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Perron Complements of Strictly Generalized Doubly Diagonally Dominant Matrices

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Abstract. As is known, Meyer introduced the concept of the Perron complements of nonnegative irreducible matrices. In addition, the Schur complements of generalized doubly diagonally dominant matrices were introduced by Liu etal. [Linear Algebra Appl., 378(2004): 231-244]. In this paper, properties of the Perron complement of strictly generalized doubly diagonally dominate matrices are presented.

Keywords: generalized doubly diagonally dominant matrix; diagonally dominant matrix; nonnegative irreducible matrix; Perron complement; Schur complement

1. Introduction

Let $A=(a_{ij})$ be a $n\times n$ matrix, and recall that *A* is (row) diagonally dominant if

$$|a_{ii}|\geq\sum_{j\neq i}|a_{ij}|, i=1,2,\cdots,n. \tag{1}$$

A is further said to be strictly diagonally dominant if all the inequalities of (1) are strict. A doubly diagonally dominant matrix (see, e.g., [6]) is a matrix such that for all $i\neq j$

$$|a_{ii}||a_{jj}|\geq\sum_{\substack{t=1\\t\neq i}}^{n}|a_{it}|\sum_{\substack{t=1\\t\neq j}}^{n}|a_{jt}|, \tag{2}$$

and that A is strictly doubly diagonally dominant if all the inequalities of (2) are strict. We call *A* a generalized doubly diagonally dominant matrix (see, e.g., [6]) if there exist proper subset γ_1,γ_2 of $\langle n\rangle:=\{1,2,\cdots,n\}$ such that $\gamma_1\cap\gamma_2=\phi,\ \gamma_1\cup\gamma_2=\langle n\rangle$ and

$$(|a_{ii}|-\alpha_i)(|a_{jj}|-\beta_j)\geq\beta_i\alpha_j. \tag{3}$$

for all $i\in\gamma_1,\ j\in\gamma_2$ where

$$\alpha_i=\sum_{\substack{t\neq i\\t\in\gamma_1}}|a_{it}|,\ \beta_i=\sum_{\substack{t\neq i\\t\in\gamma_2}}|a_{it}|.$$

We call *A* a strictly generalized doubly diagonally dominant matrix if all the inequalities of (3) are strict.

Assuming that the matrix order is $n\geq 2$, we use the same notation as in paper [6]: D_n for diagonally dominant matrices; SD_n for strictly diagonally dominant matrices; DD_n for doubly diagonally dominant matrices; SDD_n for strictly doubly diagonally dominant matrices; $GDD_n^{\gamma_1,\gamma_2}$ for generalized doubly diagonally dominant matrices; $SGDD_n^{\gamma_1,\gamma_2}$ for strictly generalized doubly diagonally dominant.

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Let $A \in Z \equiv \{(a_{ij}) \in R^{n,n} : a_{ij} \le 0, i \ne j\}$. If

$$A = aI - B, B \ge 0, a > \rho(B),$$

then A is called an M-matrix. The absolute matrix of A is defined by $|A| = (|a_{ij}|)$. The comparison matrix $\mu(A) = (\mu_{ij})$ is defined by

$$\mu_{ij} = \begin{cases} -|a_{ij}|, i \ne j, \\ |a_{ij}|, i = j. \end{cases}$$

A is called an H-matrix if $\mu(A)$ is an M-matrix. In the following we denote M-matrices and H-matrices by M_n and H_n, respectively.

Let α, β be nonempty ordered subsets of $\langle n \rangle$, both consisting of strictly increasing integers. By $A(\alpha, \beta)$ we shall denote the submatrix of A lying in rows indexed by α and columns indexed by β. If, in addition, $\alpha = \beta$, then the principal submatrix $A(\alpha, \alpha)$ is abbreviated to $A(\alpha)$.

Suppose that $\alpha \subset \langle n \rangle$. If $A(\alpha)$ is nonsingular, then the Schur complement of $A(\alpha)$ in A is given by

$$S(A/A(\alpha)) = A(\beta) - A(\beta,\alpha)\left[A(\alpha)\right]^{-1} A(\alpha,\beta), \tag{4}$$

where $\beta = \langle n \rangle \setminus \alpha$. Furthermore, the Schur complements have been well studied for various classes of matrices, including: positive definite matrices, M-matrices, inverse M-matrices and totally nonnegative matrices. A well-known result due to Carlson and Markham [1] states that the Schur complements of strictly diagonally dominant matrices are diagonally dominant. And the Schur complement of a generalized doubly diagonally dominant matrix is a generalized doubly diagonally dominant matrix (see, [6]).

