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Linearisation of Boundary Optimal Control Problems by Finite Element Method

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Abstract. Purpose of this paper is to solve a nonstandard optimal control problem governed by ordinary differential equation by finite element method. In the first part of the paper, we describe the method of discretisation of continuous optimal control problem and then linearize the problem and obtain a linear programming. By solving the linear programming, the control and state functions and the value of objective function are obtained.

Keywords: Finite element method, Ordinary differential equation, Min-max objective function, Discrete maximum principle.

1. Introduction

Optimal control is one of the most common techniques in dynamic optimization. This article is primarily aimed at instructors who teach and want to complement their teaching through simple numerical analysis. Realistic mathematical models of dynamical processes from scientific or engineering background may often have to consider different physical phenomena and therefore may lead to coupled systems of equations that include ordinary and partial differential equations as well as algebraic equations.

The flight of a hypersonic aircraft under the objective of minimum fuel consumption may serve as a typical example. The flight trajectory is described, as usual, by a system of ordinary differential equations. This system is controlled by the usual control variables of flight path optimization under various control and state variable inequality constraints. There are several papers dealing with control problems with state constraints [8], [9], [2]. In [7], Kostreva and Ward introduced a new method for solving a state constrained boundary control problem with a min-max objective function with elliptic partial differential operator $Ly = -\nabla \cdot (K\nabla y)$.

Alt and Bräutigam in [1], develop error estimates for the solution of the discrete problem. For a discretization of the state equation by the method of Finite Differences and a piecewise approximation of the control. Hinze in [6], by using the method of Finite Elements and derives upper bounds for semi-discretizations where only the state variable z is discretized and the control u is an element of L^2. This paper is concerned with the one-dimensional elliptic problem of optimal control with ordinary differential operator $Ly = -\dfrac{d}{dx}(k(x)\dfrac{dy}{dx}) + cy$.

The purpose of this article is to derive a technique for solving non-standard optimal control problems by finite element method. In fact, the problem reduces to a linear boundary control problem.

In Section 2, we state the problem in terms of optimal control theory, in subsection 2.2, an outline of the finite element method in ordinary differential equation is presented and a family of discrete linear programming problems is defined. In Section 3, the main theoretical results are stated with an application of the continuous maximum principles. A two-level linear formulation of the problem is presented in Section 4. In Section 5, numerical results are presented to demonstrate the stability and efficiency of the numerical method to optimal configuration problems.

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2. The continuous problem and its discretisation

2.1. Problem formulation

Consider the state constrained boundary optimal control problem

$$\begin{aligned} &\min_{a_1} \quad \max_{[a,b]} \; y(x) \\ &\text{s.t.} \quad Ly(x)= f, \qquad x\in(a,b), \\ &\qquad\; y(x)\geq \psi(x), \quad x\in[a,b], \\ &\qquad\; y(a)=a_1, \\ &\qquad\; y(b)=a_2. \end{aligned} \tag{2.1}$$

Let the operator L be given by

$$Ly=-\frac{d}{dx}(k(x)\frac{dy}{dx})+cy, \tag{2.2}$$

where

$$c\geq 0, \qquad on \quad (a,b).$$

Assume, furthermore that for all x in $[a,b], k(x)>0.$

Let ψ be a lower bound constraint on the state y such that $\psi\in C^2(a,b)$ and $\psi(b)\leq b_1$. Also, f is a given function and a_1 represents the boundary control. With the stated conditions and assumptions, the boundary optimal control problem (2.1) is well defined.

2.2. Discretisation

We now present the finite element discretisation scheme for problem (2.1). Let

$$V=\{v\in H^1(a,b): v(a)=v(b)=0\}, \tag{2.3}$$

be the test space. Define

$$a(y,v)=\int_a^b k(x)\frac{dy}{dx}\frac{dv}{dx}dx+\int_a^b cyvdx \tag{2.4}$$

and

$$\ell(v)=\int_a^b fvdx \tag{2.5}$$

where a is the continuous, coercive, bilinear form on V, and ℓ is continuous and linear form on V.

Let z be any element in $H^1(a,b)$ such that $z(a)=a_1$ and $z(b)=a_2$. Then the weak form of the boundary value problem (2.1) is: Find $y-z\in V$ such that

$$a(y,v)=\ell(v), \quad \forall v\in V. \tag{2.6}$$

Let T_h be a partition of (a,b) where h is the length of the longest subinterval in T_h. We associate as usual the finite element spaces

$$\begin{aligned} X_h &=\{v_h\in C(a,b): v_h\big|_T\in P_1(T), \forall T\in T_h\}, \\ V_h &=\{v_h\in X_h : v_h(a)=v_h(b)=0\} \end{aligned} \tag{2.7}$$

where P_1 is the set of all polynomials of degree less than or equal to 1.

