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(2,1)-Total Labelling of Cactus Graphs

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Abstract. A (2,1)-total labelling of a graph $G=(V,E)$ is an assignment of integers to each vertex and edge such that: (i) any two adjacent vertices of G receive distinct integers, (ii) any two adjacent edges of G receive distinct integers, and (iii) a vertex and its incident edge receive integers that differ by at least 2. The *span* of a (2,1)-total labelling is the maximum difference between two labels. The minimum span of a (2,1)-total labelling of G is called the (2,1)-total number and denoted by $\lambda_2^t(G)$.

A cactus graph is a connected graph in which every block is either an edge or a cycle. In this paper, we label the vertices and edges of a cactus graph by (2,1)-total labelling and have shown that, $\Delta+1 \le \lambda_2^t(G) \le \Delta+2$ for a cactus graph, where Δ is the degree of the graph G.

Keywords: Graph labelling; (2,1)-total labelling; cactus graph

1. Introduction

Motivated by frequency channel assignment problem Griggs and Yeh [5] introduced the $L(2,1)$-labelling of graphs. The notation was subsequently generalized to the $L(p,q)$-labelling problem of graphs. Let p and q be two non-negative integers. An $L(p,q)$-labelling of a graph G is a function c from its vertex set $V(G)$ to the set $\{0,1,\ldots,k\}$ such that $|c(x)-c(y)|\ge p$ if x and y are adjacent and $|c(x)-c(y)|\ge q$ if x and y are at distance 2. The $L(p,q)$-labelling number $\lambda_{p,q}(G)$ of G is the smallest k such that G has an $L(p,q)$-labelling c with $\max\{c(v)\,|\,v\in V(G)\}=k$.

The $L(p,q)$-labelling of graphs has been studied rather extensively in recent years [2, 8, 12, 16, 17, 18].

Whittlesey at el. [19] investigated the $L(2,1)$-labelling of incidence graphs. The incidence graph of a graph G is the graph obtained from G by replacing each edge by a path of length 2. The $L(2,1)$-labelling of the incident graph G is equivalent to each element of $V(G)\cup E(G)$ such that:

(i) any two adjacent vertices of G receive distinct integers,

(ii) any two adjacent edges of G receive distinct integers, and

(iii) a vertex and an edge incident receive integers that differ by at least 2.

This labelling is called (2,1)-total labelling of graphs which introduced by Havet and Yu [6] and generalized to the $(d,1)$-total labelling, where $d\ge 1$ be an integer. A k-$(d,1)$-total labelling of a graph G is a function c from $V(G)\cup E(G)$ to the set $\{0,1,\ldots,k\}$ such that $c(u)\ne c(v)$ if u and v are adjacent and $|c(u)-c(e)|\ge d$ if a vertex u is incident to an edge e. The $(d,1)$-total number, denoted by $\lambda_d^t(G)$, is the least integer k such that G has a k-$(d,1)$-total labelling. When $d=1$, the (1,1)-total labelling is well known as total colouring of graphs.

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Let $\Delta(G)$ (or simply Δ) denote the maximum degree of a graph G.

Havet and Yu [6] proposed the following conjecture.

Conjecture 1 $\lambda_d^t(G) \leq min\{\Delta + 2d - 1, 2\Delta + d - 1\}$.

2. Some general bounds of $(d,1)$-total labelling

It is shown in [6] that for any graph G,

(i) $\lambda_d^t(G) \leq 2\Delta + d - 1$;

(ii) $\lambda_d^t(G) \leq 2\Delta - 2log(\Delta + 2) + 2log(16d - 8) + d - 1$; and

(iii) $\lambda_d^t(G) \leq 2\Delta - 1$ if $\Delta \geq 5$ is odd.

Again in [6] it was shown that

(i) $\lambda_d^t(G) \geq \Delta + d - 1$;

(ii) $\lambda_d^t(G) \geq \Delta + d$ if G is Δ-regular;

(iii) $\lambda_d^t(G) \geq \Delta + d$ if $d \geq \Delta$; and

(iv) $\lambda_d^t(G) \leq \chi(G) + \chi'(G) + d - 2$, where $\chi(G)$ and $\chi'(G)$ are known as chromatic number and chromatic index of G respectively.

Let Mad(G) is the maximum average degree of G, Mad$(G) = max\{2|E(H)|/|V(G)|, H \subseteq G\}$. Montassier and Raspaud [15] proved that if G be a connected graph with maximum degree Δ, $d \geq 2$, then $\lambda_d^t(G) \geq \Delta - 2d - 2$ in the following cases:

(i) $\Delta \geq 2d + 1$ and Mad$(G) < \frac{5}{2}$;

(ii) $\Delta \geq 2d + 2$ and Mad$(G) < 3$;

(iii) $\Delta \geq 2d + 3$ and Mad$(G) < \frac{10}{3}$.

For a complete graph K_n, the result for $(d,1)$-total labelling is given in [6]. If n is odd then $\lambda_2^t(K_n) = min\{n + 2d - 2, 2n + d - 2\}$; if n is even then $\lambda_2^t(K_n) = min\{n + 2d - 2, 2n + d - 2\}$, $n \leq d + 5$, $\lambda_2^t(K_n) = n + 2d - 1$, $n > 6d^2 - 10d + 4$ and $\lambda_2^t(K_n) \in \{n + 2d - 2, 2n + d - 1\}$ otherwise. Then they focused in (2,1)-total labelling and shown that if $\Delta \geq 2$, then $\lambda_2^t(K_n) \leq 2\Delta + 2$ and therefore the $(d,1)$-total labelling conjecture is true when $p = 2$ and $\Delta = 3$. In fact, the bound for this special case is tight as $\lambda_2^t(K_4) = 7$ [6].

In [13], Molloy and Reed proved that the total chromatic number of any graph with maximum degree Δ is at most Δ plus an absolute constant. Moreover, in [14], they gave a similar proof of this result for sparse graphs.

In [7], it was shown that for any tree T, $\Delta + 1 \leq \lambda_2^t(T) \leq \Delta + 2$, where Δ is the maximum degree among all the vertices of the tree.

The $(d,1)$-total labelling for a few special graphs have been studied in literature, e.g., complete graphs [6], complete bipartite graphs [11], planar graphs [1], outer planar graphs [3], products of graphs [4], graphs with a given maximum average degree [15], etc. A more generalization of total colouring of graphs so called $[r,s,t]$-colouring, was defined and investigated in [9].

It is shown in [10] that for any cactus graphs, $\Delta + 1 \leq \lambda_{2,1} \leq \Delta + 3$. Now in this paper, we label the vertices and edges of a cactus graphs G by $(2,1)$-total labelling and it is shown that $\Delta + 1 \leq \lambda_2^t \leq \Delta + 2$.

Lemma 1 [6] *If* H *is a subgraph of* G *, then* $\lambda_2^t(H) \le \lambda_2^t(G)$.

3. The (2,1)-total labelling of induce sub-graphs of cactus graphs

Let $G = (V, E)$ be a given graph and U is a subset of V . The *induced subgraph* by U , denoted by $G[U]$, is the graph given by $G[U] = (U, E')$, where $E' = \{(u,v) : u, v \in U$ and $(u,v) \in E\}$.

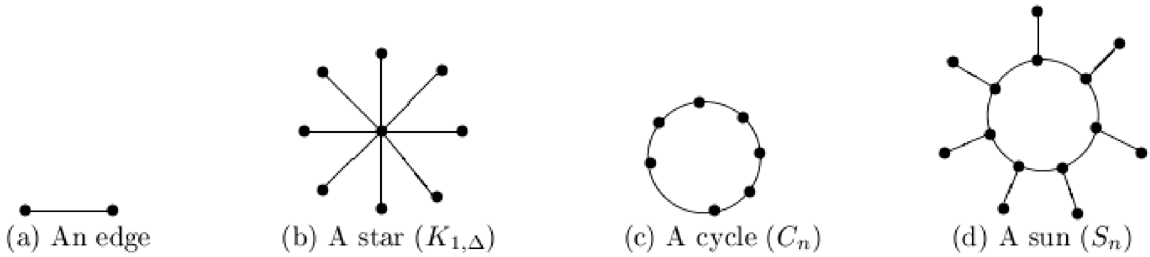


Figure 1: Some induce subgraphs of cactus graph.

The star graph $K_{1,\Delta}$ is a subgraph of $K_{n,m}$. For any star graph $K_{1,\Delta}$ one can verify the following result.

Lemma 2 *For any star graph* $K_{1,\Delta}$, $\lambda_2^t(K_{1,\Delta}) = \Delta + 2$.