A remarkable Schur formula is ([8])

$$\det(S(A/A(\alpha))) = \frac{\det A}{\det A(\alpha)} \tag{5}$$

In connection with a divide and conquer algorithm for computing the stationary distribution vector for a Markov chain, Meyer [2,3] introduced, for a $n \times n$ nonnegative and irreducible matrix A, the notion of the Perron complement. Again, Let $\alpha \subset \langle n \rangle$ and $\beta = \langle n \rangle \setminus \alpha$. Then the Perron complement of $A(\alpha)$ in A is given by

$$P(A/A(\alpha)) = A(\beta) + A(\beta,\alpha)\left[\rho(A)I - A(\alpha)\right]^{-1} A(\alpha,\beta) \tag{6}$$

where $\rho(\cdot)$ denotes the spectral radius of a matrix. Recall that as A is irreducible, $\rho(A) > \rho(A(\alpha))$, so that the expression on the right-hand side of (6) is well defined, and we observe that $\rho(A)I - A(\alpha)$ is an M-matrix and thus $(\rho(A)I - A(\alpha))^{-1} \ge 0$. Meyer [2, 3] has derived several interesting and useful properties of $P(A/A(\alpha))$, such as $P(A/A(\alpha))$ is also nonnegative and irreducible, and $\rho(P(A/A(\alpha))) = \rho(A)$. And such matrices arise in a variety of applications [9], have been studied most of the 20th century, and have received increasing attention of late (see [4,5]). For general irreducible nonnegative matrices, Johnson and Xenophotos [9] investigate when the Perron complements are primitive or just irreducible and thus answer some issues which were raised by Meyer in his earlier paper.

Some of the results in [2, 3, 4, 5] motivated this study on Perron complements of strictly generalized doubly diagonally dominant matrices. In fact, given a matrix family, it is always interesting to know whether some important properties or structures of the family of the matrices are inherited by their submatrices or by the matrices associated with the original matrices.

In addition, for any $\alpha \subset \langle n \rangle$ and for any $t \ge \rho(A)$, let the extended Perron complement at t be the matrix

$$P_t(A/A(\alpha)) = A(\beta) + A(\beta,\alpha)\left[tI - A(\alpha)\right]^{-1} A(\alpha,\beta) \tag{7}$$

which is also well defined since $t \ge \rho(A) > \rho(A(\alpha))$.

In this paper, we shall show, in Section 2, that some Perron complements of strictly generalized doubly diagonally dominant and nonnegative irreducible matrices A are strictly diagonally dominant and strictly generalized doubly diagonally dominant only if the Perron root of A satisfies some conditions.

2. Perron complements of strictly generalized doubly diagonally dominant matrices

Since the Schur complements of strictly generalized doubly diagonally dominant matrices are strictly generalized doubly diagonally dominant, it seems natural to ask: When is the Perron complement of a strictly generalized doubly diagonally dominant matrix strictly generalized doubly diagonally dominant? Then, we consider properties of the Perron complements of strictly generalized doubly diagonally dominant and nonnegative irreducible matrices.

Lemma2.1. ([2]) Let A be a nonnegative irreducible matrix with spectral radius $\rho(A)$, and let $\phi \ne \alpha \subset \langle n \rangle$ and $\beta = \langle n \rangle \setminus \alpha$. Then the Perron complement

$$P(A/A(\alpha)) = A(\beta) + A(\beta,\alpha)\left[\rho(A)I - A(\alpha)\right]^{-1} A(\alpha,\beta)$$

is also a nonnegative irreducible matrix with spectral radius $\rho(A)$.

Lemma2.2. ([6]) Let $A \in SGDD_n^{\gamma_1,\gamma_2}$, Then $\gamma_1 \subseteq \{i \in \langle n \rangle \,|\, |a_{ii}| > \sum_{t=1, t\ne i}^{n} |a_{it}|\}$ or $\gamma_2 \subseteq \{i \in \langle n \rangle \,|\, |a_{ii}| > \sum_{t=1, t\ne i}^{n} |a_{it}|\}$.

Lemma2.3. ([7]) Let $A \in SD_n$, SDD_n, or $SGDD_n$. Then $\mu(A) \in M_n$; i.e., $A \in H_n$.