Now, we assume x_i, $i=1,\dots,n+2$, denote the points at the end of all elements, where $x_{n+1}=a$, $x_{n+2}=b$, and we let $\phi_i, i=1,\dots,n+2$, denote the functions of X_h which satisfy

$$\varphi_i(x_j)=\delta_{ij}, \quad i,j=1,\dots,n+2, \tag{2.8}$$

i.e., the functions $\varphi_i, i=1,\dots,n$ or $\varphi_i, i=1,\dots,n+2$, form a basis of V_h, or of X_h. Let

$$u_h(x) = a_1\varphi_{n+1}(x), \tag{2.9}$$

and

$$g_h(x) = a_2\varphi_{n+2}(x). \tag{2.10}$$

For $z_h \in X_h$ such that $z_h(a) = u_h(a)$ and $z_h(b) = g_h(b)$, the finite-dimensional variational problem is to find $y_h - z_h \in V_h$ such that

$$a(y_h, v) = \ell(v), \quad \forall v \in V_h, \tag{2.11}$$

Since any function $v \in V_h$ may be written as a linear combination of the basis functions $\{\phi_i\}_{i=1}^{n}$, the function y_h may be represented as

$$y_h = \sum_{i=1}^{n} \alpha_i\varphi_i + a_1\varphi_{n+1} + a_2\varphi_{n+2}, \tag{2.12}$$

for some choice of the coefficients $\alpha_1,...,\alpha_n$, and a_1. We define

$$\begin{array}{lll} A = (a_{ij}), & a_{ij} = a(\varphi_i, \varphi_j), & i = 1,...,n, \quad j = 1,...,n, \\ \tilde{A} = (\tilde{a}_{i,n+1}), & \tilde{a}_{i,n+1} = a(\varphi_{n+1}, \varphi_i), & i = 1,...,n, \\ F = (f_j), & f_j = \ell(\varphi_j) - a_2 a(\varphi_j, \varphi_{n+2}) & j = 1,...,n, \end{array} \tag{2.13}$$

and $\alpha = (\alpha_1,...,\alpha_n)^T$, so the discrete variational problem reduces to solving the linear system

$$A\alpha = F - \tilde{A}a_1 \tag{2.14}$$

for some choice of a_1.

Let ψ_h be an element of X_h which approximates the continuous lower bound ψ. Let Ψ be the vector of coefficients $\psi_h(x_i)$, $i = 1,...,n$, and let $\tilde{\Psi} = \psi_h(x_{n+1})$. Then the finite-dimensional boundary control problem is stated as

$$\begin{array}{ll} \min & \max\{\alpha, a_1\} \\ s.t. & A\alpha + \tilde{A}a_1 = F \\ & \alpha \geq \Psi \\ & a_1 = \tilde{\Psi} \end{array} \tag{2.15}$$

For further discussion of the finite element method, the reader is referred to [3], [4] and [10].

3. Weak maximum principle

In this section we consider the conditions under which the weak maximum principle holds for the continuous control problem and the discrete control problem. Using a linear reformulation of the problem, it is advantageous to consider the solution on boundary rather than the entire region.

By the divergence theorem, the weak form of

$$Ly = 0 (\leq 0, \geq 0), \tag{3.1}$$

is

$$\int_a^b k(x)\frac{dy}{dx}\frac{dv}{dx}dx + \int_a^b cyvdx = 0(\leq 0, \geq 0), \quad \forall v \in H^1(a,b), \tag{3.2}$$

If $y \in H^1(a,b)$ satisfies (3.2) then y is said to satisfy (3.1) weakly. Furthermore, if $y \in H^1(a,b)$ satisfies

$$\int_a^b k(x)\frac{dy}{dx}\frac{dv}{dx}dx + \int_a^b cyvdx = 0(\leq 0, \geq 0), \quad \forall v \in V, \tag{3.3}$$

then y is said to satisfy (3.1) weakly.

Theorem. Let $y \in H^1(a,b)$ satisfy $Ly \le 0$, on (a,b). Then

$$\sup_{(a,b)} y(x) \le \sup_{\{a,b\}} y^+(x)$$

where $y^+ = \max\{y,0\}$.

