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A Full-Newton Step Primal-Dual Interior Point Algorithm for Linear Complementarity Problems*

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Abstract. In this paper, we give a full-Newton step primal-dual interior-point algorithm for monotone horizontal linear complementarity problem. The searching direction is obtained by modification of the classic Newton direction, and which also enjoys the quadratically convergent property in the small neighborhood of central path. The complexity bound is derived, which is $\mathrm{O}\left(2\sqrt{n}\log\frac{n\mu^0}{\varepsilon}\right)$.

Keywords: horizontal linear complementarity problem, interior-point algorithm, full-Newton step, complexity bound.

1. Introduction

A monotone horizontal linear complementarity problem (LCP) is to find a pair $x, s \in R^n$ such that

$$Mx + Ns = q, \quad xs = 0, \quad x, s \geq 0. \tag{1}$$

where $q \in R^m$ and $M, N \in R^{m\times n}$, moreover M and N have the column monotonic property, i.e., for any $u, w \in R^n$

$$Mu + Nw = 0 \Rightarrow u^T w \geq 0. \tag{2}$$

The formulation (1) includes linear and convex quadratic programming problems expressed by their optimality conditions in their usual format. Properties of this formulation are described in [1], where the rank $r([M, N]) = n$ has been proved under the monotonic hypothesis.

There are a variety of solution approaches for LCP which have been studied intensively. Among them, the interior-point methods (IPMs) gained much attention than other methods. Due to the close connection between LCP and linear and convex quadratic programming problems, some IPMs for linear and convex quadratic programming problems have been extended to LCP. For instance, Gonzaga et al. [2,3] studied the largest step path following algorithm for LCP and showed that the fast convergence of the simplified largest step path following algorithm. Huang et al. [4] proposed a high-order feasible IPM for LCP with $\mathrm{O}\left(\sqrt{n}\log\frac{\varepsilon^0}{\varepsilon}\right)$ iterations. Monteiro et al. [5] studied the limiting behavior of the derivatives of certain trajectories associated with the monotone LCP. Zhang [6] presented a class of infeasible IPMs for LCP and showed that the algorithm has $\mathrm{O}\left(n^2\log\frac{1}{\varepsilon}\right)$ under some mild assumptions. Some other relevant references can be found in [4,7,8].

In this paper, we give a full-Newton step IPM for LCP, the algorithm uses a modified Newton direction, which enjoys the nice property of quadratically convergent in the small neighborhood of central path. We

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derive the complexity bound for the algorithm, and the complexity result is the best-known for LCP.

The paper is organized as follows: In Section 2, the basic concepts of IPMs are given, which include central path and the classic Newton direction. In section 3, we give a scaled version of the classic Newton direction, and from which we give a modified Newton direction. The generic algorithm is described in section 4. In section 5, the properties of full-Newton step are analyzed, which include the estimation of the upper bound for dual gap and the increase of the proximity after one full-Newton step, the decrease of proximity after the parameter update is also given in this section. At the end of this section, we give a complexity result for the full-Newton step IPM. Section 6 gives a simple numerical example. Section 7 ends the paper with a conclusion.

Some notations used throughout the paper are as follows. $\|\cdot\|$ and $\|\cdot\|_\infty$ denotes the 2-norm and ∞-norm of a vector respectively. For any $x=(x_1,x_2,\ldots,x_n)^T\in R^n$, $x_{\min}$ denotes the smallest value of the components of x and xs denotes the componentwise (or Hadamard) product of the vectors x and s .

2. Preliminary

We assume the following hypotheses hold: the existence of an interior feasible solution and the existence of a strictly complementarity optimal solution.

2.1. The central path

The basic idea of the IPM is to replace the second equation in (1) by the parameterized equation $xs=\mu e$, with $\mu>0$. Thus we consider the system

$$Mx+Ns=q,\quad xs=\mu e,\quad x,s\ge 0. \tag{3}$$

The parameterized system (3) has a unique solution for each $\mu>0$. This solution is denoted as $(x(\mu),s(\mu))$ and is called the μ -center of LCP. The set of μ -centers (with μ running through all positive real numbers) gives a homotopy path, which is called the central path of LCP. If $\mu\to 0$, then the limit of the central path exists and since the limit points satisfy the complementarity condition $xs=0$, the limit yields an optimal solution for LCP, see [6,9].

