ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 5, No. 4, 2010, *pp*. 299-304

A Generalized Proof of the Smoothness of 6-Point Interpolatory Scheme

Ghulam Mustafa[1], Faheem Khan [2] and Muhammad Sadiq Hashmi [3]

Department of Mathematics, The Islamia University of Bahawalpur Pakistan

(*Received September 24, 2009, accepted October 22, 2009*)

Abstract. S. S. Siddiqi and A. Nadeem [A proof of the smoothness of the 6-point interpolatory scheme, International Journal of Computer Mathematics, 83(5-6), 503-509, 2006] proved that Weissman's 6-point subdivision scheme is C^2 for a particular value of parameter by means of Laurent polynomial method. In this work, we also use the same method to get C^2 continuity of 6-point scheme over the parametric interval. The original interval (0.0, 0.0277) presented by Weissman for continuity is contained in the interval (0.0, 0.0425] introduced in this article.

Keywords: interpolating subdivision scheme, continuity, smoothness, convergence, shape parameter, Laurent polynomial.

1. Introduction

A subdivision curve, in the field of 2D and 3D computer graphics, is a method of representing a smooth curve via the specification of a coarser piecewise linear polygon. The smooth curve can be calculated from the coarse polygon as the limit of an iterative process of subdividing each edge into smaller edges that better approximate the smooth curve. Subdivision schemes are classified into two categories: interpolating and approximating. If the control points of the original polygon and the newly generated control points after subdivision are interpolated by the limit curve then scheme is called interpolating otherwise it is called approximating.

Dyn et al. [1] introduced 4-point interpolating subdivision scheme and proved that scheme is C^1 by means of eigenanalysis. Youchun et al. [5] used the Laurent polynomial to obtain the same result. Weissman [4] introduced a 6-point interpolating scheme which gives C^2 limit function over the parametric interval (0.0, 0.02). Siddiqi and Nadeem [3] have shown that smoothness of the 6-point scheme is C^2 for particular value of parameter (i.e. 0.02) by means of Laurent polynomial method. In this article, we also take advantage of Laurent polynomial method to get C^2 continuity of the 6-point subdivision scheme over the parametric interval (0.0, 0.0425]. The original parametric interval for continuity presented by Weissman is subset of the interval introduced in our work.

2. Preliminaries

A general compact form of univariate subdivision scheme S which maps a polygon $f^k = \{f_i^k\}_{i\in Z}$ to a refined polygon $f^{k+1} = \{f_i^{k+1}\}_{i\in Z}$ is defined by

[1] Corresponding author. *E-mail address*: ghulam.mustafa@iub.edu.pk , mustafa_rakib@yahoo.com
[2] *E-mail address*: sadiq.hashmi@gmail.com
[3] *E-mail address*: fahimscholar@gmail.com
This work is supported by the Indigenous PhD Scholarship Scheme of Higher Education Commission (HEC) Pakistan.

$$\begin{cases} f_{2i}^{k+1} = \sum_{j\in Z} a_{2j} f_{i-j}^{k}, \\ f_{2i+1}^{k+1} = \sum_{j\in Z} a_{2j+1} f_{i-j}^{k}, \end{cases} \tag{2.1}$$

where the set $a=\{a_i : i\in Z\}$ of coefficients is called the mask of the subdivision scheme. A necessary condition for the uniform convergence of subdivision scheme (2.1) is that

$$\sum_{j\in Z} a_{2j} = \sum_{j\in Z} a_{2j+1} = 1. \tag{2.2}$$

For the analysis of subdivision scheme with mask a, it is very practical to consider the $z-$transform of the mask

$$a(z) = \sum_{i\in Z} a_i z^i, \tag{2.3}$$

which is usually called the *symbol/Laurent polynomial* of the scheme. From (2.2) and (2.3) the Laurent polynomial of a convergent subdivision scheme satisfies

$$a(-1)=0 \quad \text{and} \quad a(1)=2. \tag{2.4}$$

This condition guarantees the existence of a related subdivision scheme for the divided differences of the original control points and the existence of associated $a^{(1)}(z)$, which can defined as follows:

$$a^{(1)}(z) = \frac{2z}{1+z} a(z).$$

The subdivision scheme S_1 with *symbol* $a^{(1)}(z)$, is related to scheme S with *symbol* $a(z)$ by the following theorem.

