ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 6, No. 1, 2011, *pp*. 003-014

Exact Solutions for Nonlinear PDEs with the Variable Coefficients in Mathematical Physics

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(*Received October 19, 2010, accepted December 20, 2010*)

Abstract. In this article, we construct the exact solutions for nonlinear partial differential equations with the variable coefficients in the mathematical physics via the generalized time- dependent variable coefficients KdV-mKdV equation and the coupled modified KdV equations with non-uniformity terms by using a generalized (G'/G) - expansion method with the variable coefficients, where G satisfies the Jacobi elliptic equation. Many of the exact solutions in terms of Jacobi elliptic functions are obtained. The proposed method is reliable and effective and gives more new exact solutions.

Keywords: A generalized (G'/G)- expansion method with the variable coefficients, The generalized time- dependent variable coefficients KdV- mKdV equation, The coupled modified KdV equations with non-uniformity terms, The Jacobi elliptic functions.

AMS Subject Classifications: 35K99; 35P05; 35P99.

1. Introduction

In recent years, the exact solutions of nonlinear PDEs have been investigated by many authors (see for example [1-36]) who are interested in nonlinear physical phenomena. Many powerful different methods have been presented by those authors. For integrable nonlinear differential equations, the inverse scattering transform method [3], the Hirota method [8], the truncated Painleve expansion method [23,30], the Backlund transform method [14,15] and the exp- function method [5,32] are used to find the exact solutions. Among non-integrable nonlinear differential equations there is a wide class of equations that referred to as partially integrable, because these equations become integrable for some values of their parameters. There are many different methods used to find the exact solutions of these equations. The most famous algorithms are the tanh- function method [1,7,33], the Jacobi elliptic function expansion method [6,11,13,26,27], F- expansion method [2,18,31] and the generalized Riccati equation [17]. There are other methods which can be found in [10,12,17-25].

Wang et.al.[22] have introduced a simple method which is called the (G'/G)- expansion method to look for traveling wave solutions of nonlinear evolution equations, where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where λ and μ are arbitrary constants. For further references see the articles [4,28,34,35]. Recently, Zayed [29] introduced an alternative approach, which is called a generalized (G'/G)- expansion method, where $G = G(\xi)$ satisfies the Jacobi elliptic equation $[G'(\xi)]^2 = e_2 G^4(\xi) + e_1 G^2(\xi) + e_0$, $\xi = x - Vt$ and e_0, e_1, e_2, V are arbitrary constants while $' = \frac{d}{d\xi}$. The main objective of this article is using the generalized (G'/G)- expansion method with the variable coefficients to construct the exact solutions for nonlinear evolution equations in the mathematical physics via the generalized time- dependent variable coefficients KdV- mKdV equation and the coupled modified KdV equations with non-uniformity terms, where G satisfies the Jacobi elliptic

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equation. Many exact solutions in terms of Jacobi elliptic functions are obtained.

2. Description of a generalized (G'/G)- expansion method with the variable coefficients

Suppose we have the following nonlinear partial differential equation

$$F(u,u_t,u_x,u_{tt},u_{xx},u_{xt},..)=0, \tag{2.1}$$

where $u=u(x,t)$ is an unknown function, F is a polynomial in $u=u(x,t)$ and its various partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of a generalized (G'/G)- expansion method

Step 1. Suppose the solution of Eq.(2.1) can be expressed by a polynomial in (G'/G) as follows

$$u(\xi)=\sum_{i=0}^{n}\alpha_i(t)\left(\frac{G'(\xi)}{G(\xi)}\right)^i , \tag{2.2}$$

where $\xi = p(t)x+q(t)$ and $\alpha_i(t)(\mathrm{i}=0,1,2,..,\mathrm{n}),\mathrm{p(t)},\mathrm{q(t)}$ are arbitrary functions of t to be determined later while $\mathrm{G}=\mathrm{G}(\xi)$ satisfies the following Jacobi elliptic equation:

$$\mathrm{G}''(\xi)=2e_2G^3(\xi)+e_1G(\xi) \tag{2.3}$$

or

$$[\mathrm{G}'(\xi)]^2 = e_2G^4(\xi)+e_1G^2(\xi)+\mathrm{e}_0$$

and e_2,e_1,e_0 are arbitrary constants.

Step 2. The positive integer " n " can be determined by considering the homogeneous balance between the highest order partial derivative and the nonlinear terms appearing in Eq. (2.1). Therefore, we can get the value of n in (2.2).

Step 3. Substituting (2.2) into (2.1) with the conditions (2.4), we obtain polynomial in $x^i(G'(\xi)^jG^k(\xi)$, $(i,j=0,1,k=0,\pm1,\pm2,...)$. Equating each coefficient of the resulted polynomial to zero, yields a set of ordinary differential equations $\alpha_i(t)$, $(i=0,1,...,n)$, $p(t)$ and $q(t)$.

Step 4. Solving the obtained system of the differential equations with the aid of Maple or mathematica to calculate $\alpha_i(t)$, $(i=0,1,...,n)$, $p(t)$ and $q(t)$.

Step 5. Since the general solutions of Eq. (2.3) have been well known for us (see Appendix A), then substituting $\alpha_i(t)$,($i=0,1,...,n)$, $p(t)$,$q(t)$ and the general solution of Eq. (2.3) into (2.2) we have many new exact solutions of the nonlinear partial differential equation (2.1).