3.1. (2,1)-total labelling of cycles

3.1.1 (2,1)-total labelling of one cycle

Lemma 3 *For any cycle* C_n *of length* n , $\lambda_2^t(C_n) = 4 = \Delta + 2$.

Proof. Let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of the cycle C_n . We classify C_n into two groups, viz., C_{2k} , C_{2k+1} . Then the (2,1)-total labelling of vertices and edges of the cycle are as follows.

Case 1. Let $n = 2k$ (see Figure 2(a)).

$c(v_{2i}) = 0$, $c(v_{2i+1}) = 1$, $c(v_{2i}, v_{2i+1}) = 3$, for $i = 0,1,2,\ldots,k-1$; $c(v_{2i+1}, v_{2i+2}) = 4$, for $i = 0,1,2,\ldots,k-2$ and $c(v_{2k-1}, v_0) = 4$.

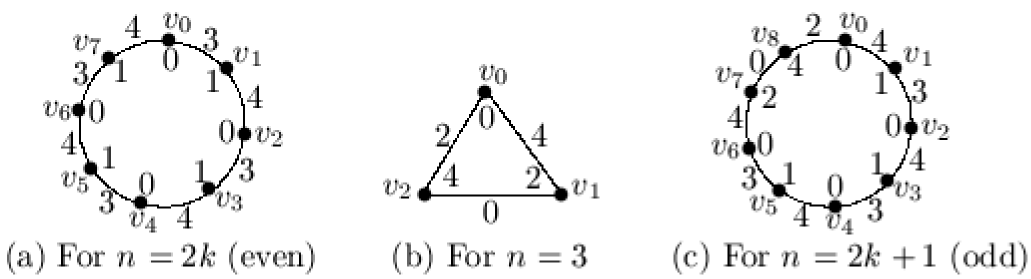


Figure 2: Illustration of Lemma 3

Case 2. Let $n = 3$ (see Figure 2(b)).

$c(v_0) = 0$, $c(v_1) = 2$, $c(v_2) = 4$, $c(v_0, v_1) = 4$, $c(v_1, v_2) = 0$ and $c(v_2, v_0) = 2$.

Case 3. Let $n = 2k + 1$ (see Figure 2(c)).

We label the vertices as $c(v_{2i}) = 0$, for $i = 0,1,2,\ldots,k-1$; $c(v_{2i+1}) = 1$, for $i = 0,1,2,\ldots,k-2$; $c(v_{2k-1}) = 2$ and $c(v_{2k}) = 4$. And we label the edges as

$c(v_{2i-1}, v_{2i}) = 3$, $c(v_{2i}, v_{2i+1}) = 4$, for $i = 0,1,2,\ldots,k-1$, $c(v_{2k-1}, v_{2k}) = 0$ and $c(v_{2k}, v_0) = 2$.

From all above cases, we conclude that, $\lambda_2^t(C_n) = 4 = \Delta + 2$. □

3.1.2 (2,1)-total labelling of two cycles

Lemma 4 If a graph $G(=C_n \cup C_m)$ contains two cycles having a common cutvertex with degree 4, then,

$$\lambda_2^t(G)=\begin{cases}6, & \text{when length of each cycle is even;}\\ 5, & \text{otherwise.}\end{cases}$$

Proof. Let G contains two cycles C_n and C_m of lengths n and m respectively. Again let v_0 be the cutvertex and $v_0, v_1, \ldots, v_{n-1}$ and $v_0, v_1', \ldots, v_{m-1}'$ be the vertices of C_n and C_m respectively. Now we label the vertices and edges of the graph as follows.

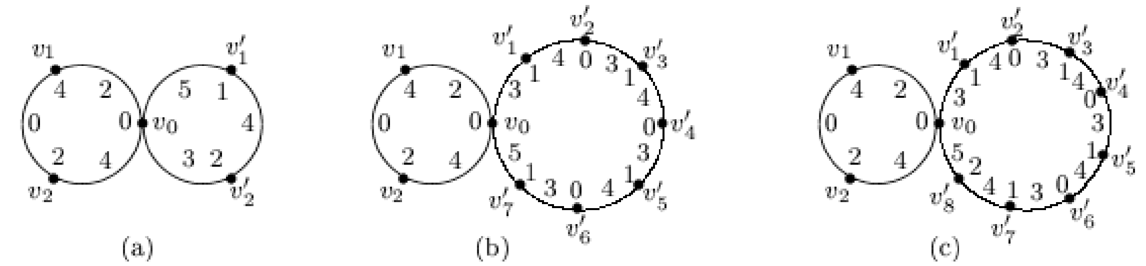


Figure 3: Illustration of Lemma 4

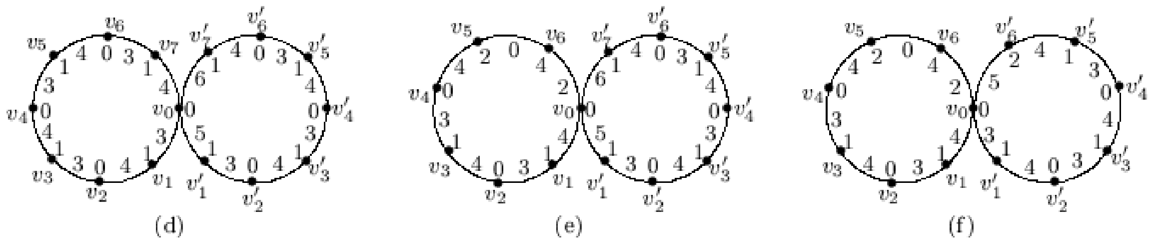


Figure 3: (continuation)

Case 1. For $n=3$, $m=3$ (shown in Figure 3(a)).

At first we label the cutvertex v_0 by 0. Then we label the vertices and edges of first C_3 (i.e., C_n) as same as given in case 2 of previous lemma. And then we label other vertices and edges as $c(v_1')=1$, $c(v_2')=2$, $c(v_0,v_1')=3$, $c(v_1',v_2')=4$ and $c(v_2',v_0)=5$.

Case 2. For $n=3$, $m=2k+i$, $i=0,1$.

We label the edges and vertices of C_3 as same as in the above case. Then we label the second cycle as follows.

When m is even, i.e., $m=2k$ (shown in Figure 3(b)), then

$c(v_{2i}')=0$, $c(v_{2i+1}')=1$, $c(v_{2i}',v_{2i+1}')=3$, for $i=0,1,2,\ldots,k-1$; $c(v_{2i+1}',v_{2i+2}')=4$, for $i=0,1,2,\ldots,k-2$; $c(v_{2k-1}',v_0)=3$ and $c(v_0,v_1')=5$.

When m is odd, i.e., $m=2k+1$ (shown in Figure 3(c)), then

we label the vertices $v_i', i=1,2,\ldots,2k-1$ and the edges $(v_i',v_{i+1}'), i=1,2,\ldots,2k-2$, (v_0,v_1'), (v_0,v_{2k}') as same as in the above except the label of the vertex v_{2k}' and the edge (v_{2k-1}',v_{2k}'). We label that vertex and that edge as $c(v_{2k}')=2$ and $c(v_{2k-1}',v_{2k}')=4$.

Case 3. For $n=2k+i$, $m=2k+i$, $i=0,1$.

When $n=2k$ (even), $m=2k$ (even) (shown in Figure 3(d)), then we label the vertices and edges of C_n as same as in case 1 of Lemma 3. Now we label all the vertices of the cycle C_n as the labelling of the vertices of the cycle C_n. Now we label the edges of C_m as follows.

$c(v_0, v'_1) = 5$, $c(v'_{2k-1}, v_0) = 6$ and $c(v'_{2i}, v'_{2i+1}) = 3$, for $i = 0,1,2,\ldots,k-1$,
$c(v'_{2i+1}, v'_{2i+2}) = 4$, for $i = 0,1,2,\ldots,k-2$.

When $n = 2k+1$ (odd), $m = 2k$ (even) (shown in Figure 3(e)), then we label the vertices and edges of C_n as same as in case 3 of previous lemma. Then we label another cycle as same as in the above subcase except the label of the edges (v_0, v'_1) and (v'_{2k-1}, v_0) and we label that edges as

$$c(v_0, v'_1) = 3 \text{ and } c(v'_{2k-1}, v_0) = 5.$$

When $n = 2k+1$ (odd), $m = 2k+1$ (odd) (Figure 3(f)), then the labelling procedure of the C_n as same as given in case 3 of Lemma 3. And then we label the cycle C_m as same as given in case 2 (for $n = 3$, $m = 2k+1$).