Theorem 2.1. *[2] Let S denote a subdivision scheme with Laurent polynomial $a(z)$ satisfying (2.2). Then there exist a subdivision scheme S_1 with the property*

$$\Delta f^k = S_1 \Delta f^{k-1},$$

where $f^k = S^k f^0$ and $\Delta f^k = \{(\Delta f^k)_i = 2^k(f_{i+1}^k - f_i^k);\ i\in Z\}$. Furthermore, S is a uniformly convergent if and only if $\frac{1}{2}S_1$ converges uniformly to zero function for all initial data f^0, in the sense that

$$\lim_{k\to 0}\left(\frac{1}{2}S_1\right)^k f^0 = 0. \tag{2.5}$$

A scheme S_1 satisfying (2.5) for all initial data f^0 is termed contractive. By Theorem 2.1, the convergence of S is equivalent to checking whether S_1 is contractive, which is then equivalent to checking whether $\left\|\left(\frac{1}{2}S_1\right)^L\right\|_\infty < 1$, for some integer $L > 0$.

Since there are two rules for computing the values at next refinement level, one with even coefficients of the mask and one with odd coefficients of the mask, we define the norm

$$\|S\|_\infty = \max\left\{\sum_{j\in Z}|a_{2j}|,\ \sum_{j\in Z}|a_{2j+1}|\ \right\},$$

and

$$\left\|\left(\frac{1}{2}S_n\right)^L\right\|_\infty = \max\left\{\sum_{j\in Z}\left|b^{[n,L]}_{i+2^L j}\right|;\quad i=0,\ 1, 2,\ .\ .\ .\ ,\ 2^L\ \text{-}1\right\}, \tag{2.6}$$

where

$$b^{[n,L]}(z) = \frac{1}{2^L}\prod_{j=1}^{L-1} a^{(n)}(z^{2j}), \qquad a^{(n)}(z) = \frac{2z}{1+z}a^{(n-1)}(z).$$

Theorem 2.2 *[2] Let* $a(z) = \left(\frac{1+z}{2}\right)^n q(z)$, *If* S_q *is convergent, then* $S_a^\infty \in C^n(R)$ *for any initial data* f^0.

3. The smoothness analysis of 6-point scheme

The Laurent polynomial of finite mask of Weissman's 6-point scheme *S* is

$$a(z) = z^{-1}(1+z)\xi_1(z)\,, \tag{3.1}$$

where

$$\xi_1(z) = wz^5 - wz^4 + \left(-\frac{1}{16} - 2w\right)z^3 + \left(\frac{1}{16} + 2w\right)z^2 + \frac{1}{2}z + \frac{1}{2} + \left(\frac{1}{16} + 2w\right)z^{-1} + \left(-\frac{1}{16} - 2w\right)z^{-2} - wz^{-3} + wz^{-4}$$

From (2.6) for $L = n = 1$ and (3.1), we have

$$b^{[1,1]}(z) = \frac{1}{2}a^{(1)}(z) = \frac{z}{1+z}a(z) = \xi_1(z). \tag{3.2}$$

For C^0 continuity of *S*, we require that the Laurent polynomial $a(z)$ satisfy (2.4), which it does, and $\left\|\frac{1}{2}S_1\right\|_\infty < 1$. The norm of scheme $\frac{1}{2}S_1$ is

$$\left\|\frac{1}{2}S_1\right\|_\infty = |w| + \left|-\frac{1}{16} - 2w\right| + \frac{1}{2} + \left|\frac{1}{16} + 2w\right| + |-w| < 1\,,$$

for $-0.10416 \le w < 0.06250$. Therefore $\frac{1}{2}S_1$ is contractive. Hence by Theorem 2.1, S is C^0.