2.2. The classic Newton direction

The search directions used in all primal-dual IPMs were computed from the linear system

$$\begin{aligned} M(x+\Delta x)+N(s+\Delta s) &= q\\ (x+\Delta x)(s+\Delta s) &= \mu e \end{aligned} \tag{4}$$

Neglecting the quadratic term $\Delta x\Delta s$ in the left-hand side expression of the second equation, we obtain the so-called classic Newton direction Δx and Δs .

$$\begin{aligned} M\Delta x+N\Delta s &= 0\\ x\Delta s+s\Delta x &= \mu e-xs \end{aligned} \tag{5}$$

The unique solution of the system (5) is guaranteed by Lemma 4.1 in [9].

3. New search direction

To describe the ideas underlying this paper, we need to consider a scaled version of the system (5) that defines the search directions.

3.1. A scaled-Newton direction

Now we introduce the scaled vector v and the scaled search directions d_x and d_s according to

$$v=\sqrt{\frac{xs}{\mu}}\ \text{ and }\ d_x=\frac{v\Delta x}{x},\ d_s=\frac{v\Delta s}{s} \tag{6}$$

According to (6), the system (5) can be rewritten as

$$\begin{aligned} \overline{M} d_x + \overline{N} d_s &= 0 \\ d_x + d_s &= v^{-1} - v \end{aligned} \tag{7}$$

where

$$\overline{M} = MV^{-1}X\ ,\ \overline{N} = NV^{-1}S\ ,\ V = diag(v)\ ,\ X = diag(x) \text{ and } S = diag(s) \tag{8}$$

The search directions d_x and d_s are obtained by solving (7), so Δx and Δs can be computed via (6).

3.2. A modified Newton direction

Rearrange the second equation in (7), we obtain

$$v^2 + v(d_x + d_s) = e$$

taking square root at both side the equation, one has

$$\left(v^2 + v(d_x + d_s)\right)^{\frac{1}{2}} = e$$

Using Taylor series at v^2, which gives the following equation

$$v + \frac{1}{2}(d_x + d_s) = e$$

rearrange the above equation and substitute the second equation in (7), one obtain the new Newton system

$$\begin{aligned} \overline{M} d_x + \overline{N} d_s &= 0 \\ d_x + d_s &= 2(e - v) \end{aligned} \tag{9}$$

Once system (9) is solved, Δx and Δs can be computed via (6) too.

It should be mentioned that the idea of equivalent algebraic transformation above was also proposed by [10]. There, the power transformation $\varphi(t) = \sqrt{t}$ was focused on xs space.

Moreover, we define a proximity measure to the central path by

$$\sigma(x, s; \mu) = \sigma(v) = \|e - v\| \tag{10}$$

Let us introduce the notation

$$p_v = d_x + d_s\ ,\quad q_v = d_x - d_s$$

then we have

$$d_x = \frac{p_v + q_v}{2}\ ,\quad d_s = \frac{p_v - q_v}{2} \text{ and } d_x d_s = \frac{p_v^2 - q_v^2}{4}. \tag{11}$$

We compare the norm of p_v and q_v by the following lemma.

Lemma 3.1. One has

$$\|q_v\| \le \|p_v\|.$$

Proof: By the monotonicity property, see (2), one has

$$M\Delta x + N\Delta s = 0 \Rightarrow \Delta x^T \Delta s \ge 0 \Rightarrow d_x^T d_s \ge 0. \tag{12}$$

Thus

$$\|q_v\|^2 = e^T (d_x - d_s)^2 = e^T (d_x + d_s)^2 - 4 d_x^T d_s \le \left\|p_v^{\ 2}\right\|,$$

the result follows.

4. Generic Primal-Dual IPMs for LCP

We investigate a full-Newton step algorithm using the modified Newton direction. It is assumed that we are given a positive primal-dual pair $x^0, s^0 > 0$ and $\mu^0 > 0$ such that $\left(x^0, s^0\right)$ is close to the μ^0-center in the sense of the proximity measure $\sigma\left(x^0, s^0; \mu^0\right)$.

In the algorithm Δx and Δs denote the modified Newton step, as defined before.

