An Iterative Conjugate Gradient Regularization Method for Image Restoration

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Abstract: Image restoration is an ill-posed inverse problem, which has been introduced the regularization method to suppress over-amplification. In this paper, we propose to apply the iterative regularization method to the image restoration problem and present a nested iterative method, called iterative conjugate gradient regularization method. Convergence properties are established in detail. Based on [6], we also simultaneously determine the regularization parameter based on the restored image at each step. Simulation results show that the proposed iterative regularization method is feasible and effective for image restoration.

Key Words: Tikhonov regularization, image restoration, blur.

1. Introduction

Image restoration problem has been extensively studied and used in several areas of science and engineering [2][6][8-11]. It calls for the recovery of an original scene from a degraded observation. For example, stellar images observed by ground-base telescopes are degraded due to atmospheric turbulence, while there are also applications where the stellar images need to be restored even if they are not observed through the atmosphere.

In most cases, the image degradation process can be modeled by a linear blur and an additive white Gaussian noise process, that is

$$y = Hx + n \qquad (1.1)$$

where y, x, n are $MN \times 1$ vectors and represent respectively the lexicographically ordered $M \times N$ pixel observed degraded image, original image, and additive noise. The matrix H represents the degradation matrix of size $MN \times MN$, which may represent a spatially invariant or a spatially varying degradation. The image restoration problem calls for applying an inverse procedure to obtain an approximation of the original image x based on the image degradation model. It is an ill-posed problem, which means that a small perturbation in the data leads to a large perturbation in the solution. Therefore, a regularization method has to be used in order to determine a useful approximation of the true image. One of the most popular regularization techniques is Tikhonov regularization. This method approximately solves (1.1) by solving the unconstrained minimization problem

$$\min_{x} M(x, \alpha) \qquad (1.2)$$

With

$$M(x, \alpha) = \|y - Hx\|_2^2 + \alpha \|x\|_2^2 \qquad (1.3)$$

where α is a positive regularization parameter. A solution of (1.2) is computed by solving its first-order conditions

$$(H^T H + \alpha I)x = H^T y \qquad (1.4)$$

This method has been extensively studied in image restoration. However for ill-posed problem, the convergence rate may be improved [4] in an iterated version of (1.4) given by

$$(H^T H + \alpha I)x_{k+1} = \alpha x_k + H^T y \qquad (1.5)$$

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We call it iterative regularization.

In this paper, we propose to apply the iterative regularization method to the image restoration problem and present a nested iterative method, called iterative conjugate gradient regularization (ICGR) method. Convergence properties are established in detail. Based on [6], we also simultaneously determine the regularization parameter based on the restored image at each step. Simulation results show that the proposed iterative regularization method is feasible and effective for image restoration.

The rest of the paper is organized as follows. In Section 2, ICGR method is introduced, along with the choice of regularization parameter. In Section 3, convergence properties are established. Experimental results are presented in Section 4 and conclusions are reached in Section 5.

2. Iterative Regularization

For ill-posed problems involving closed, densely defined linear operators, M. Hanke and C. W. Groetsch have studied the iterative regularization method [4]. Image degradation is an ill-posed problem, and the iterative regularization method could be naturally applied to this problem. For the image degradation model (1.1), iterative regularization is

$$(H^T H + \alpha I)x_{k+1} = \alpha x_k + H^T y \tag{2.1}$$

which is equivalent to the following minimization problem.

$$\min_x L(x, \alpha) \tag{2.2}$$

where

$$L(x, \alpha) = \left\| y - Hx \right\|_2^2 + \alpha \left\| x - x_k \right\|_2^2 \tag{2.3}$$

2.1. Choice of regularization parameter α

In order for the nonlinear cost function $L(x, \alpha)$ to have a global minimum, the regularization parameter α should be chosen in a proper way. It is noted that choosing a suitable regularization parameter a priori is difficult, though there are many meaningful choices of the regularization parameter. In this paper, the following properties are needed which is adapted from [6].

Property 1. α should be a function of the smoothing functional: We choose α to be proportional to $L(x, \alpha)$ which represents the regularized noise power.