By (3.2) the Laurent polynomial of scheme $\frac{1}{2}S_1$ can be written as

$$a^{(1)}(z) = 2z^{-1}(1+z)\xi_2(z)\,, \tag{3.3}$$

where

$$\xi_2(z) = wz^5 - 2wz^4 - \frac{1}{16}z^3 + \left(\frac{2}{16} + 2w\right)z^2 + \left(\frac{6}{16} - 2w\right)z + \left(\frac{2}{16} + 2w\right) - \frac{1}{16}z^{-1} - 2wz^{-2} + wz^{-3}.$$

Utilizing (2.6) for n = 2 & L = 1 and (3.3), we get

$$b^{[2,1]}(z) = \frac{1}{2} a^{(2)}(z) = \frac{z}{1+z} a^{(1)}(z) = 2\xi_2(z). \tag{3.4}$$

Now for C^1 continuity we first need $a^{(1)}(z)$ to satisfy (2.4), which it does, and for first integer value of $L >$ 0 for which $\left\| \left(\frac{1}{2} S_2 \right)^L \right\|_\infty < 1.$ We empirically calculate the norm of $\frac{1}{2} S_2$ for L and see that for

-0.01586 $\leq w \leq$ 0.05914

$$\left\| \left(\frac{1}{2} S_2 \right)^2 \right\|_\infty = \max\{\eta_1, \eta_2, \eta_3\} < 1,$$

where

$$\eta_1 = 2\left|-8w^2\right| + \left|\frac{1}{32} - \frac{3}{2} + 48w^2\right| + \left|\frac{1}{4} + \frac{7}{2} w - 16w^2\right| + \left|-\frac{1}{32} - \frac{5}{2} w - 40w^2\right| + \left|\frac{1}{2} w + 24w^2\right|,$$

$$\eta_2 = 2\left|-\frac{1}{4} w - 8w^2\right| + 2\left|\frac{1}{64} - \frac{11}{4} w + 24w^2\right| + 2\left|\frac{7}{64} + 3w - 16w^2\right|,$$

$$\eta_3 = 2\left|4w^2\right| + 2\left|\frac{7}{4} w - 8w^2\right| + 2\left|-\frac{1}{8} + 2w - 4w^2\right| + \left|\frac{1}{2} - \frac{15}{2} w + 16w^2\right|.$$

Therefore $\frac{1}{2} S_2$ is contractive. Hence by Theorem 2.2, *S* is C^1. Now from (3.4) the Laurent polynomial of scheme $\frac{1}{2} S_2$ is

$$a^{(2)}(z) = 4z^{-1}(1+z)\xi_3(z), \tag{3.5}$$

where

$$\xi_3(z) = wz^5 - 3wz^4 + \left(3w - \frac{1}{16}\right)z^3 + \left(\frac{3}{16} - w\right)z^2 + \left(\frac{3}{16} - w\right)z + \left(3w - \frac{1}{16}\right) - 3wz^{-1} + wz^{-2}.$$

With the choice of n = 3 & L = 1, we have the following from (2.6) and (3.5)

$$b^{[3,1]}(z) = \frac{1}{2} a^{(3)}(z) = \frac{z}{1+z} a^{(2)}(z) = 4\xi_3(z). \tag{3.6}$$

For C^2 continuity, we require that $a^{(2)}(z)$ satisfy (2.4), which is incidentally true, and also for first integer value of L > 0 for which $\left\| \left(\frac{1}{2} S_3 \right)^L \right\|_\infty < 1.$ We have for 0.0 $< w \leq$ 0.0277

$$\left\| \left(\frac{1}{2} S_3 \right)^2 \right\|_\infty = \max\{\gamma_1, \gamma_2, \gamma_3\} < 1,$$

where

$$\gamma_1 = 2\left|-8w^2\right| + \left|\frac{1}{32} - \frac{3}{2}w + 48w^2\right| + \left|\frac{1}{4} + \frac{7}{2}w - 16w^2\right| + \left|-\frac{1}{32} - \frac{5}{2}w - 40w^2\right| + \left|\frac{1}{2}w + 24w^2\right|,$$

$$\gamma_2 = 2\left|-\frac{1}{4}w - 8w^2\right| + 2\left|\frac{1}{64} - \frac{11}{4}w + 24w^2\right| + 2\left|\frac{7}{64} + 3w - 16w^2\right|,$$

$$\gamma_3 = 2\left|4w^2\right| + 2\left|\frac{7}{4}w - 8w^2\right| + 2\left|-\frac{1}{8} + 2w - 4w^2\right| + \left|\frac{1}{2} - \frac{15}{2}w + 16w^2\right|.$$