Generic primal-dual IPMs for LCP

Input:

A threshold parameter $\tau > 0$; an accuracy parameter $\varepsilon > 0$;

a fixed barrier update parameter $\theta : 0 < \theta < 1$.

a strictly feasible $\left(x^0, s^0\right)$ and $\mu^0 = \left(x^0\right)^T s^0 / n$ such that $\sigma\left(x^0, s^0; \mu^0\right) \le \tau$

begin

$x := x^0; s := s^0; \mu := \mu^0$

while $x^T s \ge \varepsilon$ do

begin

$x := x + \Delta x$; $s := s + \Delta s$

$\mu := (1-\theta)\mu$

end

end

---

5. Complexity analysis

In this section, we derive the complexity bound for the IPM based on the modified Newton direction.

5.1. Feasibility condition

Lemma5.1. If $\sigma(v) < 1$, and denote $x^+ = x + \Delta x$ and $s^+ = s + \Delta s$, then the iterate $\left(x^+, s^+\right)$ is strictly feasible.

Proof: For each $0 \le \alpha \le 1$, let introduce the notation $x(\alpha) = x + \alpha\Delta x$ and $s(\alpha) = s + \alpha\Delta s$. Then we have

$$x(\alpha)s\left(x(\alpha) = x + \alpha\Delta x\right) = xs + \alpha\left(s\Delta x + x\Delta s\right) + \beta^2 \Delta x \Delta s ,$$

by (6), we obtain

$$\frac{x(\alpha)s(\alpha)}{\mu} = v^2 + \alpha v\left(d_x + d_s\right) + \alpha^2 d_x d_s$$

Furthermore, from (11) we get

$$\frac{x(\alpha)s(\alpha)}{\mu} = (1-\alpha)v^2 + \alpha\left(v^2 + vp_v\right) + \alpha^2 \frac{p_v^2 - q_v^2}{4}$$

Using the second equation of (9) we find that

$$v^2 + vp_v = 2v - v^2 = e - (e-v)^2 = e - \frac{p_v^2}{4}$$

and this relation leads to

$$\frac{x(\alpha)s(\alpha)}{\mu} = (1-\alpha)v^2 + \alpha\left(e - (1-\alpha)\frac{p_v^2}{4} - \alpha\frac{q_v^2}{4}\right) \tag{13}$$

Evidently, the inequality $x(\alpha)s(\alpha) > 0$ is satisfied if

$$\left\|(1-\alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right\|_\infty < 1$$

Since

$$\left\|(1-\alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right\|_\infty \le (1-\alpha)\left\|\frac{p_v^2}{4}\right\|_\infty + \alpha\left\|\frac{q_v^2}{4}\right\|_\infty \le (1-\alpha)\left\|\frac{p_v^2}{4}\right\| + \alpha\left\|\frac{q_v^2}{4}\right\| \tag{14}$$

By Lemma 3.1, one has

$$(1-\alpha)\left\|\frac{p_v^2}{4}\right\| + \alpha\left\|\frac{q_v^2}{4}\right\| \le \frac{\left\|p_v^2\right\|}{4} \le \frac{\left\|p_v\right\|^2}{4} = \sigma(v)^2 < 1 \tag{15}$$

Hence, for each $0 \le \alpha \le 1$, we have $x(\alpha)s(\alpha) > 0$. Consequently, the sign of the continuous function of α, $x(\alpha)$ and $s(\alpha)$ remains the same for every $0 \le \alpha \le 1$. Hence $x(0) = x > 0$ and $s(0) = s > 0$ yields $x(1) = x^+ > 0$ and $s(1) = s^+ > 0$. This completes the proof.

5.2. Dual gap

In the following lemma, we analyze the effect of the full-Newton step on the dual gap.

Lemma 5.2. Let $x^+ = x + \Delta x$ and $s^+ = s + \Delta s$. Then we have

$$\left(x^+\right)^T s^+ \le \mu n$$

Proof: Observe that making the substitution $\alpha = 1$ in (13) that equation becomes

$$\frac{x^+ s^+}{\mu} = e - \frac{q_v^2}{4} \tag{16}$$

and using this equation we obtain

$$\left(x^+\right)^T s^+ = e^T\left(x^+ s^+\right) = \mu\left(e^T e - \frac{e^T q_v^2}{4}\right) = \mu\left(n - \frac{\left\|q_v\right\|^2}{4}\right) \le \mu n\,.$$

This implies the lemma.

5.3. Quadratic convergence

We first estimate the increase of the proximity after one full-Newton step.