Property 2. Extreme minimizers of $L(x, \alpha)$. The minimizer of $L(x, \alpha)$ should represent a solution between two extreme solutions: one representing the generalized inverse solution of (1.1) when the data are noiseless, and the other representing the smoothest possible solution, when the noise power becomes infinite.

Property 3. The functional $L(x, \alpha)$ should be convex for all choices of α. This requirement on convexity is obviously very important, since a local extremum of a nonlinear functional becomes a global extremum, if the functional is convex. Therefore, the iterative algorithm that will be employed for obtaining a minimizer of $L(x, \alpha)$ will not depend on the initial condition.

Based on the above properties, α can take the following form,

$$\alpha = \frac{\left\| y - Hx \right\|_2^2}{\frac{1}{\gamma} - \left\| x - x_k \right\|_2^2}$$

Also according to [6], we set $\frac{1}{\gamma} = 2\left\| y \right\|_2^2$.

2.2. ICGR method

The basic idea of our proposed ICGR method is as follows. Given a starting vector $x_0 \in R^n$, suppose that we have got approximations $x_0, x_1, ..., x_k$ to the solution x^* of the normal equation

$$H^T Hx = H^T y \tag{2.4}$$

Then the next approximation x_{k+1} to x^* is obtained by solving the following equation iteratively,

$$(H^T H + \alpha_k I)x = H^T y + \alpha_k x_k \tag{2.5}$$

with the CG method, to certain arithmetic precision. More precisely, this iterative CG regularization method can be described as follows.

Algorithm 2.1. (The ICGR method)

   1. Input the largest admissible number of outer iteration k_{max} and the outer iteration stoping tolerance ε

   2. Input the largest admissible number of inner iteration l_{max} and the inner iteration stoping tolerance δ

   3. Input the starting vector x and the regularization parameter $\alpha, k = 0$

   4. Do the following steps

     4.1. $r = y - Hx, s = H^T r, \rho^{(0)} = \|s\|_2^2$

     4.2. $z = x, l = 1$

     4.3. Do while $\sqrt{\rho^{(l-1)}} > \delta\sqrt{\rho^{(0)}}$ and $l < l_{max}$

        (a) If $l = 1$ then $\beta = 0$ and $p = s$

          else

            $\beta = \rho^{(l-1)} / \rho^{(l-2)}$ and $p = s + \beta p$

        (b) $q = Hp$

        (c) $w = H^T q + \alpha p$

        (d) $\alpha = \rho^{(l-1)} / p^T w$

        (e) $z = z + \alpha p$

        (f) $s = s - \alpha w$

        (g) $\rho^{(l)} = \|r\|_2^2$

        (h) $l = l + 1$

     4.4. EndDo

     4.5. $\text{rres} = \dfrac{\|x - x\|_2}{\|z\|_2}$

     4.6. $x = z$

     4.7. Update α

     4.8. $k = k + 1$

  5. Until $\text{rres} < \varepsilon$ or $k > k_{max}$

Notice that, at kth outer iterates, x_{k-1} is chosen as the initial approximation, which means that $\|x - x_{k-1}\|$ is zero for $x = x_{k-1}$, therefore, we take α at the kth outer iterates the following form,

$$\alpha = \frac{\|y - Hx_{k-1}\|_2^2}{2\|y\|_2^2}$$

3. Convergence Analysis

Before proving the convergence of the ICGR method, we first introduce some notations and lemmas.

For a symmetric positive definite (SPD) matrix $B \in R^{n \times n}$, we use $\Lambda(B)$ to represent its spectrum set, and $\lambda_{min}(B)$ and $\lambda_{max}(B)$ its smallest and largest eigenvalues respectively. For any $z \in R^n$, its B-norm is defined by $\|z\|_B = \sqrt{z^T B z}$. For a nonsymmetric matrix $H \in R^{n \times n}$, we use $\sigma(H)$ to represent its singular value, and $\sigma_{min}(H)$ and $\sigma_{max}(H)$ its smallest and largest singular values respectively. Clearly, for any nonsingular matrix H, $\|H\|_2 = \sigma_{max}(H)$.