Therefore $\frac{1}{2}S_3$ is contractive. Hence by Theorem 2.2, S is C^2. Now from (3.6) the Laurent polynomial of scheme $\frac{1}{2}S_3$ can be written as

$$a^{(3)}(z) = 8z^{-1}(1+z)\xi_4(z)$$

where

$$\xi_4(z) = wz^5 - 4wz^4 + \left(7w - \frac{1}{16}\right)z^3 + \left(\frac{4}{16} - 8w\right)z^2 + \left(7w - \frac{1}{16}\right)z - 4w + wz^{-1}.$$

In [4], it is shown that 6-point interpolatory subdivision scheme is C^3 continuous over the interval [0.0139, 0.0143].

***Remark* 3.1.** To get optimal range of parameter *w* for C^2 continuity, we compute the inequality $\left\|\left(\frac{1}{2}S_3\right)^L\right\|_\infty <$ 1 for L = 3, 4, . . . , 8 and get following ranges:

0.0 < w $\leq$ 0.0362, 0.0 < w $\leq$ 0.0406, 0.0 < w $\leq$ 0.0404, 0.0 < w $\leq$ 0.0418, 0.0 < w $\leq$ 0.0417 & 0.0 < w $\leq$ 0.0425 for L =3, 4, 5, 6, 7 & 8 respectively. Computation of above inequality needs the evaluation of 2^L inequalities over the parametric interval. It takes nearly 128 hours to calculate parametric range for w at L = 8 on a 2.4 Ghz Core 2 Duo computer. For computing range of parameter w at L = 9 requires evaluation of 512 inequalities (with each inequality contains modulus values of atleast six different polynomials of degree nine), which create computational complexity in computations. But from above computed ranges one can easily conclude that there will be no significant improvement in the parametric interval for $L \geq 9$. Hence the nearly optimal range of parametric interval for C^2 continuity is 0.0 < $w \leq$ 0.0425.

Theorem 3.1. The 6-point Weissman [5] interpolatory subdivision scheme is C^2 over the parametric interval (0.0, 0.0425] and C^3 over [0.0139, 0.0143].

***Remark* 3.2.** Theorem 3.1 of Siddiqi and Nadeem [3] is special case of our Theorem 3.1. It is also noted that the original parametric interval (0.0, 0.0277) for C^2 continuity introduced by Weissman is smaller than the one presented by us.

4. References

[1] N. Dyn, D. Levin, J. A. Gregory. A 4-point interpolatory subdivision scheme for curve design. *Computer Aided Geometric Design*. 1987, **4**: 257-268.

[2] N. Dyn. Interpolatory subdivision schemes and analysis of convergence and smoothness by the formalism of Laurent polynomials, in: A. Iske, E. Quak, M. S Floater (Eds). *Tutorials on Multiresolution in Geometric Modeling.* Springer. 2002, pp. 51-68 (Chapter 2 and 3).

[3] S. S. Siddiqi, A. Nadeem. A proof of the smoothness of the 6-point interpolatory scheme. *International Journal of Computer Mathematics.* 2006, **83**(5-6): 503-509.

[4] Sigalit Hed. Analysis of subdivision schemes for surface generation, M.Sc. thesis. Tel Aviv University, 1992.

[5] A. Weissman. A 6-point interpolatory subdivision scheme for curve design, M. Sc. Thesis, Tel Aviv University, 1990.

[6] T. Youchun, K. P. Ko, B.-G. Lee. A new proof of the smoothness of 4-point Deslauriers-Dubuc scheme. *Journal of Applied Mathematics and Computing.* 2005, **8**(1-2): 553-562.