearly stability, positive solutions, reaction-diffusion.

## 1. Introduction

In this paper, we consider stability of positive solutions to the coupled-system of boundary value problem

$$\begin{aligned} -\Delta u(x) &= \lambda f(u, v) & x \in \Omega, & (1) \\ -\Delta v(x) &= \mu g(u, v) & x \in \Omega, & (2) \\ Bu = 0 &= Bv & x \in \partial\Omega, & (3) \end{aligned}$$

where  $\lambda, \mu$  are positive parameters,  $\Delta$  is the Laplacian operator,  $\Omega$  is a bounded region in  $\mathbb{R}^N$  ( $N \geq 1$ ) with smooth boundary

$$Bz(x) = \alpha h(x)z + (1 - \alpha) \frac{\partial z}{\partial n} \quad (4)$$

here  $B$  is a boundary operator and  $\left(\frac{\partial}{\partial n}\right)$  denotes the outward conormal derivative,  $\alpha \in [0, 1], h: \partial\Omega \rightarrow \mathbb{R}^+$

with  $h = 1$  when  $\alpha = 1$ , i.e; the boundary condition may be of Dirichlet ( $u=0$  on  $\partial\Omega$ ) Neumann  $\frac{\partial}{\partial n} = 0$  on  $\partial\Omega$  or mixed type (robin boundary condition (for  $\alpha \neq 0, 1$ ) and  $f, g: [0, +\infty) \times [0, +\infty) \rightarrow \mathbb{R}$  are  $C^1$  functions.

Existence results of problem (1)-(3) with sign-changing weight were obtained in [1]. Also stability/instability of positive solutions in the case nonsystem has been extensively studied; the reader is referred to [2,4,5,6,7]. Imre Voros in [8] study the stability of positive stationary solutions of the coupled system of semilinear partial differential equations. But in this paper, we discuss stability of positive solutions for Systems with Unequal Diffusion Coefficients and establish every positive stationary solution of B.v.p (1)-(3) is linearly stable(unstable) provided one of the following sets of conditions hold:

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$$(A.1)_i \begin{cases} -f + u \frac{\partial f}{\partial u} + v \frac{\partial g}{\partial u} < 0 (> 0) \\ -g + u \frac{\partial f}{\partial v} + v \frac{\partial g}{\partial v} < 0 (> 0) \end{cases}$$

$$(A.2)_i \begin{cases} -\lambda f + \lambda v \frac{\partial f}{\partial v} + \mu u \frac{\partial g}{\partial v} < 0 (> 0) \\ -\mu g + \lambda v \frac{\partial f}{\partial u} + \mu u \frac{\partial g}{\partial u} < 0 (> 0) \end{cases}$$

As a reminder, if  $(u, v)$  be any positive stationary solution of (1)-(3), then the linearized equation about  $(u, v)$  consists of

$$-\Delta w - \lambda \frac{\partial f}{\partial u}(u, v)w - \lambda \frac{\partial f}{\partial v}(u, v)z = \eta w \quad x \in \Omega, \quad (5)$$

$$-\Delta z - \mu \frac{\partial g}{\partial u}(u, v)w - \mu \frac{\partial g}{\partial v}(u, v)z = \eta z \quad x \in \Omega, \quad (6)$$

$$Bw = 0 = Bz \quad x \in \partial\Omega, \quad (7)$$

where Equations (5) and (6) obtained from the formal derivative of the operator  $\Delta$ .

**Definition 1.1.** We say a positive solution  $(u, v)$  of (1)-(3) is linearly stable if all eigenvalues of (5) and (6) are strictly positive, which can be inferred if the principal eigenvalue  $\eta_1$  (i.e., eigenvalue corresponding to positive eigenfunction) be positive. Otherwise we say  $(u, v)$  is linearly unstable.

## 2. Main results

In this section, we prove the stability/instability of positive stationary solutions by analyzing linearized equation.

**Theorem 2.1** Let (A.1) holds, then any positive stationary solution of Equations (1)-(3) is linearly stable(unstable).

**Proof:** Let  $(u_0, v_0)$  be any positive solution of (1)-(3), then the linearized equation about  $(u_0, v_0)$  consists of

$$-\Delta w - \lambda \frac{\partial f}{\partial u}(u_0, v_0)w - \lambda \frac{\partial f}{\partial v}(u_0, v_0)z = \eta w \quad x \in \Omega, \quad (8)$$

$$-\Delta z - \mu \frac{\partial g}{\partial u}(u_0, v_0)w - \mu \frac{\partial g}{\partial v}(u_0, v_0)z = \eta z \quad x \in \Omega, \quad (9)$$

$$Bw = 0 = Bz \quad x \in \partial\Omega, \quad (10)$$

Let  $\eta_1$  is principal eigenvalue and also  $(\phi, \psi)(\phi, \psi \geq 0)$  is a corresponding eigenfunction.

