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Sign Idempotent Matrices and Generalized Inverses

Jingyue Zhang [1,+], T. Z. Huang [1], Zhongshan Li [2] and Xingwen Zhu [1]

[1]School of Mathematical Sciences, University of Electronic Science and Technology, 610054

[2]Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30302-4110, USA

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Abstract. A matrix whose entries consist of $\{+,-,0\}$ is called a sign pattern matrix. Let $Q(A)$ denote the set of all real $n\times n$ matrices B such that the signs of entries in B match the corresponding entries in A. For nonnegative sign patterns, sign idempotent patterns have been characterized. In this paper, we Firstly give an equi-valent proposition to characterize general sign idempotent matrices (sign idempotent). Next, we study properties of a class of matrices which can be generalized permutationally similar to specialized sign patterns. Finally, we consider the relationships among the allowance of idempotent, generalized inverses and the allowance of tripotent in symmetric sign idempotent patterns.

Keywords: sign pattern matrix; symmetric sign pattern; sign idempotent; allowance of idempotent; generalized inverses

1. Introduction

A matrix whose entries consist of $\{+,-,0\}$ is called a sign pattern matrix. A subpattern of a sign pattern ma-trix A is a sign pattern matrix obtained by replacing some of the $+$ or $-$ entries in A with 0. The sign pattern I_n is the diagonal pattern of order n with + diagonal entries. In order to study conveniently, we also ues I to denote diagonal patterns with + diagonal entries. A sign pattern is said to be sign nonsingular, if all the real matrices $B\in Q(A)$ are nonsingular. A is said to be sign singular if every matrix $B\in Q(A)$ is a singular matrix.

A permutation pattern is a square sign pattern with entries 0 and +, where the entry + occurs precisely once in each row and column. A permutational similarity of the square pattern A is a product of the form $P^T AP$, where P is a permutation pattern. A signature pattern is a diagonal sign pattern matrix each of whose diagonal entries is + or -. A generalized permutation pattern is either a permutation pattern or a signature pattern obtained by replacing some or all of the + entries in a permutation pattern with – entries. A is a symmetric sign pattern matrix if the entries of A satisfy $a_{i,j}=+(-or0)$ if and only if $a_{i,j}=+(-or0)$ for any i and j.

A matrix A is called constantly signed if it is of the form $A=\partial J$, where $\partial\in(+,-,0)$ and J is a sign pattern matrix whose entries are all positive. If A is said to be row (column) constantly signed if entres in rows (columns) have the same sign. For a sign pattern matrix A, we can define A^2 as a sign pattern matrix if no two nonzero terms in the sum $\sum_k a_{i,k}a_{k,j}$ are oppositely signed for all i and j; otherwise A^2 is not a sign pattern. if $A^2=A$, then A is called sign idempotent. A sign pattern matrix A is said to be allowed idempotent (the allow-ance of idempotent) if there exists $B\in Q(A)$, where $B^2=B$. Since an irreducible sign idempotent matrix has been characterized, for a reducible sign pattern matrix, we often assume a sign pattern matrix A is in Frobenius normal form, i.e.

[+] Corresponding author. *E-mail address*: jyzhang2046@163.com

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1k} \\ & \ddots & \vdots \\ & & A_{kk} \end{pmatrix} \tag{1}$$

where each A_{ii} is square and irreducible or A_{ii} is a 0-entry, denoted by (0).

In [1], Eschenbach defined the modified Frobenius normal form, which is a sign pattern matrix A whose form is as follow:

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1k} \\ & \ddots & A_{kk} \end{pmatrix} \tag{2}$$

where each A_{ii} is either positive or entry-wise zero.

For a sign pattern matrix A, the minimum rank of A denoted by $mr(A)$ is defined as

$$mr(A) = \min\{rank(B) | B \in Q(A)\}.$$

Now, we will give several notations about generalized inverses to denote the class of sign pattern matrices.

1. A sign pattern matrix A is said to allow idempotent (the allowance of idempotent) if there exists $B \in Q(A)$, where $B^2 = B$. We denote ID as the class of all square sign patterns which have the property.