Theorem 5.3. Let $\sigma^+ = \sigma\left(x^+, s^+; \mu\right)$ and $v^+ = \sqrt{\frac{x^+ s^+}{\mu}}$, If $\sigma(v) \le 1$. Then

$$\sigma^+ \le \frac{\sigma^2}{1 + \sqrt{1 - \sigma^2}}\,.$$

Hence, the full-Newton step is quadratically convergent.

Proof. We deduce from Lemma 5.1 that the full-Newton step is strictly feasible, thus $x^+ > 0$ and $s^+ > 0$. Observe that by (16), we have

$$\left(v^+\right)^2 = e - \frac{q_v^2}{4} \tag{17}$$

Thus

$$v^{+}{}_{\min} = \sqrt{1-\frac{\|q_v^2\|_\infty}{4}} \geq \sqrt{1-\frac{\|q_v^2\|}{4}} \geq \sqrt{1-\frac{\|p_v^2\|}{4}} \geq \sqrt{1-\frac{\|p_v\|^2}{4}} = \sqrt{1-\sigma^2} \tag{18}$$

Furthermore, (17) and (18) lead to

$$\sigma\left(v^+\right) = \left\|\frac{e-\left(v^+\right)^2}{e+v^+}\right\| \leq \frac{1}{1+v_{\min}^+}\left\|e-\left(v^+\right)^2\right\| \leq \frac{1}{1+\sqrt{1-\sigma^2}}\left\|\frac{q_v^2}{4}\right\| \leq \frac{\sigma^2}{1+\sqrt{1-\sigma^2}},$$

the last inequality follows from the fact that

$$\|q_v^2\| \leq \|q_v\|^2 \leq \|p_v\|^2$$

Consequently, we have $\sigma\left(v^+\right) \leq \sigma\left(v\right)^2$, and this implies the lemma.

5.4. Proximity changes after one iteration

After the full-Newton step, a μ-update will arise the changes of μ-centre. We assume that μ is reduced by the factor $(1-\theta)$ in each iterate.

Lemma 5.4. Let $\sigma = \sigma(x,s;\mu) < 1$ and $\mu^+ = (1-\theta)\mu$, where $0 \leq \theta \leq 1$. We have

$$\sigma\left(x^+,s^+;\mu^+\right) \leq \frac{\theta\sqrt{n}+\sigma^2}{1-\theta+\sqrt{(1-\theta)\left(1-\sigma^2\right)}}.$$

Proof. From (17) and (18) we deduce

$$\begin{aligned}
\sigma\left(x^+,s^+;\mu^+\right) &= \left\|e-\sqrt{\frac{x^+s^+}{\mu^+}}\right\| \\
&= \frac{1}{\sqrt{1-\theta}}\left\|\sqrt{1-\theta}e-v^+\right\| = \frac{1}{\sqrt{1-\theta}}\left\|\frac{(1-\theta)e-\left(v^+\right)^2}{\sqrt{1-\theta}e+v^+}\right\| \\
&\leq \frac{1}{\sqrt{1-\theta}\left(\sqrt{1-\theta}+\min\left(v^+\right)\right)}\left\|-\theta e+\frac{q_v^2}{4}\right\| \leq \frac{1}{1-\theta+\sqrt{(1-\theta)\left(1-\sigma^2\right)}}\left(\theta\sqrt{n}+\frac{\|q_v^2\|}{4}\right) \\
&\leq \frac{\theta\sqrt{n}+\sigma^2}{1-\theta+\sqrt{(1-\theta)\left(1-\sigma^2\right)}},
\end{aligned}$$

which completes the proof.

5.5. Fixing the parameter

We want to find a update parameter θ and a threshold parameter τ. Thus, after each iterate of the algorithm the property $\sigma(x,s;\mu) \leq \tau$ is maintained, and hence the algorithm is well defined.

By Lemma 5.4, it suffices if

$$\frac{\theta\sqrt{n}+\sigma^2}{1-\theta+\sqrt{(1-\theta)\left(1-\sigma^2\right)}} \leq \tau \tag{19}$$

The left side of the inequality (19) is monotonically increasing according to σ, it certainly suffices if

$$\frac{\theta\sqrt{n}+\tau^2}{1-\theta+\sqrt{(1-\theta)\left(1-\tau^2\right)}} \leq \tau \tag{20}$$

At this state, if we set $\tau = \frac{1}{2}$ and assume that $n \geq 4$, it suffices if

$$\theta = 1/2\sqrt{n} \qquad (21)$$

that the inequality (20) certainly establish. Thus the full-Newton step interior-point algorithm well defined for LCP.