Lemma 3.1. Let $B \in R^{n \times n}$ be an SPD matrix. Then for any $z \in R^n$,

$$\left\| B^{\frac{1}{2}} z \right\|_2 = \|z\|_B$$

and

$$\sqrt{\lambda_{\min}(B)} \|z\|_B \leq \|Bz\|_2 \leq \sqrt{\lambda_{\max}(B)} \|z\|_B$$

Lemma 3.2. Let $B \in R^{n \times n}$ be a SPD matrix. If the CG method is started from an initial iterate $x_0 \in R^n$, then after k-steps of iterates, it generates an approximation x_k to the solution x^* of the linear equation $Bx = y$, which satisfies

$$\left\| x_k - x^* \right\|_B \leq \min_{p_k \in P_k} \max_{z \in \Lambda(B)} |p_k(z)| \left\| x_0 - x^* \right\|_B \leq 2 \left(\frac{\sqrt{\lambda_{\max}(B)} - \sqrt{\lambda_{\min}(B)}}{\sqrt{\lambda_{\max}(B)} + \sqrt{\lambda_{\min}(B)}} \right)^k \left\| x_0 - x^* \right\|_B$$

where $P_k = \{ p \mid p \text{ is a polynomial of degree } k \text{ and } p(0) = 1 \}$ is the set of kth degree residual polynomials.

Corollary 3.1. If $H \in R^{n \times n}$ is an nonsingular and nonsymmetric matrix. If the CG method for equation (2.5) is started from an initial iterate $z_0 \in R^n$, then after m_k-steps of iterates, it generates an approximation z_{m_k} to the solution $z^{(k,*)}$ of the linear equation (2.5), which satisfies

$$\left\| z_{m_k} - z^{(k,*)} \right\|_{(H^T H + \alpha_k I)} \leq 2 \left(\frac{\sqrt{\sigma_{\max}^2(H) + \alpha_k} - \sqrt{\sigma_{\min}^2(H) + \alpha_k}}{\sqrt{\sigma_{\max}^2(H) + \alpha_k} + \sqrt{\sigma_{\min}^2(H) + \alpha_k}} \right)^{m_k} \left\| z_0 - z^{(k,*)} \right\|_{(H^T H + \alpha_k I)}$$

Theorem 3.1. Let $H \in R^{n \times n}$ be a nonsingular matrix. If the ICGR method is started from an initial iterate $x_0 \in R^n$, and applies m_k steps of CG iteration to get the next approximation x_{k+1} to the solution x^* of the normal equation $H^T H x = H^T y$. Then

$$\left\| H(x_{k+1} - x^*) \right\|_2 \leq q(m_k) \left\| H(x_k - x^*) \right\|_2 \tag{3.1}$$

where

$$q(m_k) = \frac{\alpha_k}{\sigma_{\min}^2(H) + \sigma_k} + 2\sigma_{\max}^2(H) \frac{\sqrt{\sigma_{\max}^2(H) + \alpha_k}}{(\sigma_{\max}^2(H) + \alpha_k)^{\frac{3}{2}}} \left(\frac{\sqrt{\sigma_{\max}^2(H) + \alpha_k} - \sqrt{\sigma_{\min}^2(H) + \alpha_k}}{\sqrt{\sigma_{\max}^2(H) + \alpha_k} + \sqrt{\sigma_{\min}^2(H) + \alpha_k}} \right)^{m_k} \tag{3.2}$$

Proof. Denote $c_k(\alpha_k) = \alpha_k x_k + H^T y$, $x^{(k,*)}$ the exact solution of the linear equation (2.5), i.e., it satisfies

$$(H^T H + \alpha_k I) x^{(k,*)} = c_k(\alpha_k),$$

and $z^{(k,m_k)}$ the final result of the inner CG iteration at the kth outer iterate of the regularized CGNR method. Then from Corollary 3.1 we have

$$\left\| z^{(k,m_k)} - x^{(k,*)} \right\|_{(H^T H + \alpha_k I)} \leq 2 \left(\frac{\sqrt{\sigma_{\max}^2(H) + \alpha_k} - \sqrt{\sigma_{\min}^2(H) + \alpha_k}}{\sqrt{\sigma_{\max}^2(H) + \alpha_k} + \sqrt{\sigma_{\min}^2(H) + \alpha_k}} \right)^{m_k} \left\| z^{(k,0)} - x^{(k,*)} \right\|_{(H^T H + \alpha_K I)}$$

