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A Note on the SIMPLE Preconditioner for Nonsymmetric Saddle Point Problems

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Abstract. The eigenvalue analysis of the SIMPLE preconditioner for solving two-by-two block linear equations with the (1, 2)-block being the transpose of the (2, 1)-block and the (2, 2)-block being zero was investigated in Li and Vuik [Numer Lin. Alg. Appl., 2004, 11:511-523]. In this note, we extend their ideas by allowing the (1, 2)-block to be not equal to the transpose of the (2, 1)-block and investigate the relationship of the two different formulations spectrum of the SIMPLE preconditioned matrix by using the theory of matrix eigenvalue. And also the SIMPLE type methods are given.

Keywords: SIMPLE preconditioner, eigenvalue analysis, generalized eigenvalue problem, diagonal scaling.

1. Introduction

In many cases, discretization and linearization of the PDEs often lead to the following large sparse linear algebraic system:

$$\mathcal{A}x = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = b, \tag{1.1}$$

where $A \in R^{n\times n}$ is nonsingular, and B^T, $C \in R^{m\times n}$ with $m \le n$ are of full rank, i.e., $\det(A) \neq 0$, $\text{rank}(B)=\text{rank}(C)=m$. Such systems are referred to as general saddle point problems. Saddle point systems of the form (1.1) appear in many applications and have attracted a lot of researches [1-6, 11-12]; especially, one can see [1] for a comprehensive survey.

As is known, there exist two kinds of methods to solve the linear systems: direct methods and iterative methods. Direct methods are widely employed when the size of the coefficient matrix is not too large, and are usually regarded as robust methods. The memory and the computational requirements for solving the large linear systems may seriously challenge the most efficient direct solution method available today. Naturally, it is necessary that we make the use of iterative methods instead of direct methods to solve the large sparse linear systems. Meanwhile, iterative methods are easier to implement efficiently on high performance computers than direct methods. Currently, Krylov subspace methods [13] are considered as one kind of the important and efficient iterative techniques available for solving large linear systems because the methods are cheap to be implemented and are able to exploit the sparsity of the coefficient matrix. However, in fact Krylov subspace methods are not competitive without a good preconditioner. To speed up the convergence, it is profitable to use a good preconditioner. A lot of preconditioners are presented for solving systems (1.1), such as block-diagonal preconditioners (with exact Schur complement and approximate Schur complement) [1-2, 5-6, 11-12] and constraint preconditioners [3, 4]. For a broad overview of the numerical solution of saddle point systems, one can see [1]. It is known that the better the clustered spectrum of the preconditioner Krylov subspace iteration is, the faster the method converges. It means that the eigenvalues of the preconditioned matrix play an important role in the Krylov subspace method.

In [7-10, 15], the SIMPLE-type methods were investigated for the saddle point systems with $C = B^T$

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$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{1.2}$$

The eigenvalues of the precontidioned matrices were investigated in [17-21]. Especially, in [8], the eigenvalue analysis is given for the SIMPLE preconditioned matrix for system (1.2), two different formulations spectrum of the preconditioned matrix are derived. The relationship between the two different formulations has been investigated by using the theory of matrix singular value decomposition.

In this note, we extend the SIMPLE preconditioner for solving the non-symmetric saddle point system (1.1), and give the eigenvalue analysis of the SIMPLE preconditioned matrix. The relationship between the two different formulations spectrum has been investigated by using the theory of matrix eigenvalue. From the analysis of the following proposition, we know that our results needs a weaker condition for the matrix D. At the same time, for the non-symmetric saddle point system (1.1), we also consider the diagonal scaling and the SIMPLE type methods are given.