Now multiplying (1) by  $\frac{1}{\lambda}\phi$ , (2) by  $\frac{1}{\mu}\psi$ , (8) by  $-\frac{1}{\lambda}u_0$ , (9) by  $-\frac{1}{\mu}v_0$ , and integrating by parts over  $\Omega$  and adding the two resulting expressions, we have

$$\begin{aligned} & -\eta_1 \int_{\Omega} \left( \frac{1}{\lambda} u_0 \phi + \frac{1}{\mu} v_0 \psi \right) dx \\ &= \frac{1}{\lambda} \int_{\Omega} [u_0(x) \Delta \phi - \phi(x) \Delta u_0] dx + \frac{1}{\mu} \int_{\Omega} [v_0(x) \Delta \psi - \psi(x) \Delta v_0] dx \\ &+ \int_{\Omega} \phi \left( -f(u_0, v_0) + u_0 \frac{\partial f}{\partial u}(u_0, v_0) + v_0 \frac{\partial g}{\partial u}(u_0, v_0) \right) dx \\ &+ \int_{\Omega} \psi \left( -g(u_0, v_0) + u_0 \frac{\partial f}{\partial v}(u_0, v_0) + v_0 \frac{\partial g}{\partial v}(u_0, v_0) \right) dx, \end{aligned} \quad (11)$$

But by Green identity, we get

$$\int_{\Omega} [u_0(x) \Delta \phi - \phi(x) \Delta u_0] dx = \int_{\partial\Omega} \left[ u_0(x) \frac{\partial \phi}{\partial n} - \phi(x) \frac{\partial u_0}{\partial n} \right] ds,$$

and

$$\int_{\Omega} [v_0(x)\Delta\psi - \psi(x)\Delta v_0] dx = \int_{\partial\Omega} \left[ v_0(x) \frac{\partial\psi}{\partial n} - \psi(x) \frac{\partial v_0}{\partial n} \right] ds,$$

We observe that in (4) when  $\alpha = 1$  then  $Bu_0 = u_0 = 0$  and  $Bv_0 = v_0 = 0$  for all  $s \in \partial\Omega$  and because  $(\phi, \psi)$  is eigenfnction, thus  $\phi = \psi = 0$  for all  $s \in \partial\Omega$ . Also when  $\alpha \neq 1$ , then

$$\int_{\partial\Omega} \left[ u_0(x) \frac{\partial\phi}{\partial n} - \phi(x) \frac{\partial u_0}{\partial n} \right] ds = \int_{\partial\Omega} \frac{\alpha h\phi(s)}{(1-\alpha)} (u_0 - u_0) ds = 0,$$

and

$$\int_{\partial\Omega} \left[ v_0(x) \frac{\partial\psi}{\partial n} - \psi(x) \frac{\partial v_0}{\partial n} \right] ds = \int_{\partial\Omega} \frac{\alpha h\psi(s)}{(1-\alpha)} (v_0 - v_0) ds = 0,$$

Therefore for any  $\alpha \in [0,1]$

$$\int_{\Omega} [u_0(x)\Delta\phi - \phi(x)\Delta u_0] dx = \int_{\Omega} [v_0(x)\Delta\psi - \psi(x)\Delta v_0] dx = 0. \quad (12)$$

Now by replaying (12) in (11), we have

$$\begin{aligned} -\eta_1 \int_{\Omega} \left( \frac{1}{\lambda} u_0 \phi + \frac{1}{\mu} v_0 \psi \right) dx &= \int_{\Omega} \phi \left( -f(u_0, v_0) + u_0 \frac{\partial f}{\partial u}(u_0, v_0) + v_0 \frac{\partial g}{\partial u}(u_0, v_0) \right) dx \\ &\quad + \int_{\Omega} \psi \left( -g(u_0, v_0) + u_0 \frac{\partial f}{\partial v}(u_0, v_0) + v_0 \frac{\partial g}{\partial v}(u_0, v_0) \right) dx. \end{aligned} \quad (13)$$

By hypothesis,  $u_0, v_0 > 0$  and  $\phi, \psi > 0$  in  $\Omega$ , hence if (A.1) holds then  $\eta_1 > 0 (< 0)$ , thus every positive stationary solution of (1)-(3) is linearly stable(unstable).