Notice that $z^{(k,0)} = x_k, x_{k+1} = z^{(k,m_k)}$, the above estimate immediately leads to

$$\left\| x_{k+1} - x^{(k,*)} \right\|_{(H^T H + \alpha_k I)} \leq 2 \left(\frac{\sqrt{\sigma_{\max}^2(H) + \alpha_k} - \sqrt{\sigma_{\min}^2(H) + \alpha_k}}{\sqrt{\sigma_{\max}^2(H) + \alpha_k} + \sqrt{\sigma_{\min}^2(H) + \alpha_k}} \right)^{m_k} \left\| x_k - x^{(k,*)} \right\|_{(H^T H + \alpha_K I)}$$

Define the vector $r_k = y - H x_k$ and $B(\alpha_k) = H^T H + \alpha_k I$.

Then we have

$$B(\alpha_k)x_{k+1} - c_k\alpha_k$$
$$= (H^T H + \alpha_k I)x_{k+1} - (H^T y + \alpha_k x_k)$$
$$= \alpha_k(x_{k+1} - x_k) + (H^T H x_{k+1} - H^T y)$$
$$= \alpha_k H^{-1}(r_k - r_{k+1}) - H^T r_{k+1}$$
$$= \alpha_k H^{-1} r_k - (\alpha_k H^{-1} + H^T)r_{k+1}$$
$$= \alpha_k H^{-1} r_k - H^{-1}(\alpha_k I + HH^T)r_{k+1}$$

So,

$$H^{-1}(\alpha_k I + HH^T)r_{k+1} = \alpha_k H^{-1} r_k + (c_k(\alpha_k) - B(\alpha_k)x_{k+1})$$

Therefore,

$$r_{k+1} = (HH^T + \alpha_k I)^{-1} H(\alpha_k H^{-1} r_k + (c_k(\alpha_k) - B(\alpha_k)x_{k+1}))$$
$$= (HH^T + \alpha_k I)^{-1}(\alpha_k r_k + H(c_k(\alpha_k) - B(\alpha_k)x_{k+1})) \tag{3.3}$$

On the other hand,

$$\left\| c_k(\alpha_k) - B(\alpha_k)x_{k+1} \right\|_2$$
$$= \left\| c_k(\alpha_k) - (H^T H + \alpha_k I)x_{k+1} \right\|_2$$
$$= \left\| (H^T H + \alpha_k I)((H^T H + \alpha_k I)^{-1} c_k(\alpha_k) - x_{k+1}) \right\|_2$$
$$= \left\| (H^T H + \alpha_k I)(x^{(k,*)} - x_{k+1}) \right\|_2$$
$$\leq \sqrt{\sigma_{max}^2(H) + \alpha_k} \left\| x^{(k,*)} - x_{k+1} \right\|_{(H^T H + \alpha_k I)}$$
$$\leq 2\sqrt{\sigma_{max}^2(H) + \alpha_k}\left(\frac{\sqrt{\sigma_{max}^2(H) + \alpha_k} - \sqrt{\sigma_{min}^2(H) + \alpha_k}}{\sqrt{\sigma_{max}^2(H) + \alpha_k} + \sqrt{\sigma_{min}^2(H) + \alpha_k}}\right)^{m_k} \left\| x^{(k,*)} - x_k \right\|_{(H^T H + \alpha_k I)}$$
$$\leq 2\frac{\sqrt{\sigma_{max}^2(H) + \alpha_k}}{\sqrt{\sigma_{min}^2(H) + \alpha_k}}\left(\frac{\sqrt{\sigma_{max}^2(H) + \alpha_k} - \sqrt{\sigma_{min}^2(H) + \alpha_k}}{\sqrt{\sigma_{max}^2(H) + \alpha_k} + \sqrt{\sigma_{min}^2(H) + \alpha_k}}\right)^{m_k} \left\| (H^T H + \alpha_k I)(x^{(k,*)} - x_k) \right\|_2$$
$$= 2\frac{\sqrt{\sigma_{max}^2(H) + \alpha_k}}{\sqrt{\sigma_{min}^2(H) + \alpha_k}}\left(\frac{\sqrt{\sigma_{max}^2(H) + \alpha_k} - \sqrt{\sigma_{min}^2(H) + \alpha_k}}{\sqrt{\sigma_{max}^2(H) + \alpha_k} + \sqrt{\sigma_{min}^2(H) + \alpha_k}}\right)^{m_k} \left\| c_k(\alpha_k) - (H^T H + \alpha_k I)x_k \right\|_2$$
$$= 2\frac{\sqrt{\sigma_{max}^2(H) + \alpha_k}}{\sqrt{\sigma_{min}^2(H) + \alpha_k}}\left(\frac{\sqrt{\sigma_{max}^2(H) + \alpha_k} - \sqrt{\sigma_{min}^2(H) + \alpha_k}}{\sqrt{\sigma_{max}^2(H) + \alpha_k} + \sqrt{\sigma_{min}^2(H) + \alpha_k}}\right)^{m_k} \left\| H^T r_k \right\|_2 \tag{3.4}$$