Here the degree of the cutvertex v_0 is 4. Then from all the above cases, it follows that

$$\lambda_2^t(G) = \begin{cases} 6, & \text{both cycles are of even length;} \\ 5, & \text{otherwise.} \end{cases}$$ □

3.1.3 (2,1)-total labelling of three cycles

Lemma 5 *Let G be a graph contains three cycles and they have a common cutvertex v_0 with degree $\Delta = 6$, then*

$$\lambda_2^t(G) = \begin{cases} \Delta + 2, & \textit{when three cycles are of even lengths;} \\ \Delta + 1, & \textit{otherwise.} \end{cases}$$

Proof. Let C_n, C_m and C_l be three cycles and $v_0, v_1, \ldots, v_{n-1}$; $v_0, v'_1, \ldots, v'_{m-1}$; $v_0, v''_1, \ldots, v''_{l-1}$ be the vertices of them. They joined with a common cutvertex v_0 with degree $\Delta(=6)$. The labelling procedure of two cycles are given in previous lemma. Now according to the previous lemma we have to label the vertices and edges of the remaining cycle C_l. When we label C_l, there are three cases arise, viz., $l = 3$, $l = 2k$ (even) and $l = 2k+1$ (odd). Here the label of the cutvertex is 0. Then we label the third cycle as follows.

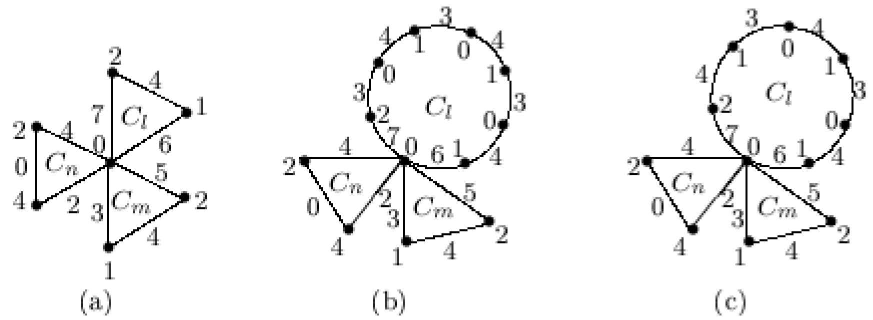


Figure 4: Illustration of some cases of Lemma 5

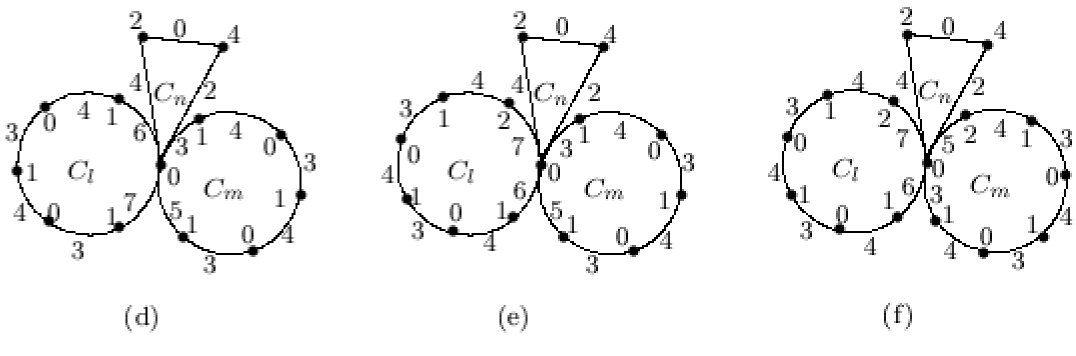


Figure 3: (continuation)

Case 1. When $l=3$, then we relabel (v_0,v_1''), v_1'', (v_1'',v_2''), v_2'' and (v_2'',v_0) by 6, 1, 4, 2 and 7 respectively.

Case 2. When $l=2k$ (even), then we label the vertices of C_{2k} as

$c(v_{2i}'')=0$, for $i=1,2,\ldots,k-1$;

$c(v_{2i+1}'')=1$, for $i=0,1,\ldots,k-2$;

and $c(v_{2k-1}'')=2$.

And the edges as

$c(v_{2i}'',v_{2i+1}'')=3$, for $i=1,2,\ldots,k-1$;

$c(v_{2i+1}'',v_{2i+2}'')=4$, for $i=0,1,\ldots,k-2$;

$c(v_0'',v_1'')=6$ and $c(v_{2k-1}'',v_0)=7$.

If the cycle C_l attach with two cycles of even lengths then the label of two edges incident on v_0 of C_l are different. And the labels are

$c(v_0,v_1'')=7$ and $c(v_{2k-1}'',v_0)=8$ respectively.

Case 3. When $l=2k+1$ (odd), then the labels of the vertices and edges of C_l are same as the labelling of the cycle C_m given in case 2 (for $n=3$ and $m=2k+1$) of lemma 4 except the labels of two edges (v_0,v_1'') and (v_{2k-1}'',v_0). And we relabel these two edges as

$c(v_0'',v_1'')=6$ and $c(v_{2k-1}'',v_0)=7$ respectively.

Here we see that the values of λ_2^t are 7 and 8.

Therefore we conclude that,

$$\lambda_2^t(G)=\begin{cases}\Delta+2, & \text{when three cycles are of even lengths;}\\ \Delta+1, & \text{otherwise.}\end{cases}$$ □

3.1.4 (2,1)-total labelling of finite number of cycles

We can extend the lemmas 4 and Lemma 5 for the finite number of cycles when they are joined at a common cutvertex.

Lemma 6 *If a graph G contains finite number of cycles of finite lengths and if they are joined with a common cutvertex with degree Δ, then,*

$$\lambda_2^t(G)=\begin{cases}\Delta+2, & \textit{when all cycles are of even lengths;}\\ \Delta+1, & \textit{otherwise.}\end{cases}$$

Proof. Let us consider a graph G contains n number of cycles of length 3 (triangles). The n triangles joined with a common cutvertex say v_0 with degree $\Delta=2n$, then we have to prove that $\lambda_2^t(G)=\Delta+1$. Let $T_0,T_1,\ldots,T_{n-1}$ be the n number of triangles and v_0 be the cutvertex (see Figure 5). Then G is equivalent to $\bigcup_{v_0} T_i$. Again let v_{ij}, $i=1,2$ and $j=0,1,\ldots,n-1$, be the vertices of G. We label the vertices v_{1j}, v_{2j} and (v_{1j},v_{2j}), for $j=1,2,\ldots,n-1$, using the same procedure of labelling of v_1', v_2' and the edge (v_1',v_2') of C_3 in case 1 of Lemma 3. Then we label the remaining two edges as

$$c(v_0,v_{ij})=\begin{cases}2j+2, & \text{if } i=1;\\ 2j+3, & \text{if } i=2, \text{ for } j=0,1,...,n-1.\end{cases}$$

Then the (2,1)-total number of G is $2n+1$ which is exactly equal to $\Delta+1$.

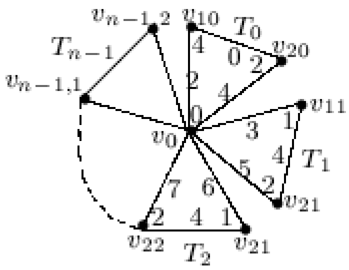


Figure 5: The graph contains n triangles

Now we consider the graph G which contains n number of cycles of length 3 and m number of cycles of length 4. They joined with a cutvertex with degree $\Delta = 2(n+m)$. Then the λ_2^t-value for that graph is $\Delta+1$.

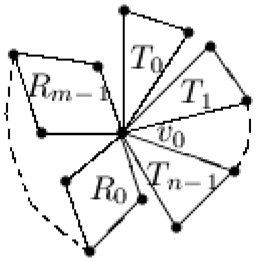


Figure 6: The graph contains n C_3's and m C_4's

Let $T_0, T_1, \ldots, T_{n-1}$ be the n number of cycles of length 3 and $R_0, R_1, \ldots, R_{m-1}$ be the n number of cycles of length 4 (shown in Figure 6). They are joined with a common cutvertex say v_0. Let v_{ij}, $i = 1, 2$ and $j = 0, 1, \ldots, n-1$, be the vertices of all T_i's and v_0, v'_{kp}, $k = 1, 2$ and $p = 0, 1, \ldots, m-1$, be the vertices of all R_p's. Now the labelling of vertices of all R_p's are same as the labelling of vertices of even number of cycles. Then we label the edges as follows:

$c(v'_{1p}, v'_{2p}) = 4$, $c(v'_{2p}, v'_{3p}) = 3$, for $p = 0, 1, \ldots, m-1$ and then we label the edges (v_0, v'_{kp}), for $k = 1, 3$ and $p = 0, 1, \ldots, m-1$ as follows:

$$c(v_0, v'_{kp}) = \begin{cases} 2n + 2(p+1), \text{ if } k = 1; \\ 2n + 2(p+1) + 1, \text{ if } k = 2, \text{ for } p = 0, 1, \ldots, m\text{-}1. \end{cases}$$

We have $c(v_0, v'_{3,m-1}) = 2n + 2m + 1 = \Delta + 1$.