Lemma2.4. ([5]) Let A be any $n \times n$ irreducible nonnegative matrix, and fix any nonempty set $\beta \subset \langle n \rangle$. Then for any $\phi \ne \gamma_1, \gamma_2 \subset \beta$, with $\gamma_1 \cup \gamma_2 = \beta$ and $\gamma_1 \cap \gamma_2 = \phi$, we have

$$P_t(A/\beta) = P_t(P_t(A/\gamma_1)/\gamma_2),$$

for any $t \in [\rho(A), \infty)$.

We are now in a position to state the main results for Perron complements of strictly generalized doubly diagonally dominant matrices.

Theorem2.5. Let A be any $n \times n$ nonnegative irreducible matrix with spectral radius $\rho(A)$ and $A \in SGDD_n^{\gamma_1,\gamma_2}$. Then, for $\rho(A) \ge 2|a_{ii}|$, if $i \in \gamma_1$ then $P(A/A(\gamma_1))$ is a strictly diagonally dominant and nonnegative irreducible matrix.

Proof. Let $\gamma_1 = \{i_1, i_2, \cdots, i_k\}$ and $\gamma_2 = \{j_1, j_2, \cdots, j_l\}$, where $k + l = n$. By Lemma 2.2, without loss of generality, we assume

$$\gamma_1 \subseteq \{i \in \langle n \rangle \,|\, |a_{ii}| > \sum_{\substack{t=1 \\ t\ne i}}^{n} |a_{it}|\} \tag{8}$$

and let $i \in \gamma_1$. By $\rho(A) \ge 2|a_{ii}|$, $i \in \gamma_1$ and by (8), we get

$$\rho(A) - \sum_{\substack{t=1 \\ t\in\gamma_1}}^{k} |a_{it}| \ge |a_{ii}| - \sum_{\substack{t\ne i \\ t\in\gamma_1}}^{k} |a_{it}| > 0. \tag{9}$$

Thus,

$$\frac{1}{\rho(A) - \sum_{\substack{t=1 \\ t\in\gamma_1}}^{k} |a_{it}|} \le \frac{1}{|a_{ii}| - \sum_{\substack{t\ne i \\ t\in\gamma_1}}^{k} |a_{it}|}.$$

Since A is irreducible and nonnegative, we obtain

$$\sum_{\substack{t=1 \\ t\in\gamma_2}}^{l} |a_{it}| \neq 0.$$

So,

$$\frac{\sum_{\substack{t=1 \\ t\in\gamma_2}}^{l}|a_{it}|}{\rho(A)-\sum_{\substack{t=1 \\ t\in\gamma_1}}^{k}|a_{it}|} \le \frac{\sum_{\substack{t=1 \\ t\in\gamma_2}}^{l}|a_{it}|}{|a_{ii}|-\sum_{\substack{t\neq i \\ t\in\gamma_1}}^{k}|a_{it}|} \le \max_{i\in\gamma_1}\frac{\sum_{\substack{t=1 \\ t\in\gamma_2}}^{l}|a_{it}|}{|a_{ii}|-\sum_{\substack{t\neq i \\ t\in\gamma_1}}^{k}|a_{it}|}.$$

Thus,

$$\max_{i\in\gamma_1}\frac{\sum_{\substack{t=1 \\ t\in\gamma_2}}^{l}|a_{it}|}{\rho(A)-\sum_{\substack{t=1 \\ t\in\gamma_1}}^{k}|a_{it}|} \le \max_{i\in\gamma_1}\frac{\sum_{\substack{t=1 \\ t\in\gamma_2}}^{l}|a_{it}|}{|a_{ii}|-\sum_{\substack{t\neq i \\ t\in\gamma_1}}^{k}|a_{it}|}. \tag{10}$$

Denote

$$x=\left(\rho(A)I-A(\alpha)\right)^{-1}\begin{pmatrix}\sum_{s=1}^{l}a_{i_1 j_s}\\ \vdots \\ \sum_{s=1}^{l}a_{i_k j_s}\end{pmatrix} \tag{11}$$

or

$$\left(\rho(A)I-A(\alpha)\right)x=\begin{pmatrix}\sum_{s=1}^{l}a_{i_1 j_s}\\ \vdots \\ \sum_{s=1}^{l}a_{i_k j_s}\end{pmatrix}.$$