By using (3.4) and taking 2-norm on both sides of (3.3), we get

$$\left\| r_{k+1} \right\|_2 \leq \alpha_k \left\| (H^T H + \alpha_k I)^{-1} \right\|_2 \left\| r_k \right\|_2 + \left\| (HH^T + \alpha_k I)^{-1} \right\|_2 \left\| H \right\|_2 \left\| c_k(\alpha_k) - B(\alpha_k)x_{k+1} \right\|_2$$
$$\leq \frac{\alpha_k}{\sigma_{min}^2(H) + \alpha_k}\left\| r_k \right\|_2 + 2\sigma_{max}(H)\frac{\sqrt{\sigma_{max}^2(H) + \alpha_k}}{(\sigma_{min}^2(H) + \alpha_k)^{\frac{3}{2}}}\left(\frac{\sqrt{\sigma_{max}^2(H) + \alpha_k} - \sqrt{\sigma_{min}^2(H) + \alpha_k}}{\sqrt{\sigma_{max}^2(H) + \alpha_k} + \sqrt{\sigma_{min}^2(H) + \alpha_k}}\right)^{m_k} \left\| H^T r_k \right\|_2$$
$$\leq \left(\frac{\alpha_k}{\sigma_{min}^2(H) + \alpha_k} + 2\sigma_{max}^2(H)\frac{\sqrt{\sigma_{max}^2(H) + \alpha_k}}{(\sigma_{min}^2(H) + \alpha_k)^{\frac{3}{2}}}\left(\frac{\sqrt{\sigma_{max}^2(H) + \alpha_k} - \sqrt{\sigma_{min}^2(H) + \alpha_k}}{\sqrt{\sigma_{max}^2(H) + \alpha_k} + \sqrt{\sigma_{min}^2(H) + \alpha_k}}\right)^{m_k}\right)\left\| r_k \right\|_2$$

(3.1) is proved.

4. Experimental Results

In this section we present the results of two image restoration test problems, in order to illustrate the

performance of the proposed iterative regularization method.

The first is Satellite, which is an image restoration test problem that was developed at the US Air Force Phillips Laboratory, Lasers and Imaging Directorate, Kirtland Air Force Base, New Mexico. The image is a computer simulation of a field experiment showing a satellite as taken from a ground based telescope, and therefore represents an example of atmospheric blurring. The true and blurred, noisy images have 256×256 pixels, and are shown in Figure 1. Figure 3(a) displays the restored image. It has a relative error of 0.2697, and its computation requires 45 iterations.

In the second example, we use the Grain test image which is contained in the RestoreTools package [9]. The true and blurred, noisy images have 256×256 pixels, and are shown in Figure 2. Figure 3(b) displays the restored image. It has a relative error of 0.0683, and its computation requires 27 iterations.