2. A sign pattern matrix A is said to allow tripotent (the allowance of tripotent) if there exists $B \in Q(A)$, where $B^3 = B$. We denote T as the class of all square sign patterns which have the property.

3. A sign pattern matrix A is said to allow (1)-inverse if there exists $B, C \in Q(A)$, where $BCB = B$. We denote G as the class of all square sign patterns which have the property.

Let F be the class of all square sign pattern matrices A such that A are generalized permutationally similar to a matrix of the form

$$F(I_r, A_2, A_3) = \begin{pmatrix} I_r & A_2 \\ A_3 & A_3A_2 \end{pmatrix}, \tag{3}$$

where A_2A_3 is a subpattern of I_r.

In this paper, we investigate general sign idempotent matrices and generalized inverses. In [3], for non-negative sign patterns, sign idempotent patterns have been characterized. In section 2, we give an equivalent proposition to characterize general sign idempotent patterns. Moreover, we will demonstrate that any symme-tric idempotent sign pattern matrix has the form of $\begin{pmatrix} I_r & A_2 \\ A_2^T & A_2^TA_2 \end{pmatrix}$, where $A_2A_2^T$ is a subpattern of I_r. In [2], for nonnegative sign patterns, which are permutationally similar to $\begin{pmatrix} T & A_2 \\ A_3 & A_3TA_2 \end{pmatrix}$, where A_2A_3 is diagonal, and $T^2 = I$, have have been characterized. In section 3, we extend its properties, several properties of F are characterized. Finally, some equivalent properties are given in symmetric sign idempotent patterns.

2. An equivalent form of sign idempotent matrices

From [3], we know any nonnegative sign idempotent pattern matrix can be permutationally similar to a block matrix. In this section, we generalize Theorem 3.2 of [3] to general sign patterns. Furthermore, we obtain an equivalent proposition for general sign idempotent matrices.

Lemma 1. [2] Each of the classes T, ID and G is closed under the following operations:

1. signature similarity;
2. permutation similarity;
3. transposition.

Lemma 2. [2] If A is an $n\times n$ sign pattern matrix such that $A^2=A$, then $A\in ID$ if and only if $A\in G$.

The following three lemmas from [1] are very useful to study idempotent matrices.

Lemma 3. [1] Suppose A is an $n\times n$ reducible sign pattern in Frobenius normal form (1). If A_{ii} and A_{ij} are positive blocks, then A is sign idempotent only if A_{ij} is constantly signed.

Lemma 4. [1] Suppose A is an $n\times n$ reducible sign pattern in Frobenius normal form (1). If A_{ii} is positive and $A_{jj}=(0)$, then A is sign idempotent only if A_{ij} is column constantly signed.

Lemma 5. [1] Suppose A is an $n\times n$ reducible sign pattern in Frobenius normal form (1). If $A_{ii}=(0)$ and A_{jj} is positive, then A is sign idempotent only if A_{ij} is row constantly signed.

Lemma 6. [11] Let A=[a_{ij}] be a sign idempotent pattern matrix. Then a_{ii}=0 or + for all $i\in(1,2,\cdots,n)$.

Theorem 7. Suppose A is an $n\times n$ sign idempotent pattern matrix, and A_k is a $k\times k$ principal submatrix of A, where $1\le k\le n$. If there is no zero diagonal entry in A_k, then A_k is a sign idempotent pattern matrix.

Proof. On the one hand, since A_k is a principal submatrix of A, thus A may be permutationally similar to a form

$$\begin{pmatrix} A_k & A_2 \\ A_3 & A_4 \end{pmatrix} \tag{4}$$

Since A is an $n\times n$ square sign idempotent pattern matrix, it is clear that (4) is also a sign idempotent matrix by Lemma 1. Thus we obtain

$$A_k^2+A_2A_3=A_k, \tag{5}$$

Therefore, A_k^2 must be a subpattern of A_k.