2. SIMPLE preconditioner and SIMPLE(R) iterative methods

2.1. Spectralanalysis of the SIMPLE preconditioned matrices

In this section, we will analyse the eigenvalue of the SIMPLE preconditioned matrix. Before the main results are given, we give some notions. Let $\sigma(A)$ denote the set of all eigenvalues of matrix A, and assume that the diagonal entries of A are not equal to zero. We define a matrix P as follows

$$P = M_R B_R^{-1}, \tag{2.1}$$

where

$$B_R = \begin{pmatrix} I & -D^{-1}B \\ 0 & I \end{pmatrix},\ M_R = \begin{pmatrix} A & 0 \\ C & R \end{pmatrix},\ D = \mathrm{diag}(A),\ R = -CD^{-1}B.$$

As the reference [8], we also call the preconditioner P^{-1} as SIMPLE preconditioner. For SIMPLE preconditioner, we have the following result:

Proposition 1. *If the right preconditioner P^{-1} is defined by (2.1), then the preconditioned matrix is*

$$T = \mathcal{A} P^{-1} = \begin{pmatrix} I-(I-AD^{-1})BR^{-1}CA^{-1} & (I-AD^{-1})BR^{-1} \\ 0 & I \end{pmatrix}. \tag{2.2}$$

Therefore, the spectrum of the SIMPLE preconditioned matrix T is

$$\sigma(T) = \{1\} \cup \sigma(I-(I-AD^{-1})BR^{-1}CA^{-1}). \tag{2.3}$$

Proof. By simple calculation, it can be concluded that

$$M_R^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -R^{-1}CA^{-1} & R^{-1} \end{pmatrix},$$

and

$$\begin{aligned} T &= \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}\begin{pmatrix} I & -D^{-1}B \\ 0 & I \end{pmatrix}\begin{pmatrix} A^{-1} & 0 \\ -R^{-1}CA^{-1} & R^{-1} \end{pmatrix} \\ &= \begin{pmatrix} I-(I-AD^{-1})BR^{-1}CA^{-1} & (I-AD^{-1})BR^{-1} \\ 0 & I \end{pmatrix} \end{aligned}$$

Hence, the form of the spectrum of T is described by (2.3). □

In fact, by multiplying with matrices A^{-1} and A from the left- and right-hand side of the matrix $I-(I-AD^{-1})BR^{-1}CA^{-1}$, respectively, we get

$$\begin{aligned} \sigma(I-(I-AD^{-1})BR^{-1}CA^{-1}) &= \sigma(I-(A^{-1}-D^{-1})BR^{-1}C) \\ &= \sigma(I-D^{-1}(D-A)A^{-1}BR^{-1}C) \\ &= \sigma(I-JA^{-1}BR^{-1}C) \end{aligned}$$

where the matrix $J = D^{-1}(D - A)$ is the *Jacobi* iteration matrix of the matrix A. By the above analysis, we have the following proposition:

Proposition 2. *For the SIMPLE preconditioned matrix* T *,*

(1) 1 is an eigenvalue with multiplicity at least of m;

(2) the remaining eigenvalues are $1-\mu_i, i=1,2,...,n$ *,where* μ_i *is the i-th eigenvalue of*

$$ZEx = \mu x, \tag{2.4}$$

where

$$E = BR^{-1}C \in R^{n\times n}, Z = JA^{-1} \in R^{n\times n}.$$

If J is non-singular, (2.4) is identical to the generalized eigenvalue problem

$$Ex = \mu Z^{-1}x.$$

Next, we give another eigenvalue formulation of the preconditioned matrix T, and also give the corresponding eigenvectors. Consider the following eigenvalue problem

$$Tx = \mathcal{A} P^{-1}x = \lambda x.$$

Note that the above equation is equivalent to the following generalized eigenvalue problem

$$\mathcal{A} x = \lambda Px,$$

where $\mathcal{A} = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ and $P = M_R B_R^{-1} = \begin{pmatrix} A & AD^{-1}B \\ C & 0 \end{pmatrix}$. It can also be written as

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix} = \lambda \begin{pmatrix} A & AD^{-1}B \\ C & 0 \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix}, \tag{2.5}$$

i.e.,

$$Au + Bp = \lambda(Au + AD^{-1}Bp), \tag{2.6}$$

$$Cu = \lambda Cu. \tag{2.7}$$

We solve the above equations for two different cases.