Both the numerical experiments have been carried out on a Pentium IV PC using Matlab 7.1. In both the experiments, the initial iterate x_0 has been chosen as the blurry and noisy image. The outer iteration stopping tolerance $\varepsilon = 1.0e - 3$. That is the iteration is terminated when $\dfrac{\|x_k - x_{k-1}\|_2}{\|x_{k-1}\|_2} < 1.0e - 3$. The inner iteration is set as $l_{\max} = 20$ and δ is set as $\sqrt{eps}$, that is $\delta = \sqrt{2.2204e - 16}$.

Table 1 shows the numerical results of the proposed algorithm applied to the test problems. The second column of the table gives the relative error $\dfrac{\|x_{computed} - x_{real}\|_2}{\|x_{real}\|_2}$ between the computed solution and the real one. The third column reports the number of the outer iterates.

Table 1. Numerical results for the test problems

Test problem	Relative error	Inerations
Satellite	0.2697	45
Grain	0.0683	27
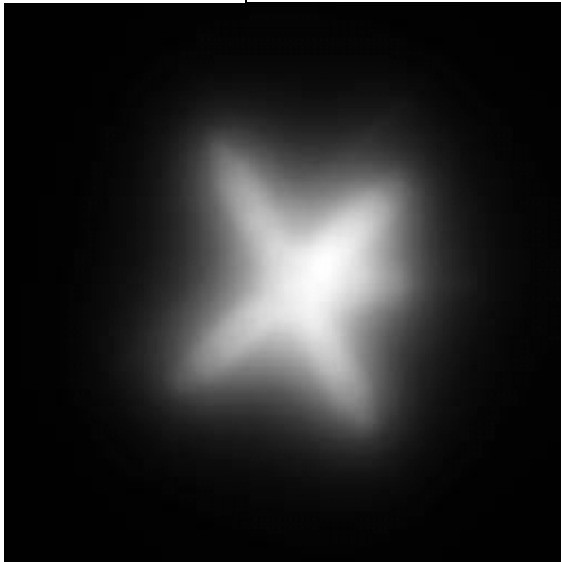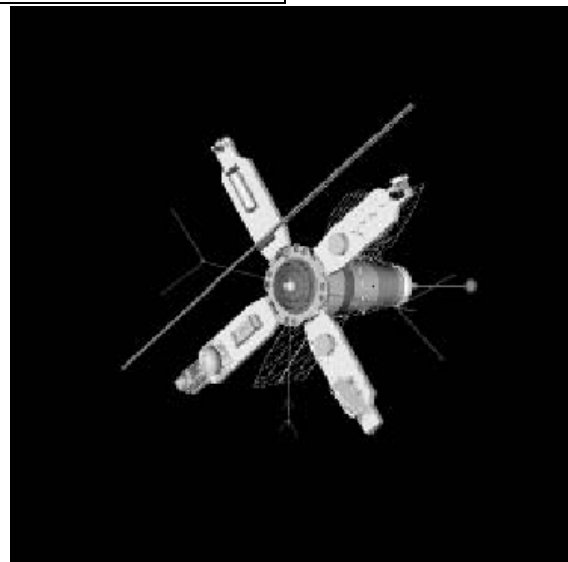
a.Original image   b. Blurred image