**Theorem 2.2** Let (A.2) holds, then any positive stationary solution of Equations (1)-(3) is linearly stable(unstable).

**Proof:** Let  $(u_0, v_0)$  be any positive solution of (1)-(3). Now multiply (1) by  $\psi$ , (2) by  $\phi$ , (8) by  $-v_0$ , (9) by  $-u_0$ , and integrate by parts over  $\Omega$  and add the two resulting expressions, then use the same way as for Theorem 2.1 to get

$$\begin{aligned} -\eta_1 \int_{\Omega} (v_0 \phi + u_0 \psi) dx &= \int_{\Omega} \psi \left( -\lambda f(u_0, v_0) + \lambda v_0 \frac{\partial f}{\partial v}(u_0, v_0) + \mu u_0 \frac{\partial g}{\partial v}(u_0, v_0) \right) dx \\ &\quad + \int_{\Omega} \phi \left( -\mu g(u_0, v_0) + \lambda v_0 \frac{\partial f}{\partial u}(u_0, v_0) + \mu u_0 \frac{\partial g}{\partial u}(u_0, v_0) \right) dx. \end{aligned}$$

Now by hypothesis, it is easy to see that if (A.2) holds then  $\eta_1 > 0 (< 0)$ , and the result follows.

**Corollary 2.3** Let system be cooperative (i.e.  $\frac{\partial f}{\partial v}, \frac{\partial g}{\partial u} > 0$ ), and assume either

(B.1):  $f(u, v)/u, g(u, v)/v$  corresponding with respect to  $u, v$  are strictly increasing,

(B.2):  $-\lambda f + \mu u \frac{\partial g}{\partial v} > 0, -\mu g + \lambda v \frac{\partial f}{\partial u} > 0$

Then any positive stationary solution of Equations (1)-(3) is linearly unstable.

**Proof:** Proceeding as in the proof of Theorem 2.1, and then since  $f(u, v)/u, g(u, v)/v$  corresponding with respect to  $u, v$  are strictly increasing, we have

$$\frac{d}{du} \left( \frac{f(u, v)}{u} \right) (u_0, v_0) = \frac{u_0 \frac{\partial f}{\partial u}(u_0, v_0) - f(u_0, v_0)}{u_0^2} > 0,$$

and

$$\frac{d}{dv} \left( \frac{g(u, v)}{v} \right) (u_0, v_0) = \frac{v_0 \frac{\partial g}{\partial v} (u_0, v_0) - g(u_0, v_0)}{v_0^2} > 0.$$

Hence the integrand on the right-hand side (13) is positive, and thus  $\eta_1 < 0$ . Also it can be easily shown that if (B.2) holds then  $\eta_1 < 0$ . This completes the proof.

**Corollary 2.4** Let system be competitive (i.e.  $\frac{\partial f}{\partial v}, \frac{\partial g}{\partial u} < 0$ ), and assume either

(C.1):  $f(u, v)/u, g(u, v)/v$  corresponding with respect to  $u, v$  are strictly decreasing,

(C.2):  $-\lambda f + \mu u \frac{\partial g}{\partial v} < 0, -\mu g + \lambda v \frac{\partial f}{\partial u} < 0$

Then any positive stationary solution of Equations (1)-(3) is linearly stable.

**Proof:** The proof proceeds in the same way as for Theorem 2.1 and Corollary 2.3.

### 3. Application

Consider the system

$$\begin{cases} -\Delta u(x) = \lambda f(v) & x \in \Omega, \\ -\Delta v(x) = \mu g(u) & x \in \Omega, \\ u = 0 = v & x \in \partial\Omega, \end{cases} \quad (I)$$

where  $\Omega$  is the open unit ball in  $R^N$ ; ( $N \geq 1$ ), with a smooth boundary  $\partial\Omega$ ,  $\lambda, \mu$  are positive parameters. See [3] where the authors study the uniqueness of positive solutions. Let  $u$  be any positive solution to problem (I). Of (A.1) and (A.2) it is easy to see that if either

$$(I.1): -f + v \frac{\partial g}{\partial u} < 0 (> 0), -g + u \frac{\partial f}{\partial v} < 0 (> 0), \text{ or}$$

$$(I.2): f(v)/v \text{ and } g(u)/u \text{ corresponding with respect to } v \text{ and } u \text{ are strictly decreasing (increasing),}$$

hold then any positive solution of Equation (I) is linearly stable (unstable).

### 4. References

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