(1) If $\lambda = 1$, (2.7) is an identical equation for any u. Equation (2.6) shows that

$$A(D^{-1} - A^{-1})Bp = 0.$$

If the matrix $D^{-1} - A^{-1}$ is non-singular, it follows from $\text{rank}(B)=m$, we know that $p = 0$. Therefore, the eigenvectors corresponding to eigenvalue 1 are

$$x_i = \begin{pmatrix} u_i \\ 0 \end{pmatrix} \in R^{n+m}, u_i \in R^n, i = 1,2,...,n,$$

where $\{u_i\}_i^n$ is a basis of R^n.

(2) If $\lambda \neq 1$, equations (2.6) and (2.7) show that

$$u = -\frac{1}{\lambda - 1} A^{-1}(\lambda AD^{-1}Bp - Bp) \text{ and } Cu = 0.$$

We insert u into $Cu = 0$, and get the following equation

$$Sp = \lambda Rp,$$

where $S = -CA^{-1}B \in R^{m\times m}$ is the Schur complement of the matrix $\mathcal{A}$.

By the above analysis, the following proposition is deduced.

Proposition 3. *For the SIMPLE preconditioned matrix* T *,*

(1) 1 is an eigenvalue with (algebraic and geometric) multiplicity of n;

(2) the remaining eigenvalues are defined by the generalized eigenvalue problem

$$Sp = \lambda Rp, \tag{2.8}$$

In [8], the relationship of the two different formulations spectrum are investigated by using the theory of matrix singular value decomposition for the symmetric saddle point systems (1.2), but for the nonsymmetric case, the relationship of the two different spectrum can't be resolved. Moreover, in [8], the diagonal entries of $D = \mathrm{diag}(A)$ must be positive. But, in this note, the diagonal entries of D are only not equal to zero.

Now, we investigate the relation between both spectral formulations for the nonsymmetric case. By Propositions 2 and 3, we know that the two different generalized eigenvalue problems (2.4) and (2.8) have been derived to describe the spectrum of T. In fact the two generalized eigenvalue problems are closely related. In the remaind of this section, we will analyse the relation of the two different formulations spectrum. First, we give a lemma for later use.

Lemma 1. *[14, Theorem 1.3.20] Suppose that $M \in R^{m\times n}$ and $N \in R^{n\times m}$ with $m \le n$. Then NM has the same eigenvalues as MN, counting multiplicity, together with an additional $n-m$ eigenvalues equal to 0.*

By (2.4), note that $ZE = Z^{n\times n}B^{n\times m}(R^{-1})^{m\times m}C^{m\times n} \in R^{n\times n}, (R^{-1})^{m\times m}C^{m\times n}Z^{n\times n}B^{n\times m} \in R^{m\times m}$. Using Lemma 1, we have

$$\sigma(ZE) = \{0\} \bigcup \sigma(R^{-1}CZB),$$

where the eigenvalue 0 with multiplicity of $n-m$, $R^{-1}CZB = R^{-1}CD^{-1}(D-A)A^{-1}B = I - R^{-1}S$. By (2.8)

$$\sigma(R^{-1}CZB) = \sigma(I - R^{-1}S) = \bigcup\{1-\lambda_i\}.$$

These relations motivate the following proposition.