Let $x_v=\max\{x_1,x_2,\cdots,x_k\}$, where x_i is the i th component of x. We have

$$\begin{aligned}\sum_{s=1}^{l}a_{i_v j_s} &= \left(\rho(A)-a_{i_v i_v}\right)x_v+\sum_{\substack{t=1 \\ t\neq v}}^{k}\left(-a_{i_v i_t}\right)x_t \\ &\ge (\rho(A)-a_{i_v i_v}+\sum_{\substack{t=1 \\ t\neq v}}^{k}\left(-a_{i_v i_t}\right))x_v \\ &= (\rho(A)-\sum_{t=1}^{k}a_{i_v i_t})x_v.\end{aligned}$$

By (9), we have

$$x_v\le\frac{\sum_{t=1}^{l}|a_{i_v j_t}|}{\rho(A)-\sum_{t=1}^{k}|a_{i_v i_t}|}\le\max_{i\in\gamma_1}\frac{\sum_{t=1}^{l}|a_{i_v j_t}|}{\rho(A)-\sum_{t=1}^{k}|a_{i_v i_t}|}. \tag{12}$$

Note that A is an irreducible and nonnegative matrix, then

$$\rho(A)>\rho\left(A(\alpha)\right),$$

so that $\rho(A)I - A(\alpha)$ is an M-matrix. Then we have

$$(\rho(A)I - A(\gamma_1))^{-1} \geq 0 \text{ and } a_{ij} \geq 0. \quad (13)$$

Since $A \in SGDD_n^{\gamma_1,\gamma_2}$, we have, by (3), for any $i \in \gamma_1$, $j \in \gamma_2$,

$$\left(|a_{ii}| - \sum_{\substack{t \neq i \\ t \in \gamma_1}} |a_{it}| \right)\left(|a_{jj}| - \sum_{\substack{t \neq j \\ t \in \gamma_2}} |a_{jt}| \right) > \sum_{t \in \gamma_1} |a_{jt}| \sum_{t \in \gamma_2} |a_{it}|.$$

By (9), we get

$$|a_{jj}| - \sum_{\substack{t \neq j \\ t \in \gamma_2}} |a_{jt}| > \frac{\sum_{t \in \gamma_2} |a_{it}|}{|a_{ii}| - \sum_{\substack{t \neq i \\ t \in \gamma_1}} |a_{it}|} \sum_{t \in \gamma_1} |a_{jt}|.$$

So, for $j \in \gamma_2$,

$$|a_{jj}| - \sum_{\substack{t \neq j \\ t \in \gamma_2}} |a_{jt}| > \max_{i \in n_1} \frac{\sum_{t \in \gamma_2} |a_{it}|}{|a_{ii}| - \sum_{\substack{t \neq i \\ t \in \gamma_1}} |a_{it}|} \sum_{t \in \gamma_1} |a_{jt}|. \quad (14)$$

Denote the (t,s)-entry of $P(A/A(\gamma_1))$ by $a'_{j_t j_s}$ .Then, for $t = 1, 2, \cdots, l$, we have

$$|a'_{j_t j_t}| - \sum_{\substack{s=1 \\ s \neq t}}^{l} |a'_{j_t j_s}|$$

$$= \left| a_{j_t j_t} + \left(a_{j_t i_1}, \cdots, a_{j_t i_k}\right)\left(\rho(A)I - A(\gamma_1)\right)^{-1} \begin{pmatrix} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{pmatrix} \right| - \sum_{\substack{s=1 \\ s \neq t}}^{l} \left| a_{j_t j_s} + \left(a_{j_t i_1}, \cdots, a_{j_t i_k}\right)\left(\rho(A)I - A(\gamma_1)\right)^{-1} \begin{pmatrix} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s} \end{pmatrix} \right|$$

$$\geq \left[|a_{j_t j_t}| - \left(|a_{j_t i_1}|, \cdots, |a_{j_t i_k}|\right) \left| \left(\rho(A)I - A(\gamma_1)\right)^{-1} \right| \begin{pmatrix} |a_{i_1 j_t}| \\ \vdots \\ |a_{i_k j_t}| \end{pmatrix} \right]$$

$$- \sum_{\substack{s=1 \\ s \neq t}}^{l} \left[|a_{j_t j_s}| + \left(|a_{j_t i_1}|, \cdots, |a_{j_t i_k}|\right) \left| \left(\rho(A)I - A(\gamma_1)\right)^{-1} \right| \begin{pmatrix} |a_{i_1 js}| \\ \vdots \\ |a_{i_k j_s}| \end{pmatrix} \right]$$