On the other hand, from Lemma 6, the diagonal entries of A must be + or 0, A_k is the principal submatrix of A with no zero entry. So I_k must be the subpattern of A_k. Hence,

$$A_k+I_k=A_k,\ A_k^2+A_k=A_k^2, \tag{6}$$

So A_k is a subpattern of A_k^2.

From (5) and (6), we can obtain $A_k^2=A_k$. □

For example, let

$$A=\begin{pmatrix} + & - & - & + & + \\ 0 & + & + & 0 & - \\ 0 & + & + & 0 & - \\ + & - & - & + & + \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is a sign idempotent pattern matrix, and

$$A_4=\begin{pmatrix} + & - & - & + \\ 0 & + & + & 0 \\ 0 & + & + & 0 \\ + & - & - & + \end{pmatrix},\ A_3=\begin{pmatrix} + & + & 0 \\ + & + & 0 \\ - & - & + \end{pmatrix}$$

are principal submatrices of A, then both A_4 and A_3 are sign idempotent pattern matrices by Theorem 7.

From Lemma 1.3 of [1], C. A. Eschenbach obtained an irreducible sign idempotent pattern is entrywise nonzero. According to Theorem 7 and Lemma 6, we can get the following proposition:

Corollary 8. Suppose *A* is an $n \times n$ irreducible sign pattern matrix, and A_k is a $k \times k$ principal submatrix of *A*, where $1 \le k \le n$. Then *A* is sign idempotent if and only if A_k is a sign idempotent pattern matrix.

For example, let

$$A = \begin{pmatrix} + & - & - & + & - & + & + \\ - & + & + & - & + & - & - \\ - & + & + & - & + & - & - \\ + & - & - & + & - & + & + \\ - & + & + & - & + & - & - \\ + & - & - & + & - & + & + \\ + & - & - & + & - & + & + \end{pmatrix}$$

is sign idempotent, thus any principal submatrices of *A* is sign idempotent from Corollary 8.

Next, we will give one proposition to characterize the general sign idempotent pattern by extending Theorem 3.2 in [3], which characterizes the nonnegative sign pattern.

Theorem 9. Let A be a square sign pattern matrix with $mr(A) = r$. Then A is sign idempotent if and only if A is generalized permutationally similar to a sign pattern of the form

$$\begin{pmatrix} A_1 & A_1A_2 \\ A_3A_1 & A_3A_1A_2 \end{pmatrix}, \tag{7}$$

where A_1 is an $r \times r$ sign nonsingular and sign idempotent matrix, and A_2A_3 is a subpattern of A_1.

Proof. ⇒: Assume that *A* is a square sign pattern matrix in the Frobenius normal form (1). Since *A* is sign idempotent, from [1], we know any irreducible sign idempotent matrix can be signature similarly to a positive matrix. We can assume *A* is generalized permutationally similar to the form of (2). Further, from Lemma 3-5 we know the blocks in the strictly upper triangular part of *A* are constantly signed or row (column) cons-tantly signed, the blocks in the strictly upper triangular part of the Frobenius normal form (2) are uniformly signed +, - or 0. Hence, any two rows or columns of each fixed irreducible positive block are identical. We can choose a row (column) of the irreducible positive block to represent the irreducible positive block.

Let *t* be the number of nonzero irreducible components, and i_j be the first row of the *j*th nonzero irreducible compotent, $1 \le j \le t$. Let A_1 be the pricinpal submatrix of *A* with row (column) index set $(i_1 \ldots i_t)$. Obviously A_1 is an upper triangular with + diagonal entries, so A_1 is a sign nonsingular. From $A^2 = A$ and Theorem 7, we obtain A_1 is also a sign idempotent matrix.