Lastly we prove that if a graph contains n number of cycles of length 4 and all the cycles joined with a cutvertex then the value of λ_2^t is $\Delta + 2$.

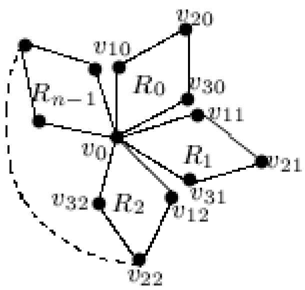


Figure 7: The graph contains n number of cycles of length 4

Let us denote the n number of cycles of length 4 by $R_0, R_1, \ldots, R_{n-1}$ (see Figure 7), joined with a common cutvertex say v_0. Again let v_0, v_{ji}, $j=1,2,3$ and $i=0,1,\ldots,n-1$ be the vertices of R_i's. We label all the vertices of each cycle as same as the label of the vertices of even cycle. And $c(v_{1i},v_{2i})=4$, $c(v_{2i},v_{3i})=3$, for $i=0,1,\ldots,n-1$. Then we label the edges which are incident to the cutvertex v_0 as

$$c(v_0,v_{ji})=\begin{cases}2(i+1)+1, \text{ if } j=1;\\ 2(i+1)+2, \text{ if } j=2, \text{ for } i=0,1,...,n\text{-}1.\end{cases}$$

We have $c(v_0,v'_{3,n-1})=2(n-1+1)+2=2n+2=\Delta+2$.

By using the above results, the general form can be proved by mathematical induction. That is, if a graph G contains finite number of cycles of finite lengths, then

$$\lambda_2^t(G)=\begin{cases}\Delta+2, \text{ when all cycles are of even lengths;}\\ \Delta+1, \text{ otherwise.}\end{cases}$$ □

Lemma 7 *If a graph G contains finite number of cycles of any length and finite number of edges joined with a common cutvertex of degree Δ, then $\lambda_2^t(G)=\Delta+1$.*

Proof. At first we prove that if a graph G contains n number of cycles of length 3, m number of cycles of length 4, p number of edges and they are joined with a common cutvertex with degree Δ ($=2n+2m+p$), then the value of λ_2^t will be $\Delta+1$. Let v_i'', $i=0,1,\ldots,p-1$ be the other end vertices of each edge. We label all v_i'''s as $c(v_i'')=1$, for $i=0,1,\ldots,p-1$. Then according to the previous lemma we label the edges (v_0,v_i''), for $i=0,1,\ldots,p-1$ as

$c(v_0,v_i'')=2n+2m+p-1+2=2(n+m)+p+1=\Delta+1$.

Again let us consider that the graph G contains n number of cycles of length 4 and p number of edges joined with a cutvertex with degree $\Delta=2n+p$. Then we have to prove that $\lambda_2^t(G)=\Delta+1$.

Now we label the vertex v_0' and the edge (v_0,v_0') by 4 and 2 respectively. Then according to the previous lemma we label the edges as $c(v_0,v_j')=2n+2+j$, for $j=0,1,\ldots,p-1$.

Then we have $c(v_0,v'_{p-1})=2n+2+p-1=2n+p+1=\Delta+1$.

By the above results, generally we conclude that if a graph contains finite number of cycles of any length and finite number of edges, then $\lambda_2^t(G)=\Delta+1$. □

Lemma 8 *Let G be a graph, contains a cycle of any length and finite number of edges and they have a common cutvertex v_0. If Δ be the degree of the cutvertex, then $\lambda_2^t(G)=\Delta+2$, if the cycle is of even length and $\Delta+1$, otherwise.*

Proof. We consider that G contains an cycle C_n of length n and p number of edges. Let $v_0,v_1,\ldots,v_{n-1}$ are the vertices of C_n and $v_0',v_1',\ldots,v_{p-1}'$ are the end vertices of all edges, joined with the cutvertex. Let Δ be the degree of G, then $\Delta=2+p$. Then we label the vertices and edges of G as follows.

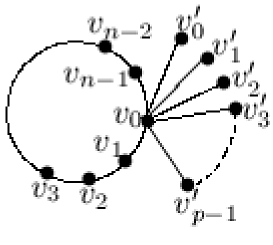


Figure 8: Illustration of Lemma 8

Case 1. Let $n=2k$ (even).

Here $c(v_0) = 0$, then we label all the endvertices of the edges as $c(v'_i) = 1$, for $i = 0, 1, \ldots, p-1$. Now we label the edges (v_0, v'_j) as $c(v_0, v'_j) = 5 + j$ for $j = 0, 1, \ldots, p-1$.

Now $c(v_0, v'_{p-1}) = p + 4 = \Delta + 2$.

Case 2. Let $n = 3$ and $n = 2k+1$ (odd).

Here we label the first edge (v_0, v'_0) by 3. Then the labelling procedure of all endvertices are same as given in the above case. And we label the remaining edges as follows

$c(v_0, v'_k) = 4 + k$, $k = 1, 2, \ldots, p-1$.

Here $c(v_0, v'_{p-1}) = 3 + k = \Delta + 1$.

From the above two cases we see that $\lambda_2^t(G) = \Delta + 2$, if the cycle is of even length and $\Delta + 1$, otherwise.
□

3.2. (2,1)-labelling of sun

Let us consider the sun S_{2n} of $2n$ vertices. This graph is obtained by adding an edge to each vertex of a cycle C_n. So C_n is a subgraph of S_{2n}. The result for any sun S_{2n} is given below.

Lemma 9 For any sun S_{2n}, $\lambda_2^t(S_{2n}) = 5 = \Delta + 2$.

Proof. Let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of C_n and v_i is adjacent to v_{i+1} and v_{i-1}. To complete S_{2n}, we add an edge (v_i, v'_i) to the vertex v_i, i.e., v'_i's are the pendent vertices. To label this graph we consider the following three cases.

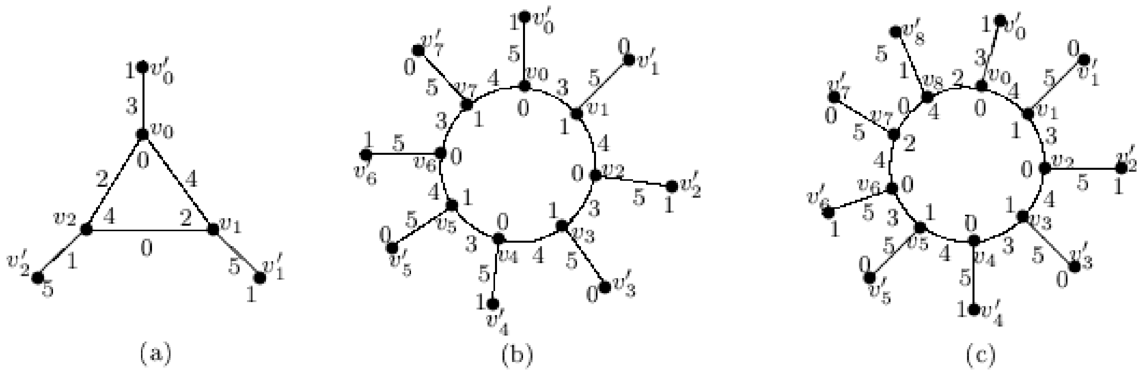


Figure 9: Illustration of Lemma 9

Case 1. Let $n = 3$ (shown in Figure 9(a)).

We label the cycle C_3 according to the Case 2 of Lemma 3. Then we label other vertices and edges as follows:

$c(v'_0) = 1$, $c(v'_1) = 5$, $c(v'_2) = 0$, $c(v_0, v'_0) = 3$, $c(v_1, v'_1) = 1$ and $c(v_2, v'_2) = 5$.

Case 2. Let $n = 2k$ (even) (see Figure 9(b)).

We label the cycle C_n as per Case 1 of Lemma 3. And we label other vertices and edges of S_{2n} as follows:

$c(v'_{2i}) = 1$, $c(v'_{2i+1}) = 0$ for $i = 0, 1, \ldots, k-1$ and $c(v_i, v'_i) = 5$ for $i = 0, 1, \ldots, n-1$.