Proof. We use the idea of Gilbarg and Trudinger [5], (3.3) and $Ly \le 0$ implies that

$$\int_a^b k(x)\frac{dy}{dx}\frac{dv}{dx}dx + \int_a^b cyvdx \le 0, \tag{3.4}$$

for all $v \in V$. So

$$\int_a^b k(x)\frac{dy}{dx}\frac{dv}{dx}dx \le 0, \tag{3.5}$$

for all $v \in V$ such that $yv \ge 0$. Taking $v = \max\{y-l,0\}$ where $l = \sup\{y^+(a), y^+(b)\}$, then $v \in V$, $v(x) \ge 0$ almost everywhere, and $v = 0$ on boundary. In a week derivative sense,

$$\frac{dv}{dx} = \begin{cases} \dfrac{dy}{dx} & if \quad y > l, \\ 0 & if \quad y \le l. \end{cases} \tag{3.6}$$

Combining (3.5), (3.6),

$$\int_a^b k(x)\frac{dy}{dx}\frac{dv}{dx}dx = \int_{\{x\in(a,b):y\le l\}} k(x)\frac{dy}{dx}\frac{dv}{dx}dx + \int_{\{x\in(a,b):y>l\}} k(x)\frac{dy}{dx}\frac{dv}{dx}dx \le 0$$

$$= \int_{\{x\in(a,b):y>l\}} k(x)\frac{dy}{dx}\frac{dv}{dx}dx$$

implies

$$\int_{\{x\in(a,b):y>l\}} k(x)(\frac{dy}{dx})^2 dx \le 0$$

Since $k(x) > 0$, either y is a constant or the set $\{x \in (a,b) : y > l\}$ has measure 0. In the other word,

$$\sup_{(a,b)} y(x) \le \sup_{\{a,b\}} y(x)$$

4. Existence and uniqueness of a solution

In [7], the existence of a solution to the state constrained boundary control problem is shown under twice differentiable conditions on the state constraint. In general, min-max problems do not have unique solutions; however, if a constant control is assumed, the min-max boundary control problem has a unique solution with implications for the convergence of the discrete boundary control problems. The weak maximum principle and discrete maximum principle are shown to hold under certain conditions and may be applied to the solutions of the boundary control problem in order to reduce the number of constraints. The continuous optimality conditions are also stated in an algebraic setting.

5. Linear Subproblems

The subproblems may be reformulated as linear programming problems. Let $s = \max\{\alpha, a_1\}$ and e be a n vector of 1's. The min-max objective function is replaced by the objective function $\min s$ and constraints $\alpha \le es$ and $a_1 \le s$. Adding $n+1$ constraints to the problem, the weak maximum principles applies to the discrete subproblems. Assume that

$$f \le 0, \quad \max_{(a,b)} \psi > a_2.$$

Under these conditions, the constraints $\alpha \le es$ are redundant. Thus, when the weak maximum principle holds, a valid linear formulation of the discrete boundary control problem is

$$\begin{aligned} \min \quad & s \\ s.t. \quad & s - a_1 \ge 0, \\ & A\alpha + \tilde{A}a_1 = F, \\ & \alpha \ge \Psi, \\ & a_1 \ge \tilde{\Psi}. \end{aligned} \tag{5.1}$$

6. Numerical results

In order to illustrate our technique to solve the optimal control of system governed by ordinary differential equation, we consider following example:

$$\begin{aligned} \min_{a_1} \quad & \max_{(0,2)} \; y(x) \\ \text{s.t.} \quad & Ly(x) = -4x^2, \quad x \in (0,2), \\ & y(x) \ge x^2 - x, \quad x \in [0,2], \\ & y(0) = 0_1, \\ & y(2) = a_1. \end{aligned} \tag{6.1}$$

where $Ly = -\frac{d}{dx}(x^2 \frac{dy}{dx}) + 2y$. Note that the maximum value of ψ is 2 and occurs at boundary $x = 2$ which is the control point.

By using the above procedure, the linearisation and discretisation of model (6.1) as the following form:

$$\begin{aligned} \min \quad & s \\ s.t. \quad & s - a_1 \ge 0, \\ & A\alpha + \tilde{A}a_1 = F, \\ & \alpha \ge \Psi, \\ & a_1 \ge \tilde{\Psi}. \end{aligned} \tag{6.2}$$

where

$$A = \begin{bmatrix} 0.8 & -0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.4 & 2 & -1.2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.2 & 4 & -2.4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2.4 & 6.8 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 10.4 & -6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6 & 14.8 & -8.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -8.4 & 20 & -11.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -11.2 & 26 & 14.4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -14.4 & 32.8 \end{bmatrix}$$

and

$$\tilde{A} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -18 \end{bmatrix}, \qquad F = \begin{bmatrix} -0.037333333 \\ -0.133333333 \\ -0.293333334 \\ -0.517333333 \\ -0.805333333 \\ -1.157333333 \\ -1.573333333 \\ -2.053333333 \\ -2.597333334 \end{bmatrix}$$

Now, by solving this problem, the optimal value of the objective function equal to 2, and also the optimal control on $x = 2$ is equal to 2.

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