$$= a_{j_t j_t} - \sum_{\substack{s=1 \\ s \neq t}}^{l} a_{j_t j_s} - \left(a_{j_t i_1}, \cdots, a_{j_t i_k}\right)\left(\rho(A)I - A(\gamma_1)\right)^{-1} \begin{pmatrix} \sum_{s=1}^{l} a_{i_1 j_s} \\ \vdots \\ \sum_{s=1}^{l} a_{i_k j_s} \end{pmatrix} \text{(by(13))}$$

$$\geq a_{j_t j_t} - \sum_{\substack{s\neq t\\ s=1}}^{l} a_{j_t j_s} - \left(a_{j_t i_1}, \cdots, a_{j_t i_k}\right) \begin{pmatrix} \max\limits_{i_v \in \gamma_1} \dfrac{\sum_{t=1}^{l} a_{i_v j_t}}{\rho(A) - \sum_{t=1}^{k} a_{i_v i_t}} \\ \vdots \\ \max\limits_{i_v \in \gamma_1} \dfrac{\sum_{t=1}^{l} a_{i_v j_t}}{\rho(A) - \sum_{t=1}^{k} a_{i_v i_t}} \end{pmatrix} (by(11),(12))$$

$$= \left|a_{j_t j_t}\right| - \sum_{\substack{s\neq t\\ s=1}}^{l} \left|a_{j_t j_s}\right| - \max_{i_v \in \gamma_1} \frac{\sum_{t=1}^{l} a_{i_v j_t}}{\rho(A) - \sum_{t=1}^{k} a_{i_v i_t}} \sum_{s=1}^{k} \left|a_{j_t i_s}\right| (by(13))$$

$$\geq \left|a_{j_t j_t}\right| - \sum_{\substack{s\neq t\\ s=1}}^{l} \left|a_{j_t j_s}\right| - \max_{i_v \in \gamma_1} \frac{\sum_{\substack{t=1\\ j_t \in \gamma_2}}^{l} \left|a_{i_v j_t}\right|}{\left|a_{i_v i_v}\right| - \sum_{\substack{t\neq v\\ i_t \in \gamma_1}}^{k} \left|a_{i_v i_t}\right|} \sum_{s=1}^{k} \left|a_{j_t i_s}\right| (by(10))$$

$$> 0.(by(14))$$

It follows that $P\left(A/A(\gamma_1)\right)$ is a strictly diagonally dominant matrix. By Lemma 2.1, we have that the matrix $P\left(A/A(\gamma_1)\right)$ is nonnegative irreducible. This completes the proof.

Remark2.6. According to Theorem 2.5., we can give a similar result for $i \in \gamma_2$.

We now give a example to illustrate the result of Theorem 2.5.

Example2.7. Let

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 2 & 3 & 1 \\ 2 & 0 & 2 \end{pmatrix}.$$

Obviously, $A \in SGDD_3^{\gamma_1,\gamma_2}$, where $\gamma_1 = \{1\}, \gamma_2 = \{2,3\}$ and the matrix A is nonnegative irreducible. And,

$$\rho(A) = 4.2695 \geq 2\,|a_{ii}|, i \in \gamma_1.$$

Then,

$$P\left(A/A(\gamma_1)\right) = \begin{pmatrix} 3.8813 & 1 \\ 0.8813 & 2 \end{pmatrix}$$

is strictly diagonally dominant.

From Theorem 2.5, we have a trivial result about the extended Perron complements of strictly generalized doubly diagonally dominant matrices.

Corollary2.8. Let A be any $n \times n$ nonnegative irreducible matrix with spectral radius $\rho(A)$ and $A \in SGDD_n^{\gamma_1,\gamma_2}$. Then, for any $t \in \left[\rho(A), \infty\right)$ and $\rho(A) \geq 2\,|a_{ii}|$, if $i \in \gamma_1$ then $P_t\left(A/A(\gamma_1)\right)$ is a strictly diagonally dominant and nonnegative irreducible matrix.

In Theorem 2.5, we have been obtained that the Perron complement of a strictly generalized doubly diagonally dominant matrix is strictly diagonally dominant only if the condition above holds. Furthermore, we have the following theorem in which we show that under the assumptions in Theorem 2.5, the Perron complement is strictly generalized doubly diagonally dominant.

Theorem2.9. Let A be any nonnegative $n\times n$ irreducible matrix with spectral radius $\rho(A)$ and $A\in SGDD_n^{\gamma_1,\gamma_2}$. Then, for $\rho(A)\ge 2|a_{ii}|$, if $i\in\gamma_1$ and any proper subset γ of γ_1 then $P\left(A/A(\gamma)\right)$ is a strictly generalized doubly diagonally dominant and nonnegative irreducible matrix.