Case 3. Let $n = 2k+1$ (odd) (see Figure 9(c)).

Here the labelling procedure of the cycle C_{2k+1} is same as the Case 3 of Lemma 3. Now the labelling of other vertices and edges are as follows:

$c(v'_{2i})=1$, $c(v'_{2i+1})=0$ for $i=0,1,\ldots,k-1$, $c(v'_{n-1})=5$, $c(v_i,v'_i)=5$ for $i=1,2,\ldots,n-1$, $c(v_0,v'_0)=3$ and $c(v_{n-1},v'_{n-1})=1$.

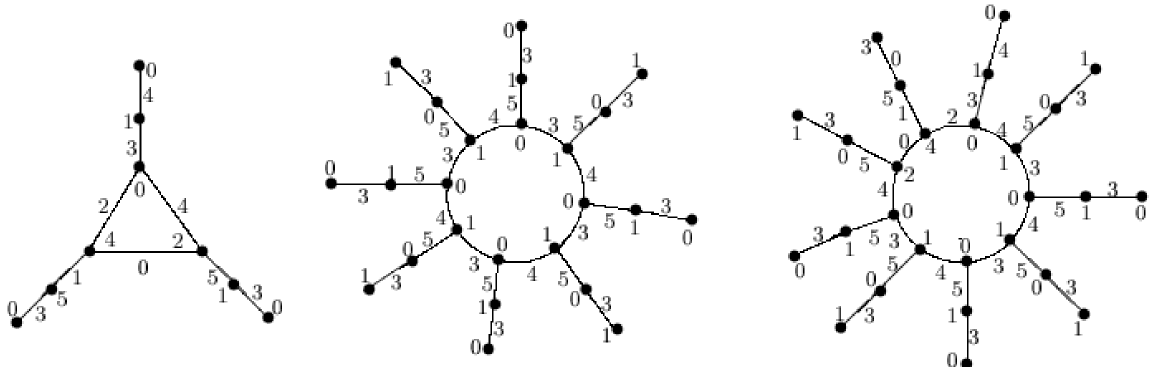
Here we see that (2,1)-total number for that graph is 5.

Hence $\lambda_2^t(S_{2n})=5=\Delta+2$. □

Lemma 10 *Let G be a graph obtained from S_{2n} by adding an edge to each of the pendent vertex of S_{2n}, then*

$$\lambda_2^t(S_{2n})=\Delta+2=5.$$

Proof. Follows from Figure 10. □



(a) For $n=3$ (b) For $n=2k$ (even) (c) For $n=2k+1$ (odd)

Figure 10: Illustration of Lemma 10

Lemma 11 *Let a graph G contains two cycles of any length and they are joined by an edge. If $\Delta(=3)$ be the degree of G , then,*

$$\lambda_2^t(G)=5=\Delta+2.$$

Proof. Let the graph G contains two cycles C_n and C_m with vertices $v_0,\ v_1,\ \ldots,v_{n-1}$ and $v'_0,\ v'_1,\ \ldots,v'_{m-1}$ respectively. And the cycles are joined by an edge (v_0,v'_0) . The degree of the graph is $\Delta(=3)$ Now we label the vertices and edges of the graph as follows.

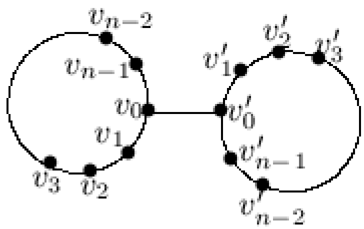


Figure 11: The graph G

Case 1. Let $n=3$, $m=3$.

First we label the vertices and edges of C_3 as same as given in case 2 of Lemma 3. Now we label the edge (v_0,v'_0) by 3 and then we label the other cycles as follows.

$c(v'_0)=1,\ c(v'_1)=0,\ c(v'_2)=2$,

$c(v'_0,v'_1)=4$, $c(v'_1,v'_2)=3$, $c(v'_2,v'_0)=5$.

Case 2. Let $n=3$, $m=2k+i$, $i=0,1$.

We label the vertices and edges of C_3 as same as given in the above case. Then we label the edge

(v_0, v_0') by 5 and other cycles as follows.

When m is even, i.e., $m = 2k$, then

$c(v_{2i}') = 1$, $c(v_{2i+1}') = 0$, for $i = 0, 1, \ldots, k-1$,

$c(v_{2i}', v_{2i+1}') = 3$, for $i = 0, 1, \ldots, k-1$,

$c(v_{2i+1}', v_{2i+2}') = 4$, for $i = 0, 1, \ldots, k-2$

and $c(v_{n-1}', v_0') = 4$.

When m is odd, i.e., $m = 2k+1$, then we label the vertices and edges of C_m as same as given in the above subcase except the label of the vertex v_{m-1}', i.e., v_{2k}' and the edge (v_{m-2}', v_{m-1}'), i.e., (v_{2k-1}', v_{2k}'). We label the vertex and the edge as follows.

$c(v_{2k}') = 2$ and $c(v_{2k-1}', v_{2k}') = 5$.

Case 3. Let $n = 2k + i$, $m = 2k + i$, $i = 0, 1$.

When $n = 2k$ and $m = 2k$, then we label the cycle C_n as same as given in Case 1 of Lemma 3. Then we label the edges (v_0, v_0') by 5 and the cycle C_m as same as in the subcase (when m is even) in Case 2 of this lemma.

When $n = 2k$ and $m = 2k+1$, then we label the edges and vertices of C_m as same as given in the subcase (when m is odd) of the above case.

When $n = 2k+1$ and $m = 2k+1$, then we label the vertices and edges of C_n as same as given in Case

Finally, we get $\lambda_2^t(G) = 5 = \Delta + 2$. □

Corollary 1 *Let a graph G contains two cycles of any lengths and they are joined by two edges. If Δ be the degree of the graph G, then*

$$\lambda_2^t(G) = \Delta + 2.$$

Lemma 12 *Let a graph G contains a cycle of any length and each vertex of the cycle contain another cycle of any length, then*

$$\lambda_2^t(G) = 6 = \Delta + 2.$$

Proof. At first we take the main cycle are of two types, viz., C_{2k}, i.e., even and C_{2k+1}, i.e., odd. Let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of C_n.

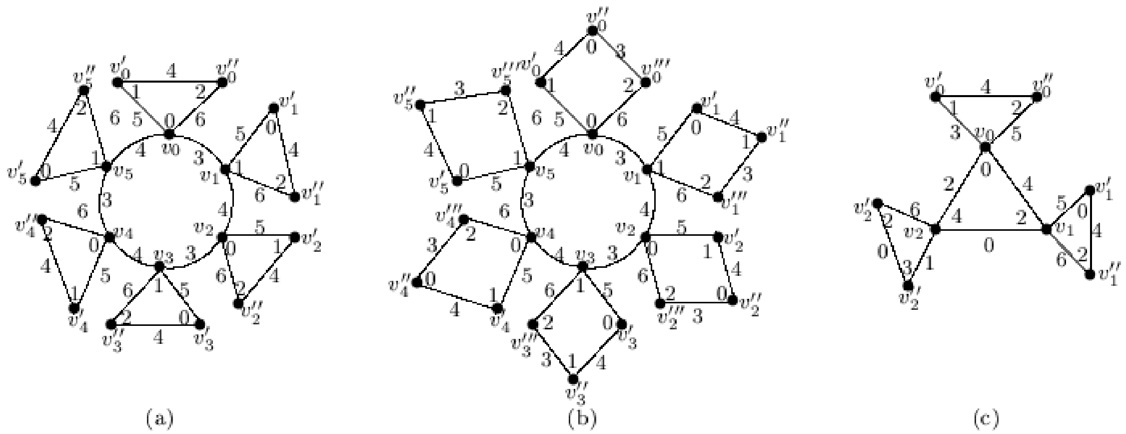


Figure 12: Illustration of Lemma 12

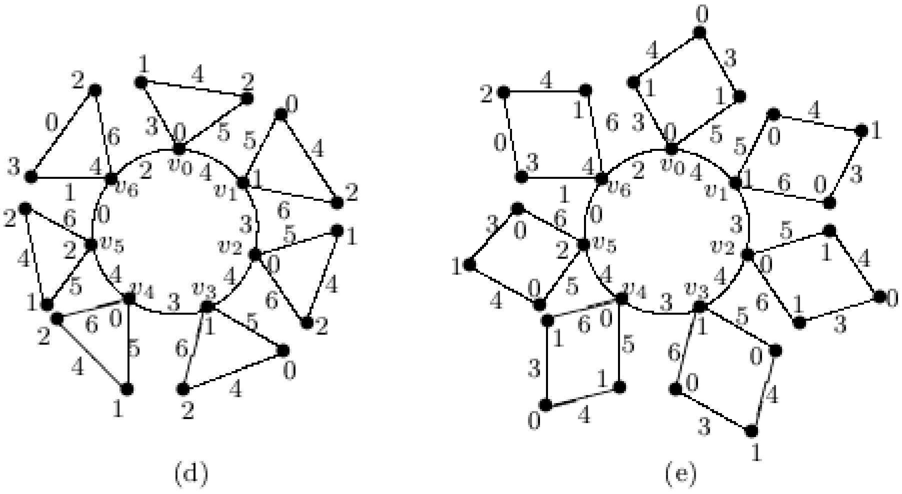


Figure 12: (continuation)

Case 1. Let $n = 2k$ (even).