Proposition 4. *For the two generalized eigenvalue problem (2.4) and (2.8), suppose that $\mu_i \in \sigma(ZE)$, $i = 1,2,...,n$, and $\lambda_i \in \sigma(R^{-1}S), i = 1,2,...,m$, the relationship between the two problems is that $\mu_i = 0$ is an eigenvalue of (2.4) with multiplicity of $n-m$, which can be denoted as $\mu_{m+1} = \mu_{m+2} = ... = \mu_n = 0$, and that $\lambda_i = 1-\mu_i$, $i = 1, 2, ..., m$ holds for the remaining m eigenvalues.*

Remark. 1. Proposition 5 in [8], the diagonal entries of $D = \mathrm{diag}(A)$ are required to be positive. However, in this note, the diagonal entries of D are only not equal to zero. The assumption is weaker than the result of [8]. Additional, the analysis of Proposition 4 is different from the analysis of [8] (using different tools), here, it is very simple using the theory of matrix eigenvalue. In fact, the above Propositions 1-4 can be regarded as the extension of the Propositions 2-5 of [8].

2. If $D = \mathrm{diag}(A)$, and A is strongly diagonally dominant, the SIMPLE preconditioner will be effective, also see [16]. In fact, D is not necessary the diagonal entries of A, in this case, the diagonal entries of A can equal to zero. If we choose D such that the eigenvalue of the generalized eigenvalue problem of (2.8) are close to unit, a Krylov subspace method such as GMRES method [13] will converge quickly.

2.2. The diagonal scaling for non-symmetric saddle point matrices

In references [8-10], a diagonal scaling strategy is proposed for a practical implementation of the SIMPLE preconditioning. In this note, for non-symmetric saddle point systems (1.1), scaling the coefficient matrix $\mathcal{A}$ is also considered by (left) multiplying with the diagonal matrix

$$\hat{D} = \begin{pmatrix} D^{-1} & 0 \\ 0 & D_R^{-1} \end{pmatrix}$$

where

$$D = \mathrm{diag}(A), \quad D_R = \mathrm{diag}(R) \quad \text{and} \quad R = -CD^{-1}B.$$

After this scaling, the coefficient matrix becomes

$$\bar{A} = \hat{D}\,\mathcal{A} = \begin{pmatrix} D^{-1}A & D^{-1}B \\ D_R^{-1}C & 0 \end{pmatrix}.$$

Hence,

$$\bar{D} = \mathrm{diag}(D^{-1}A) = I \in R^{n\times n}, \quad \bar{R} = -(D_R^{-1}C)\bar{D}^{-1}(D^{-1}B) \in R^{m\times m}$$

and

$$\overline{B}=\begin{pmatrix} I & -D^{-1}B \\ 0 & I \end{pmatrix},\quad \overline{M}=\begin{pmatrix} D^{-1}A & 0 \\ D_R^{-1}C & \overline{R} \end{pmatrix},\quad \overline{M}^{-1}=\begin{pmatrix} A^{-1}D & 0 \\ -\overline{R}^{-1}D_R^{-1}CA^{-1}D & \overline{R}^{-1} \end{pmatrix}.$$

Define the SIMPLE preconditioner as $\overline{P}^{-1}=\overline{B}\overline{M}^{-1}$.

Now, the SIMPLE preconditioned matrix is

$$\begin{aligned}\overline{T}=\overline{A}\overline{B}\overline{M}^{-1}&=\begin{pmatrix} D^{-1}A & D^{-1}B \\ D_R^{-1}C & 0 \end{pmatrix}\begin{pmatrix} I & -D^{-1}B \\ 0 & I \end{pmatrix}\begin{pmatrix} A^{-1}D & 0 \\ -\overline{R}^{-1}D_R^{-1}CA^{-1}D & \overline{R}^{-1} \end{pmatrix}\\ &=\begin{pmatrix} D^{-1}A & (I-D^{-1}A)D^{-1}B \\ D_R^{-1}C & -D_R^{-1}CD^{-1}B \end{pmatrix}\begin{pmatrix} A^{-1}D & 0 \\ -R^{-1}CA^{-1}D & R^{-1}D_R \end{pmatrix}\\ &=\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}.\end{aligned}$$