Proof. Let $\gamma_1=\{i_1,i_2,\cdots,i_k\}$ and $\gamma_2=\{j_1,j_2,\cdots,j_l\}$, where $k+l=n$. By Lemma 2.2, without loss of generality, we assume

$$\gamma_1\subseteq\{i\in\langle n\rangle \,|\, |a_{ii}|>\sum_{\substack{t=1\\t\ne i}}^{n}|a_{it}|\} \tag{15}$$

and let $i\in\gamma_1$.We show the theorem in two steps:

$$\gamma \text{ is a singleton};\quad \gamma\subseteq\gamma_1.$$

(i) Consider the case that γ contains only one element. Assume $\gamma\subseteq\gamma_1$.

If $\gamma=\{i_1\}=\gamma_1$: by Theorem 2.5, we have

$$P\left(A/A(\gamma)\right)\in SD_{n-1}=SGDD_{n-1}^{\phi,\gamma_2}.$$

If $\gamma=\{i_1\}\subset\gamma_1$: for any fixed $j_u\in\gamma_2$ and $i_s\in\gamma_1-\{i_1\}$, let

$$A_1=\begin{pmatrix}\rho(A)-|a_{i_1i_1}| & -\sum\limits_{\substack{t\ne1\\i_t\in\gamma_1}}|a_{i_1i_t}| & -\sum\limits_{j_v\in\gamma_2}|a_{i_1j_v}|\\ -|a_{i_si_1}| & |a_{i_si_s}|-\sum\limits_{\substack{t\ne1,s\\i_t\in\gamma_1}}|a_{i_si_t}| & -\sum\limits_{j_v\in\gamma_2}|a_{i_sj_v}|\\ -|a_{j_ui_1}| & -\sum\limits_{\substack{t\ne1\\i_t\in\gamma_1}}|a_{j_ui_t}| & |a_{j_uj_u}|-\sum\limits_{\substack{v\ne u\\j_v\in\gamma_2}}|a_{j_uj_v}|\end{pmatrix}. \tag{16}$$

By (4), we have

$$S\left(A_1/A_1(1)\right)=\begin{pmatrix}|a_{i_si_s}|-\sum\limits_{\substack{t\ne1,s\\i_t\in\gamma_1}}|a_{i_si_t}|-\sum\limits_{\substack{t\ne1\\i_t\in\gamma_1}}\left|\dfrac{a_{i_si_1}a_{i_1i_t}}{\rho(A)-a_{i_1i_1}}\right| & -\sum\limits_{j_v\in\gamma_2}|a_{i_sj_v}|-\sum\limits_{j_v\in\gamma_2}\left|\dfrac{a_{i_si_1}a_{i_1j_v}}{\rho(A)-a_{i_1i_1}}\right|\\ -\sum\limits_{\substack{t\ne1\\i_t\in\gamma_1}}|a_{j_ui_t}|-\sum\limits_{\substack{t\ne1\\i_t\in\gamma_1}}\left|\dfrac{a_{j_ui_1}a_{i_1i_t}}{\rho(A)-a_{i_1i_1}}\right| & |a_{j_uj_u}|-\sum\limits_{\substack{v\ne u\\j_v\in\gamma_2}}|a_{j_uj_v}|-\sum\limits_{j_v\in\gamma_2}\left|\dfrac{a_{i_si_1}a_{i_1j_v}}{\rho(A)-a_{i_1i_1}}\right|\end{pmatrix}.$$

Since $A\in SGDD_n^{\gamma_1,\gamma_2}$, we have, by (3),

$$\left(|a_{i_1i_1}|-\sum_{\substack{t\ne1\\i_t\in\gamma_1}}|a_{i_1i_t}|\right)\left(|a_{j_uj_u}|-\sum_{\substack{v\ne u\\j_v\in\gamma_2}}|a_{j_uj_v}|\right)>\sum_{j_v\in\gamma_2}|a_{i_1j_v}|\left(|a_{j_ui_1}|+\sum_{\substack{t\ne1\\i_t\in\gamma_1}}|a_{j_ui_t}|\right).$$

and

$$\left(|a_{i_si_s}|-\sum_{\substack{t\ne1,s\\i_t\in\gamma_1}}|a_{i_si_t}|-|a_{i_si_1}|\right)\left(|a_{j_uj_u}|-\sum_{\substack{v\ne u\\j_v\in\gamma_2}}|a_{j_uj_v}|\right)>\sum_{j_v\in\gamma_2}|a_{i_sj_v}|\left(|a_{j_ui_1}|+\sum_{\substack{t\ne1\\i_t\in\gamma_1}}|a_{j_ui_t}|\right).$$