When each vertex of C_n contains the cycles of length 3 (shown in Figure 12(a)).

Let v_0, v'_0, v''_0 ; v_1, v'_1, v''_1 ;…; $v_{n-1}, v'_{n-1}, v''_{n-1}$ are the vertices of the cycles of length 3. Now the labelling of the cycle C_n is same as the labelling procedure of the cycle of even length. Then we label the other vertices and edges as follows:

$c(v'_{2i}) = 1$, $c(v'_{2i+1}) = 0$ for $i = 0,1,\ldots,k-1$ and $c(v''_{2i}) = 2$ for $i = 0,1,\ldots,n-1$. $c(v_i, v'_i) = 5$, $c(v'_i, v''_i) = 4$, $c(v''_i, v_i) = 6$ for $i = 0,1,\ldots,n-1$.

When each vertex of C_n contains the cycles of length 4 (see Figure 12(b)).

Let v_0, v'_0, v''_0, v'''_0 ; v_1, v'_1, a''_1, v'''_1 ;…; $v_{n-1}, v'_{n-1}, v''_{n-1}, v'''_{n-1}$ be the vertices of all the cycles of length 4. We label the cycles as follows:

$c(v'_{2i}) = 1$, $c(v''_{2i}) = 0$ and $c(v'''_{2i}) = 1$ for $i = 0,1,\ldots,k-1$;

$c(v'_{2i+1}) = 0$, $c(v''_{2i+1}) = 1$ and $c(v'''_{2i+1}) = 0$ for $i = 0,1,\ldots,k-1$;

$c(v_i, v'_i) = 5$, $c(v'_i, v''_i) = 4$, $c(v''_i, v'''_i) = 3$ and $c(v'''_i, v_i) = 6$ for $i = 0,1,\ldots,n-1$.

Case 2. Let $n = 2k+1$ (odd).

When $n = 3$ and all cycles are of length 3 (see Figure 12(c)).

The labelling procedure of the cycle C_n is same as given in case 2 of Lemma 3. Now we label the other vertices and edges as follows:

$c(v'_0) = 1$, $c(v''_0) = 2$, $c(v_0, v'_0) = 3$, $c(v'_0, v''_0) = 4$, $c(v''_0, v_0) = 5$;

$c(v'_1) = 0$, $c(v''_1) = 1$, $c(v_1, v'_1) = 5$, $c(v'_1, v''_1) = 4$, $c(v''_1, v_1) = 6$;

$c(v'_2) = 3$, $c(v''_2) = 2$, $c(v_2, v'_2) = 1$, $c(v'_2, v''_2) = 0$, $c(v''_2, v_2) = 6$.

When each vertex of C_n contains the cycles of length 3 (shown in Figure 12(d)).

The labelling procedure for the vertices v'_i , v''_i and the edges (v_i, v'_i) , (v'_i, v''_i) , (v''_i, v_i) for $i = 1,2,\ldots,2k-2$ are same as the labelling of the graph which contains a cycle of even length and each vertices of the cycle contain cycles of length 3 given in case 1. And the labelling of v'_i, v''_i, (v_i, v'_i), (v'_i, v''_i), (v''_i, v_i) for $i = 0, 2k-2, 2k$ as same as the labelling of the above graph for $i = 0,1,2$ respectively.

When all the cycles are of length 4 except the main cycle (shown in Figure 12(e)).

We label the vertices and edges v_i', v_i'', v_i''', (v_i, v_i'), (v_i', v_i''), (v_i'', v_i''') and (v_i''', v_i) for $i = 1, 2, \ldots, 2k-1$ as same as the labelling procedure of the graph which contains a cycle of even length and each vertex contains another cycle of length 4 except the label of the vertex v_{2k-1}. We label this vertex as $c(v_{2k-1}) = 2$. For $i = 0, 2k$, we label the remaining vertices and edges of the graph as follows:

$c(v_0') = 1$, $c(v_0'') = 0$, $c(v_0''') = 1$, $c(v_0, v_0') = 3$, $c(v_0', v_0'') = 4$, $c(v_0'', v_0''') = 3$, $c(v_0''', v_0) = 5$;

$c(v_{2k}') = 3$, $c(v_{2k}'') = 2$, $c(v_{2k}''') = 1$, $c(v_{2k}, v_{2k}') = 1$, $c(v_{2k}', v_{2k}'') = 0$, $c(v_{2k}'', v_{2k}''') = 4$, $c(v_{2k}''', v_{2k}) = 6$.

Here we see that the minimum label number is 6 which is exactly equal to $\Delta + 2$.

Finally, we conclude that if a graph G contains a cycle of any length and each vertex of the cycle contains another cycle of any length then,

$\lambda_2^t(S_{2n}) = \Delta + 2 = 5$, Δ be the degree of the graph. □

An edge is nothing but P_2, so $\lambda_2^t(G) = 3$.

3.3. (2,1)-labelling of paths

Lemma 13 *For any path P_n of length n,*

$$\lambda_2^t(P_n) = 4 = \Delta + 2.$$

Proof. Let $v_0, v_1, \ldots, v_{n-2}, v_{n-1}$ be the vertices of the path P_n of length n (shown in Figure 13). We classify the path into two cases, viz., even and odd.

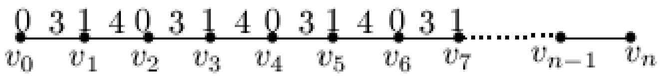


Figure 13: (2,1)-total labelling of path P_n

Case 1. When $n = 2k$, i.e., the path is even.

We label the vertices and edges of P_n according to the following rules.

$c(v_{2i}) = 0$, for $i = 0, 1, \ldots, k-1$;

$c(v_{2i+1}) = 1$, for $i = 0, 1, \ldots, k-1$;

$c(v_{2i}, v_{2i+1}) = 3$, for $i = 0, 1, \ldots, k-1$;

and $c(v_{2i+1}, v_{2i+2}) = 4$, for $i = 0, 1, \ldots, k-1$.

Case 2. When $n = 2k+1$, i.e., the path is odd.

The labelling of the vertices and edges of the path is same as in the above case, only the label of the last vertex v_{2k} and last edge (v_{2k-1}, v_{2k}) are different. We label that vertex and edge as follows:

$c(v_{2k}) = 1$ and $c(v_{2k-1}, v_{2k}) = 3$.

From all above cases we see that $\lambda_2^t(G) = 4 = \Delta + 2$. □

3.4. (2,1)-total labelling of caterpillar graph

Now, we label another important subclass of cactus graphs called caterpillar graph.

Definition 1 *A caterpillar C is a tree where all vertices of degree ≥ 3 lie on a path, called the backbone of C. The hairlength of a caterpillar graph C is the maximum distance of a non-backbone vertex to the backbone.*

Lemma 14 For any caterpillar graph G, $\lambda_2^t(G) = \Delta + 2$, where Δ is the degree of the caterpillar graph.

Proof. Let P_n be the backbone of length n of the caterpillar graph G and $v_0, v_1, \dots, v_{n-2}, v_{n-1}$ be the vertices of P_n. We label the vertices and edges of the path by using the previous lemma. Let v_k be a vertex on the path P_n with degree k. Then $k-2$ different paths (other than backbone) are originated from v_k of variable lengths. We denote such paths by P_j^{ki}, where $i\,(=0,1,\dots,k-2)$ represents the i th path originated from the vertex k and j is the length of the path. Let us take the first path P_m^{k1} and $v_k, v_1^1, v_2^1 \dots, v_{m-1}^1$ be the vertices of it. We label all the vertices of P_m^{k1} by 0 or 1 and label all the edges adjacent to v_k by $5,6,7,\dots,k+2$ because the label of the edges incident on the vertex v_k of the path P_n are either 3 and 4 respectively. We label the first edge of P_m^{k1} by 5 and other edges of P_m^1 by using the labelling procedure given in the previous lemma. All the labels are allowed to label the vertices of the remaining portion of the path P_m^{k1}. Now we take the second path P_l^{k2}. Here also the labelling procedure for the path is same as given in Lemma 13 except the label of the edge incident on the vertex v_k. We label of the edge by 6 and so on. Lastly, we label the first edge of the $(k-2)$ th path incident on the vertex v_k by $k+2$. Here $\Delta = k$, so the value of λ_2^t is $\Delta+2$. Similar method apply to all paths joined with the vertices of the path P_n.