By doing simple calculation, it can be concluded that

$$\begin{aligned}T_{11}&=I-(I-D^{-1}A)D^{-1}BR^{-1}CA^{-1}D=I-D^{-1}(I-AD^{-1})BR^{-1}CA^{-1}D\\ T_{12}&=(I-D^{-1}A)D^{-1}BR^{-1}D_R=D^{-1}(I-AD^{-1})BR^{-1}D_R\\ T_{21}&=D_R^{-1}CA^{-1}D+D_R^{-1}CD^{-1}BR^{-1}CA^{-1}D=0\\ T_{22}&=-D_R^{-1}CD^{-1}BR^{-1}D_R=I.\end{aligned}$$

Hence, it follows that

$$\overline{T}=\begin{pmatrix} I-D^{-1}(I-AD^{-1})BR^{-1}CA^{-1}D & D^{-1}(I-AD^{-1})BR^{-1}D_R \\ 0 & I \end{pmatrix}. \tag{2.9}$$

From equations (2.2) and (2.9), we find the spectra of matrices T and $\overline{T}$ are same.

2.3. SIMPLE type iterative methods

The SIMPLE-type methods as iterative methods for solving symmetric saddle point (1.2) were discussed in references [9, 10, 14]. In fact, the SIMPLE method can also solve nonsymmetric saddle point (1.1). It can be deemed to solve the system $\mathcal{A}B_R y=b,\ x=B_R y$ with definition (2.1). The following iteration (SIMPLE method) is gained by using the splitting $\mathcal{A}B_R=M_R-N_R$.

$$x_{k+1}=x_k+B_R M_R^{-1}(b-\mathcal{A}x_k).$$

SIMPLE method

1. Choose an initial estimate p^*.
2. Solve $Au^*=b_1-Bp^*$.
3. Solve $R\delta_p=b_2-Cu^*$.
4. Compute $u=u^*-D^{-1}B\delta_p$ and $p=p^*+\delta_p$.
5. If not converged, take $p^*=p$ and go to 2.

We can also obtain iteration

$$x_{k+1}=x_k+M_L^{-1}B_L(b-\mathcal{A}x_k)$$

by using the splitting $B_L\mathcal{A}=M_L-N_L$ with

$$B_L=\begin{pmatrix} I & 0 \\ -CD^{-1} & I \end{pmatrix},\quad M_L=\begin{pmatrix} A & B \\ 0 & R \end{pmatrix}.$$

When u_k is known, p_{k+1} and u_{k+1} are calculated as follows:

SIMPLER method

1. Solve $Rp^* = b_2 - CD^{-1}((D-A)u_k + b_1)$.
2. Solve $Au^* = b_1 - Bp^*$.
3. Solve $R\delta_p = b_2 - Cu^*$.
4. Compute $u_{k+1} = u^* - D^{-1}B\delta_p$ and $p_{k+1} = p^* + \delta_p$.

Note that the SIMPLER method can also be described as a classical iterative method. If B_R, M_R, B_L and M_L are chosen as above, then the SIMPLER method can be given by

$$x_{k+1} = x_k + P_E(b - \mathcal{A}\, x_k)$$

where $P_E = B_R M_R^{-1} B_L^{-1} E B_R^{-1} M_L^{-1} B_L$, E is the block diagonal part of the matrix $M_L + M_R - \mathcal{A}$.

It is well known that the SIMPLE(R) method often needs much iteration before an accurate solution is obtained. To reduce the large computation times of the SIMPLE(R) method, a Krylov subspace acceleration of the SIMPLE(R) method is necessary.

3. Conclusion

In this paper, we have discussed the SIMPLE preconditioner for the non-symmetric saddle point problems. We have shown the relationship of the two different formulations spectrum of the SIMPLE preconditioned matrices by using the theory of matrix eigenvalue. At the same time, the diagonal scaling of the non-symmetric matrix has been considered, and the SIMPLE(R) iterative methods are given.

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