Finallay, from $A^2 = A$, we know that any row or column with 0 diagonal entry can be written as a combination of laster (earlier) rows (columns). So, such a row(column) depends only on rows (columns) with + diagonal entries. Therefore, we can assume that *A* can be generalied permutationally similar to a pattern of the form

$$\begin{pmatrix} A_1 & A_1A_2 \\ A_3A_1 & A_3A_1A_2 \end{pmatrix}, \tag{8}$$

where A_1 is a $k \times k$ sign nonsingular matrix. It is obvious that $mr(A) = r$, that is $r = k$. Because $A^2 = A$, we can get that

$$A_1^2 + A_1A_2A_3 = A_1 \tag{9}$$

so $A_1A_2A_3A_1$ is a subpattern of A_1. Since I_r is a pattern of A_1, we have that A_2A_3 is a subpattern of A_1.

⇐: The proof of the sufficiency is clear. □

From Theorem 9, any sign idempotent pattern matrix A can be generalized permutationally similar to the form of (7), which can be rewritten as

$$\begin{pmatrix} A_1 & A_1A_2 \\ A_3A_1 & A_3A_1A_2 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_3 \end{pmatrix}\begin{pmatrix} A_1 & A_1A_2 \end{pmatrix}, \tag{10}$$

which is in fact the minimum rank factorization of A.

Next, we consider symmetric sign pattern matrices. As we know, if A is a symmetric sign pattern matrix, then the blocks in the strictly upper triangular part of the Frobenius normal form of A are zero.

Corollary 10. Let A be a symmetric sign pattern matrix, with $mr(A) = r$. Then A is sign idempotent if and only if A is generalized permutationally similar to a pattern of the form

$$\begin{pmatrix} I_r & A_2 \\ A_2^T & A_2^TA_2 \end{pmatrix}, \tag{11}$$

where $A_2A_2^T$ is a subpattern of I_r.

Proof. The proof of this corollary is parallel of that of Theorem 9. □

3. Generalized inverses and minimum rank factorization

In [2], Eschenbach, Hall and Li have studied the properties of $\begin{pmatrix} T & A_2 \\ A_3 & A_3TA_2 \end{pmatrix}$, where A_2A_3 is diagonal, and $T^2 = I$. They only discuss the properties of nonnegative sign pattern matrices. From Theorem 4.4 of [3], we also know a nonnegative sign pattern matrix A which can be permutationally equivalent to $\begin{pmatrix} I_r & A_2 \\ A_3 & A_3A_2 \end{pmatrix}$ has many good properties, such as generalized inverses, minimum rank factorization. In this section, we mainly discuss the properties of F, several similar properties are characterized to F. Furthermore, we generalize several properties for symmetric idempotent matrices.

Motivated by Theorem 2.6 of [2], we give the following theorem about F.

Theorem 11. Let A be an $n \times n$ sign pattern matrix, with $mr(A) = r$. If $A \in F$, then $A \in TP$.

Proof. Since $A \in F$, we can choose $B_1 \in Q(I_r)$, $B_2 \in Q(A_2)$, $B_3 \in Q(A_3)$, so

$$B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_3B_2 \end{pmatrix} \in Q(A), \tag{12}$$

By direct multiplication,

$$B^3 = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}, \tag{13}$$

Where

$$Q_1 = B_1^3 + B_2B_3B_1 + B_1B_2B_3 + B_2B_3B_1^{-1}B_2B_3,$$
$$Q_2 = B_1^2B_2 + B_2B_3B_2 + B_1B_2B_3B_1^{-1}B_2 + \left(B_2B_3B_1^{-1}\right)^2B_2,$$
$$Q_3 = B_3B_1^2 + B_3B_1^{-1}B_2B_3B_1 + B_3B_2B_3 + \left(B_3B_1^{-1}B_2\right)^2B_3,$$
$$Q_4 = B_3B_1B_2 + B_3B_1^{-1}B_2B_3B_2 + B_3B_2B_3B_1^{-1}B_2 + \left(B_3B_1^{-1}B_2\right)^3.$$

Hence, $A \in TP$ provided that $B_1 = Q_1$, $B_2 = Q_2$, $B_3 = Q_3$, $B_3B_2 = Q_4$,