By $\rho(A)\ge 2|a_{ii}|$, $i\in\gamma_1$ and (15), we get

$$\rho(A)-|a_{i_1i_1}|\ge|a_{i_1i_1}|>0$$

or

$$\rho(A)-|a_{i_1i_1}|-\sum_{\substack{t=1\\ i_t\in\gamma_1}}|a_{i_1i_t}|\geq|a_{i_1i_1}|-\sum_{\substack{t\neq 1\\ i_t\in\gamma_1}}|a_{i_1i_t}|>0. \tag{17}$$

Thus,

$$\left(\rho(A)-|a_{i_1i_1}|-\sum_{\substack{t\neq 1\\ i_t\in\gamma_1}}|a_{i_1i_t}|\right)\left(|a_{j_uj_u}|-\sum_{\substack{v\neq u\\ j_v\in\gamma_2}}|a_{j_uj_v}|\right)>\sum_{j_v\in\gamma_2}|a_{i_1j_v}|\left(|a_{j_ui_1}|+\sum_{\substack{t\neq 1\\ i_t\in\gamma_1}}|a_{j_ui_t}|\right).$$

So, we have

$$A_1\in SGDD_3^{\gamma_1,\gamma_2},\gamma_1=\{1,2\},\gamma_2=\{3\}.$$

By Lemma 2.3, we have $A_1=\mu(A_1)\in M_n$. Moreover,

$$\det A_1>0. \tag{18}$$

Denote the (t,s)-entry of $P(A/A(\gamma))$ by $a'_{i_tj_s}$. Then, for $i_t,j_s\in\langle n\rangle/\gamma$, we have

$$\left(|a'_{i_si_s}|-\sum_{\substack{t\neq 1,s\\ i_t\in\gamma_1}}|a'_{i_si_t}|\right)\left(|a'_{j_uj_u}|-\sum_{\substack{v\neq u\\ j_v\in\gamma_2}}|a'_{j_uj_v}|\right)-\sum_{j_v\in\gamma_2}|a'_{i_sj_v}|\sum_{\substack{t\neq 1\\ i_t\in\gamma_1}}|a'_{j_ui_t}|$$

$$=\left(|a_{i_si_s}+\frac{a_{i_si_1}a_{i_1i_s}}{\rho(A)-a_{i_1i_1}}|-\sum_{\substack{t\neq 1,s\\ i_t\in\gamma_1}}|a_{i_si_t}+\frac{a_{i_si_1}a_{i_1i_t}}{\rho(A)-a_{i_1i_1}}|\right)\times\left(|a_{j_uj_u}+\frac{a_{j_ui_1}a_{i_1j_u}}{\rho(A)-a_{i_1i_1}}|-\sum_{\substack{v\neq u\\ j_v\in\gamma_2}}|a_{j_uj_v}+\frac{a_{j_ui_1}a_{i_1j_v}}{\rho(A)-a_{i_1i_1}}|\right)$$

$$-\sum_{j_v\in\gamma_2}|a_{i_sj_v}+\frac{a_{i_si_1}a_{i_1j_v}}{\rho(A)-a_{i_1i_1}}|\sum_{\substack{t\neq 1\\ i_t\in\gamma_1}}|a_{j_ui_t}+\frac{a_{j_ui_1}a_{i_1i_t}}{\rho(A)-a_{i_1i_1}}|$$

$$\geq\left(|a_{i_si_s}|-|\frac{a_{i_si_1}a_{i_1i_s}}{\rho(A)-a_{i_1i_1}}|-\sum_{\substack{t\neq 1,s\\ i_t\in\gamma_1}}|a_{i_si_t}|-\sum_{\substack{t\neq 1,s\\ i_t\in\gamma_1}}|\frac{a_{i_si_1}a_{i_1i_t}}{\rho(A)-a_{i_1i_1}}|\right)$$

$$\times\left(|a_{j_uj_u}|-|\frac{a_{j_ui_1}a_{i_1j_u}}{\rho(A)-a_{i_1i_1}}|-\sum_{\substack{v\neq u\\ j_v\in\gamma_2}}|a_{j_uj_v}|-\sum_{\substack{v\neq u\\ j_v\in\gamma_2}}|\frac{a_{j_ui_1}a_{i_1j_v}}{\rho(A)-a_{i_1i_1}}|\right)$$