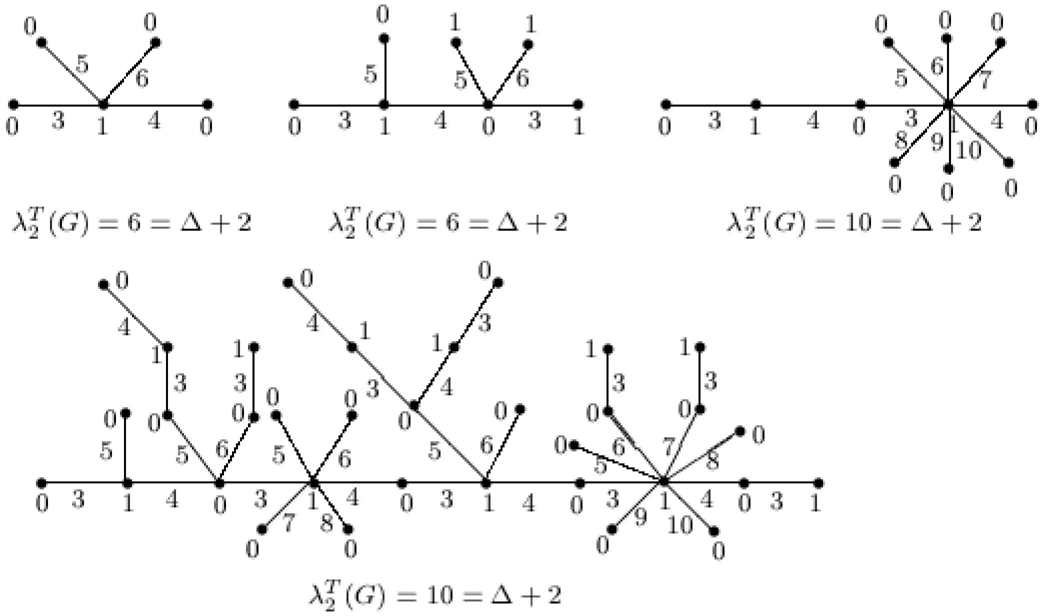


Figure 14: Labelling of caterpillar graphs

Therefore, we conclude that, for any caterpillar graph, $\lambda_2^t(G) = \Delta + 2$.

The proof of lemma 14 is illustrated in Figure 14.

4. (2,1)-total labelling of lobster

Another subclass of cactus graphs is the lobster graph. The definition of lobster graph is given below.

Definition 2 *A lobster is a tree having a path (of maximum length) from which every vertex has distance at most k, where k is an integer.*

The maximum distance of the vertex from the path is called the diameter of the lobster graph. For the above definition k is the diameter. There are many types of lobsters given in literature like diameter 2, diameter 4, diameter 5, etc. Figure 16 shows a lobster of diameter 4.

Lemma 15 *For any lobster G, $\lambda_2^t(G) = \Delta + 2$, where Δ is the degree of the lobster.*

Proof. Assume that P_n be a path of length n of the lobster graph G and $v_0, v_1, \dots, v_{n-1}$ be the vertices of it. Let us consider a vertex v_k on P_n from which p number of trees be originated. Let $T_1, T_2, \dots, T_p$ be such trees. Without lose of generality let the label of the vertex v_k be 0. Again, let Δ_i, $i=1,2,\ldots,p$ be the degrees of these trees. We know that $\lambda_2^t(T_i)$ is Δ_i+2 (if $\Delta_i \ge 4$) [7].

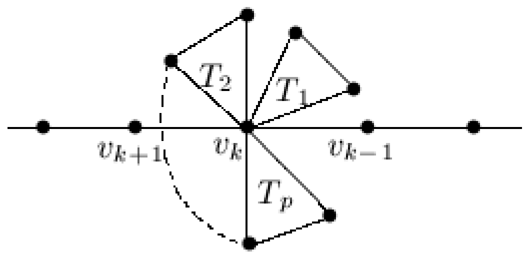


Figure 15: Illustration of Lemma 15

Now we label the edge of the tree T_i ($i=1,2,\ldots,p$) originated from v_k by $i+4$. Let v_{k-1} and v_{k+1} be two adjacent vertices of v_k on P_n. We label these vertices v_{k-1} and v_{k+1} by 1 (or 0) because the label of v_k can be assigned to 0 (resp. 1). And we label the edges (v_k, v_{k-1}) and (v_k, v_{k+1}) by 3 and 4 respectively. So we see that there are no extra labels are required to label the edges incident on v_k of the path P_n. So, the value of λ_2^t of the lobster is $\Delta+2$, where $\Delta = max\{\Delta_1, \Delta_2, \ldots, \Delta_p\}$. □

Figure 16 is an example of 4-diameter lobster and the proof of Lemma 15 is illustrated here.

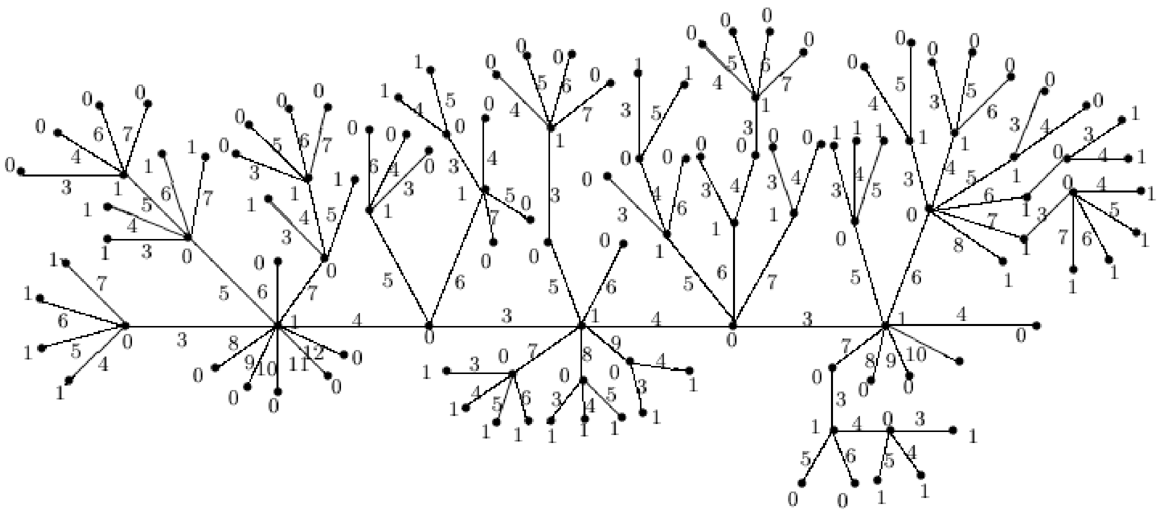


Figure 16: (2,1)-total labelling of 4-diameter lobster

Lemma 16 *Let G_1 and G_2 be two cactus graphs. If $\Delta_1+1 \le \lambda_2^t(G_1) \le \Delta_1+2$ and $\Delta_2+1 \le \lambda_2^t(G_2) \le \Delta_2+2$, then $\Delta+1 \le \lambda_2^t(G) \le \Delta+2$, G is the union of two graphs G_1 and G_2, they have only one common vertex v and max$\{\Delta_1, \Delta_2\}$ $\le \Delta \le \Delta_1+\Delta_2$.*

Proof. Let G_1 and G_2 be two cactus graphs and Δ_1, Δ_2 be the degrees of them. Now if we merge two cactus graphs G_1 and G_2 with the vertex v then we get a new cactus graphs G ($= G_1 \bigcup_v G_2$). Let Δ be the degree of new cactus graph G and it can be shown that $\max\{\Delta_1, \Delta_2\} \le \Delta \le \Delta_1+\Delta_2$. For the graph G_1, $\Delta_1+1 \le \lambda_2^t(G_1) \le \Delta_1+2$ and G_2, $\Delta_2+1 \le \lambda_2^t(G_2) \le \Delta_2+2$. Now we have to prove that the lower and upper bounds of λ_2^t will preserve for the new cactus graph G. Let u and v be two vertices of that graphs and u_0, u_1; v_0, v_1 be the adjacent vertices of u and v respectively. Let x be the label of u, then the label of u_0 and u_1 may be $x+1$ and $x+1$ or $x+4$. And the label of the edges (u,u_0) and (u,u_1) may be $x+3$

and $x+4$ or $x+1$ respectively. Similarly, if y be the label of v, then the label of v_0 and v_1 may be $y+1$ and $y+1$ or $y+4$. And the label of the edges (v,v_0) and (v,v_1) may be $y+3$ and $y+4$ or $y+1$ respectively.