Figure 1: Original and blurred satellite image

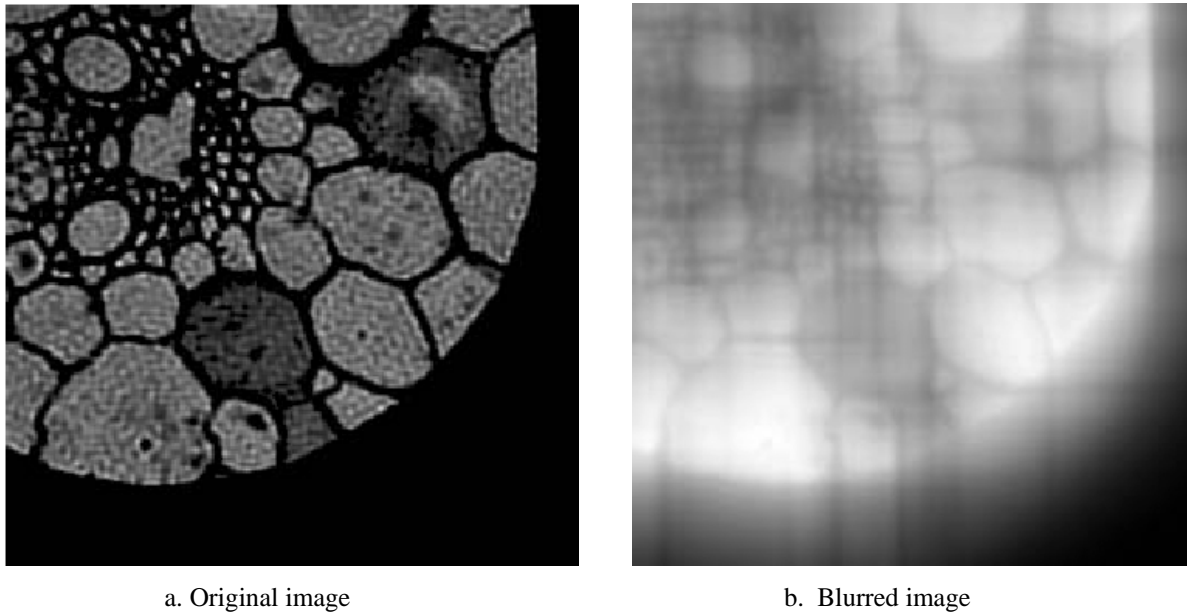a. Original image

b. Blurred image

Figure 2: Original and blurred Grain image
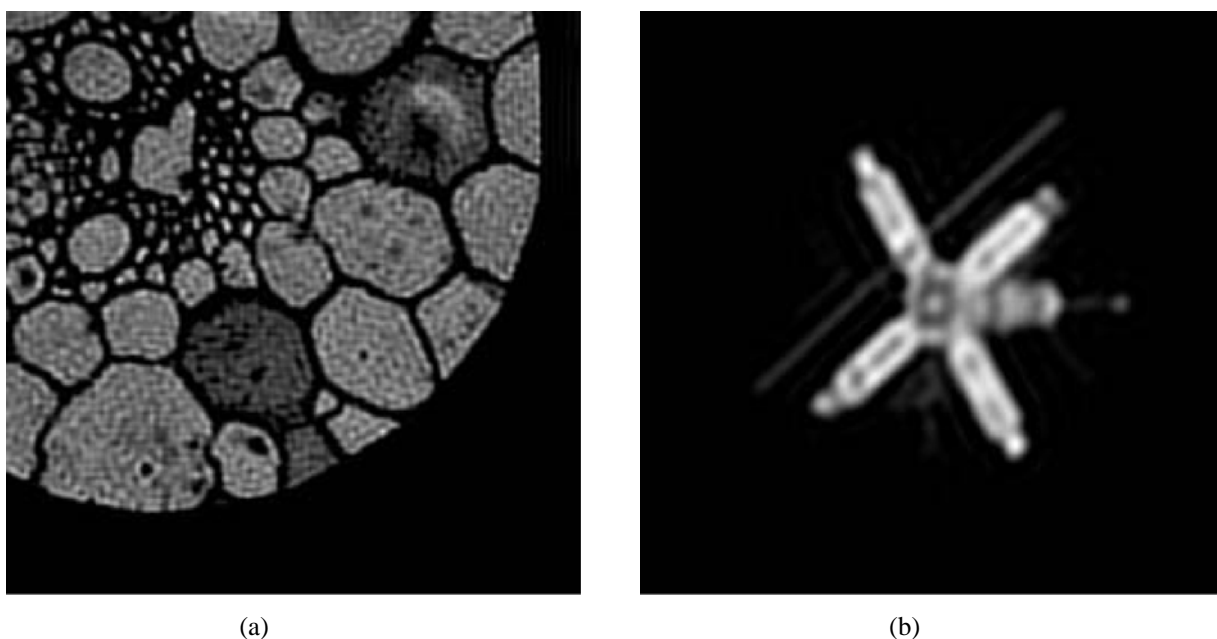

(a)

(b)

Figure 3: Restored Satellite image (a) and restored Grain image (b)

5. Conclusion

We propose an iterative conjugate gradient regularization method to the image restoration problem. Experiments have been done for two image restoration problems, the results illustrate that the proposed method has good performance.

6. References

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