$$-\left(\sum_{j_v\in\gamma_2}|a_{i_sj_v}|+\sum_{j_v\in\gamma_2}|\frac{a_{i_si_1}a_{i_1j_v}}{\rho(A)-a_{i_1i_1}}|\right)\left(\sum_{\substack{t\neq 1\\ i_t\in\gamma_1}}|a_{j_ui_t}|+\sum_{\substack{t\neq 1\\ i_t\in\gamma_1}}|\frac{a_{j_ui_1}a_{i_1i_t}}{\rho(A)-a_{i_1i_1}}|\right)$$

$$=\left(|a_{i_si_s}|-\sum_{\substack{t\neq 1,s\\ i_t\in\gamma_1}}|a_{i_si_t}|-\sum_{\substack{t\neq 1\\ i_t\in\gamma_1}}|\frac{a_{i_si_1}a_{i_1i_t}}{\rho(A)-a_{i_1i_1}}|\right)\times\left(|a_{j_uj_u}|-\sum_{\substack{v\neq u\\ j_v\in\gamma_2}}|a_{j_uj_v}|-\sum_{j_v\in\gamma_2}|\frac{a_{j_ui_1}a_{i_1j_v}}{\rho(A)-a_{i_1i_1}}|\right)$$

$$-\left(\sum_{j_v\in\gamma_2}|a_{i_sj_v}|+\sum_{j_v\in\gamma_2}|\frac{a_{i_si_1}a_{i_1j_v}}{\rho(A)-a_{i_1i_1}}|\right)\left(\sum_{\substack{t\neq 1\\ i_t\in\gamma_1}}|a_{j_ui_t}|+\sum_{\substack{t\neq 1\\ i_t\in\gamma_1}}|\frac{a_{j_ui_1}a_{i_1i_t}}{\rho(A)-a_{i_1i_1}}|\right)$$

$$=\det(S(A_1/A_1(1))$$

$$=\frac{\det A_1}{\det A_1(1)}(\text{by }(5))$$

$$=\frac{\det A_1}{\rho(A)-|a_{i_1i_1}|}$$
$$>0.(by(17),(18))$$

So, we obtain

$$P(A/A(i_1))\in SGDD_{n-1}^{\gamma_1-\{i_1\},\gamma_2},\ i_1\in\gamma_1.$$

(ii) If γ contains two elements. Assume $\gamma=\{i_1,i_2\}\subseteq\gamma_1$. By Lemma 2.4, we have

$$P(A/A(\gamma))=P(P(A/A(i_1))/A(i_2))\in SGDD_{n-2}^{\gamma_1-\{i_1,i_2\},\gamma_2}.$$

If γ contains more than two elements and $\gamma\subseteq\gamma_1$, by induction, we get

$$P(A/A(\gamma))\in SGDD.$$

Remark2.10. According to Theorem 2.9., we can give a similar result for $i\in\gamma_2$.

The following example can illustrate the result of Theorem 2.9.

Example2.11. Let

$$A=\begin{pmatrix}3&1&1&0\\1&3&0&1\\2&1&4&1\\1&2&1&4\end{pmatrix}.$$

Obviously, $A\in SGDD_4^{\gamma_1,\gamma_2}$, where $\gamma_1=\{1,2\},\gamma_2=\{3,4\}$ and the matrix A is nonnegative irreducible. And,

$$\rho(A)=6.3028\geq 2|a_{ii}|,i\in\gamma_1.$$

Then,

$$P(A/A(\gamma))=\begin{pmatrix}4.7676&1.5352\\1.5352&4.7676\end{pmatrix}.$$

where $\gamma=\{1,2\}\subseteq\gamma_1$, is strictly generalized doubly diagonally dominant.

From Theorem 2.9, we remark that it is easy to verify that the extended Perron complement of a strictly generalized doubly diagonally dominant matrix is a strictly generalized doubly diagonally dominant only if the condition above holds.

Finally we remark that we may slightly relax the “Strict”, condition so that our theorems hold for some matrices. This is done by a usual trick-continuity argument. We omit further discussions on this. Moreover, for a nonnegative matrix A, the Perron complement can be applied to research properties of A utilizing the relation between the matrix and its Perron complement. So, given a matrix family, we can consider further discussions on Perron complements of the family of the matrices.

3. References

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