Assume that the label of u be fixed and let it be 0, i.e., $x=0$, and the label y of v lies between 0 to Δ_2+2. That is, the label difference between x and y is one of the integer $0,1,\ldots,\Delta_2+2$.

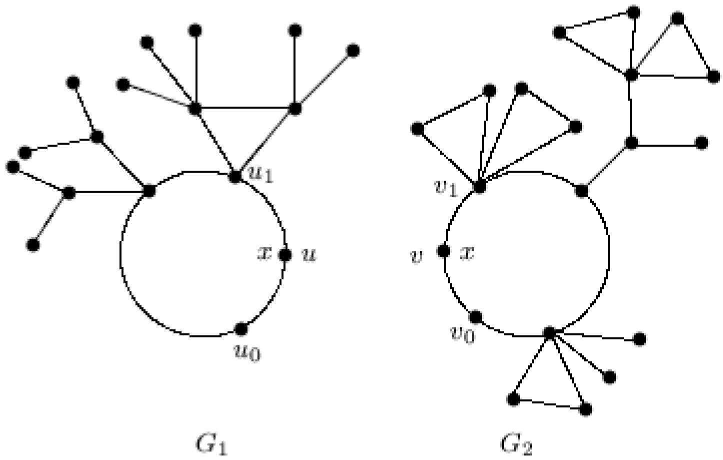


Figure 17:

Let the label of the vertices u and v be same, i.e., $x=y$ (Figure 17). If we join two cactus graphs at v, then the label of v remains unchanged and the labels of adjacent vertices v_0 and v_1 will change to $x+1$ and $x+1$ or $x+2$. And the labels of the edges (v,v_0) and (v,v_1) will change to $x+5$ and $x+6$ or $x+4$ and $x+5$. If we increase the label numbers by 1 of all the vertices and edges of G_2 except v then there are at least one vertex or edge in which we adjust the labelling to preserve the lower and upper bounds of λ_2^t.

When the label difference between x and y is 1, i.e., $y=x+1$ (see Figure 18), then without loss of generality we assume that the label numbers of adjacent vertices of u are $x+1$ and $x+1$ or $x+4$. And the label of the edges (u,u_0) and (u,u_1) are $x+3$ and $x+4$ or $x+1$. Now the label numbers of adjacent vertices of v are x or $x+2$ and x or $x+2$ or $x+3$ respectively. And for the edges (v,v_0) and (v,v_1), $x+3$ or $x+4$ and $x+4$ or x respectively. Now if we increase the label numbers by 1 of all the vertices and edges of G_2 except v then we get at least one vertex or edge in which we adjust the labelling to preserve the lower and upper bounds of λ_2^t, i.e. the λ_2^t-value of new cactus graph can't be less than $\Delta+1$ and greater than $\Delta+2$.

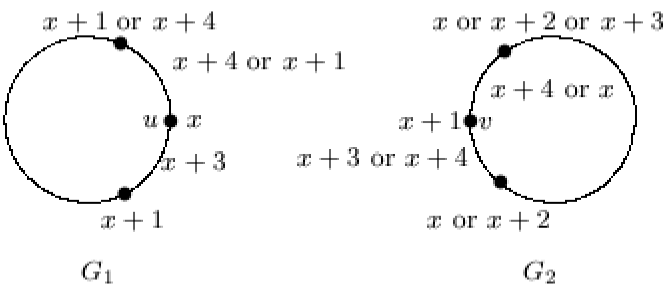


Figure 18:

Similarly, for the label differences $2,3,\ldots,\Delta_2+2$, the lower and upper bounds of λ_2^t for the new cactus graph will preserve. □

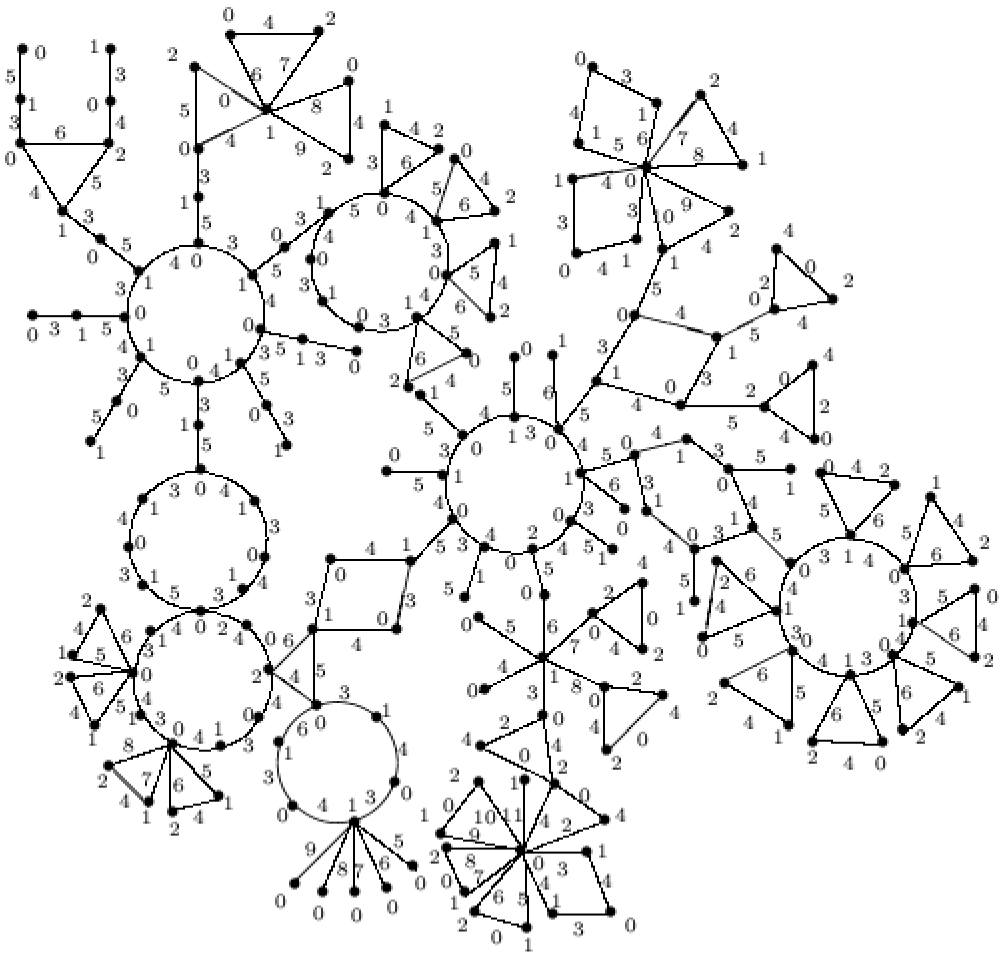


Figure 19: (2,1)-total labelling of cactus graphs

The (2,1)-labelling of all subgraphs of cactus graphs and their combinations are discussed in the previous lemmas. From these results we conclude that the λ_2^t-value of any cactus graph can not be more than $\Delta+2$ and less than $\Delta+1$. Hence we have the following theorem.

Theorem 1 If Δ is the degree of a cactus graph G, then

$$\Delta+1 \le \lambda_2^t(G) \le \Delta+2.$$

The graph of Figure 19 is an example of a cactus graph, contains all possible subgraphs and its (2,1)-total labelling.

5. Conclusion

The bounds of $(2,1)$-total labelling of a cactus graph and various subclass viz., cycle, sun, star, tree, caterpillar and lobster are investigated. The bounds of $\lambda_2^t(G)$ for these graphs are $\lambda_2^t(C_n)=4$ and for sun, star, caterpillar and lobster it is $\Delta+2$. For the cactus graph the bound for λ_2^t is $\Delta+1 \le \lambda_2^t(G) \le \Delta+2$, where Δ is the maximum degree of the cactus graph G.

6. References

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