Firstly, B_1 must be equal to Q_1, that is

$$B_1 = B_1^3 + B_2B_3B_1 + B_1B_2B_3 + B_2B_3B_1^{-1}B_2B_3, \tag{14}$$

Now, we show that we can choose proper entries so that (14) holds. Because A_2A_3 is a subpattern of I_r, so $B_2B_3 = diag(b_1...b_r)$, let x be the (i,i) entry of B_1. In order for (14) hold, we need

$$x - x^3 = 2b_i x + b_i^2 x^{-1}. \quad (15)$$

It is easy to choose x and b_i to get the above equation. We can make the following choices get (14):

$$x_1 = 1, b_1 = 0, x_2 = \frac{1}{2}, b_2 = \frac{1}{4}, x_3 = \frac{1}{3}, b_3 = \frac{6}{27}.$$

Analogously, we can obtain values of x_i and b_i, where $\forall i \in (4,5\cdots,r)$.

So

$$\begin{aligned} Q_2 &= B_1^2B_2 + B_2B_3B_2 + B_1B_2B_3B_1^{-1}B_2 + \left(B_2B_3B_1^{-1}\right)^2 B_2 \\ &= \left(B_2B_3B_1 + B_1B_2B_3 + B_2B_3B_1^{-1}B_2B_3\right)B_1^{-1}B_2 + B_1B_2 \\ &= B_1B_2 + \left(B_1 - B_1^3\right)B_1^{-1}B_2 \\ &= B_2. \end{aligned}$$

Similarly, $Q_3 = B_3$, $Q_4 = B_4$. Hence, we can choose proper entries such that $A \in TP$. □

Corrollary 12. Let A be an $n \times n$ sign pattern matrix, with $mr(A) = r$. If $A \in F$, then the followings are equivalent:

1. $A \in G$;
2. A allows a $(1,2)$-inverse;
3. $A \in ID$.

Proof. It is easy to see the equivalence, so we omit the proof. □

We denote F^* as the symmetric form of F. In section 2, we know any symmetric sign idempotent pattern matrix has the form of F^*. So from Theorem 11 and Corollary 12, we may obtain the following properties for symmetric sign idempotent pattern matrices.

Theorem 13. Let A be a symmetric idempotent sign pattern matrix, with $mr(A) = r$. Then the followings are equivalent:

1. $A \in G$;
2. A allows (1,2) inverse;
3. $A \in ID$;
4. $A \in TP$.

Proof. The equivalence in the above theorem is obvious so that we omit the proof. □

Motivated by Theorem 4.4 of [3], we obtain a minimum factorization for more general sign pattern matrices.

Theorem 14. Let A be a symmetric sign pattern matrix, $mr(A) = r$. Then the followings are equivalent:

1. A is a sign idempotent matrix;
2. $A \in F^*$;
3. $A = HH^T$, where H is an $n \times r$ sign pattern matrix and H contains some row-permutation of I_r as a submatrix, $mr(H) = r$, and some columns of H^T are orthogonal.

Proof. $1 \Rightarrow 2$: It is easy from Corollary 10.

$2 \Rightarrow 3$: Suppose 2 holds, henceforth, there exists permutation matrix P,

$$P^T AP = \begin{pmatrix} I_r & A_2 \\ A_2^T & A_2^T A_2 \end{pmatrix}, \quad (17)$$

where $A_2A_2^T$ is a subpattern of I_r. So

$$A = P\begin{pmatrix} I_r \\ A_2^T \end{pmatrix}\begin{pmatrix} I_r & A_2 \end{pmatrix} P^T , \tag{18}$$

Let $H = P\begin{pmatrix} I_r \\ A_2^T \end{pmatrix}$. A_2 has orthogonal rows. That is to say H^T has some orthogonal columns. It is clear that 3 holds.

3 $\Rightarrow$ 1: Since H contains some row-permutation of I_r as a submatrix, let $H = P\begin{pmatrix} I_r \\ A_2^T \end{pmatrix}$ for some permutation pattern P and some sign pattern A_2 such that $A_2 A_2^T$ is a subpattern of I_r. Then we can obtain 1. □

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