5.6. Complexity bound

Lemma 5.5. If the barrier parameter μ has the initial value μ^0 and is repeatedly multiplied by $(1-\theta)$, with $0 \leq \theta \leq 1$, then after at most

$$\left\lceil \frac{1}{\theta} \log \frac{n\mu^0}{\varepsilon} \right\rceil$$

iterations, we have $x^T s \leq \varepsilon$.

Proof. At the initial point, one has $(x^0)^T s^0 = n\mu^0$, after one iterate, by Lemma 5.2, the dual gap

$$(x^1)^T s^1 = (1-\theta) n\mu^0,$$

thus, after k iterates, the dual gap satisfies

$$(x^k)^T s^k = (1-\theta)^k n\mu^0.$$

So, it suffices if

$$(1-\theta)^k n\mu^0 \leq \varepsilon.$$

Taking logarithm gives

$$k \log(1-\theta) + \log n + \log \mu^0 \leq \log \varepsilon.$$

Since

$$\log(1-\theta) \leq -\theta.$$

It certainly suffices if

$$-k\theta + \log n + \log \mu^0 \leq \log \varepsilon,$$

this gives

$$k \geq \frac{1}{\theta} \log \frac{n\mu^0}{\varepsilon},$$

this completes the proof.

The following theorem holds trivially.

Theorem 5.6.Setting $\tau = \frac{1}{2}$ and $\theta = 1/2\sqrt{n}$, the initial dual gap is $(x^0)^T s^0 = n\mu^0$, the full-Newton step primal-dual IPMs for LCP has the complexity bound

$$\mathrm{O}\left(2\sqrt{n} \log \frac{n\mu^0}{\varepsilon}\right)$$

Proof. Substitute (21) in Lemma 5.5, the result follows.

6. A simple numerical experiment

In general, though there exists $(x^0, s^0) > 0$ for the LCP problem is strictly feasible, we don't know the value of $\left(x^0, s^0\right)$. Thus we should modify the system (5) as follows

$$\begin{aligned} M\Delta x + N\Delta s &= q - Mx - Ns \\ x\Delta s + s\Delta x &= 2\left(\mu\left(xs\right)^{\frac{1}{2}} - xs\right) \end{aligned} \tag{22}$$

We consider the following example:

$$M = \begin{pmatrix} 0.0368 & 0.0188 & 0.0920 & 0.0211 & 0.0332 & 0.0162 \\ 0.0188 & 0.0393 & 0.0634 & 0.0176 & 0.0300 & 0.0248 \\ 0.0920 & 0.0634 & 0.4293 & 0.0617 & 0.1355 & 0.1124 \\ 0.0211 & 0.0176 & 0.0617 & 0.0203 & 0.0239 & 0.0107 \\ 0.0332 & 0.0300 & 0.1355 & 0.0239 & 0.0513 & 0.0480 \\ 0.0162 & 0.1248 & 0.0124 & 0.0107 & 0.0480 & 0.0824 \end{pmatrix},\ q = \begin{pmatrix} 0.1630 \\ -0.2820 \\ 0.4500 \\ -0.3560 \\ 0.2420 \\ -0.2489 \end{pmatrix}$$

and $N = -E$.

Without loss of generality, we choose $x^0 = s^0 = e$ as the initial point. Setting $\varepsilon = 10^{-8}$, $\tau = \frac{1}{2}$ and $\theta = 1/2\sqrt{n}$. After 90 iterates, an optimal solution of the example is given by

$$x^* = \left(0.4169, 0.0000, 0.0000, 0.0000, 4.4476, 0.0000\right)^T$$

and

$$s^* = \left(0.0000, 0.4233, 0.1910, 0.4711, 0.0000, 0.4691\right)^T.$$

7. Conclusions

In this paper, we gave a full-Newton step IPM for LCP, the method has the quadratically convergent property in the small neighborhood of central path. The complexity bound is the best-known results for LCP.

Although the simple numerical example shows that the algorithm approximates the optimal solution after finite number of iterations, from a practical perspective they are not efficient. This is because they always perform according to their worst-case theoretical complexity bounds.

Our further research include to find an IPM for LCP with damped-Newton step and to get the same complexity bound as that with full-Newton step.

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