3. Some applications of the generalized (G'/G)- expansion method

In this section, we apply the generalized (G'/G) - expansion method with the variable coefficients to construct the exact solutions for the generalized time- dependent variable coefficients KdV- mKdV equation and the coupled modified KdV equations with non-uniformity terms which are very important nonlinear evolution equations in mathematical physics.

Example 1. The generalized time- dependent variable coefficients KdV- mKdV equation

We start with the generalized time- dependent variable coefficients KdV- mKdV equation[16] in the following form:

$$u_t-6f_0(t)uu_x-6f_1(t)u^2u_x+f_2(t)u_{xxx}-f_3(t)u_x+f_4(t)(Au+xu_x)=0, \tag{3.1}$$

where $f_0(t)$, $f_1(t)$, $f_2(t)$, $f_3(t)$, $f_4(t)$ are arbitrary functions of t and A is a constant. This equation describes the propagation of weakly nonlinear waves in a KdV- typed medium that is characterized by a varying dispersion and nonlinear coefficients. Suppose that the solution of Eq. (3.1) can be expressed by a polynomial in (G'/G) as Eq.(2.2). Considering the homogeneous balance between the highest order partial derivative u_{xxx} and the nonlinear term uu_x in (3.1), we deduce that n = 2. Thus, the exact solution of Eq.(3.1) has the following form:

$$u(\xi)=\alpha_2(t)\left(\frac{G'(\xi)}{G(\xi)}\right)^2+\alpha_1(t)\left(\frac{G'(\xi)}{G(\xi)}\right)^2+\alpha_0(t), \tag{3.2}$$

where $\xi = p(t)x+q(t)$ and $\alpha_i(t),(i=0,1,2), p(t), q(t)$ are arbitrary functions of t to be determined later. Substituting (3.2) along with Eq. (2.4) into (3.1) and collecting all terms with the same power of $x^i(G'(\xi))^j G^k(\xi)$, $(i, j=0,1, k=0,\pm1,\pm2,...)$. Equating the coefficients of this polynomial to zero, we get the following system of the ordinary differential equations:

$$30 f_1\alpha_2^2 e_0^3\alpha_1 p=0,$$

$$6 f_1\alpha_1^3 e_0^2 p+36 f_1\alpha_0\alpha_1\alpha_2 e_0^2 p+18 f_0\alpha_1\alpha_2 e_0^2 p-6 f_2\alpha_1 p^3 e_0^2+60 f_1\alpha_2^2 e_0^2 e_1\alpha_1 p=0,$$

$$-\alpha_1 e_0\frac{dp}{dt}-f_4\alpha_1 p e_0=0,$$

$$f_4 A\alpha_2 e_0+6 f_1\alpha_0^2\alpha_1 p e_0+6 f_1\alpha_1^3 e_1 p e_0+e_0\frac{d\alpha_2}{dt}+18 f_0\alpha_1\alpha_2 e_0 p e_1-\alpha_1 e_0\frac{dq}{dt}+36 f_1\alpha_0\alpha_1\alpha_2 e_0 p e_1$$
$$+30 f_1\alpha_2^2 e_0^2 e_2\alpha_1 p+30 f_1\alpha_2^2 e_1^2\alpha_1 p e_0+f_3\alpha_1 p e_0+6 f_0\alpha_0\alpha_1 p e_0-4 f_2\alpha_1 p^3 e_0 e_1=0,$$

$$\frac{d\alpha_0}{dt}+f_4 A\alpha_0+\frac{d\alpha_2}{dt}e_1+f_4 A\alpha_2 e_1=0,$$

$$-6 f_1\alpha_0^2\alpha_1 e_2 p-6 f_0\alpha_0\alpha_1 e_2 p-30 f_1\alpha_2^2 e_1^2\alpha_1 e_2 p-36 f_1\alpha_0\alpha_1\alpha_2 e_1 p e_2+e_2\frac{d\alpha_2}{dt}-18 f_0\alpha_1\alpha_2 e_1 p e_2$$
$$-f_3\alpha_1 e_2 p-6 f_1\alpha_1^3 e_1 e_2 p+f_4 A\alpha_2 e_2-30 f_1\alpha_2^2 e_0 e_2^2\alpha_1 p+4 f_2\alpha_1 p^3 e_2 e_1+\alpha_1 e_2\frac{dq}{dt}=0,$$

$$\alpha_1 e_2\frac{dp}{dt}+f_4\alpha_1 e_2 p=0,$$

$$6 f_2\alpha_1 p^3 e_2^2-60 f_1\alpha_2^2 e_1 e_2^2\alpha_1 p-6 f_1\alpha_1^3 e_2^2 p-36 f_1\alpha_0\alpha_1\alpha_2 e_2^2 p-18 f_0\alpha_1\alpha_2 e_2^2 p=0,$$

$$-30 f_1\alpha_2^2\alpha_1 e_2^3 p=0;$$

$$-12 f_1\alpha_2^3 e_2^3 p=0,$$

$$-24 f_1\alpha_0\alpha_2^2 e_2^2 p-24 f_1\alpha_2^3 e_1 e_2^2 p-12 f_0\alpha_2^2 e_2^2 p+24 f_2\alpha_2 e_2^2 p^3-24 f_1\alpha_2\alpha_1^2 e_2^2 p=0,$$

$$-24 f_1\alpha_0\alpha_2^2 e_1 e_2 p-6 f_0\alpha_1^2 e_2 p-12 f_0\alpha_2^2 e_1 e_2 p-12 f_1\alpha_0^2\alpha_2 e_2^2 p+8 f_2\alpha_2 p^3 e_1 e_2$$
$$-12 f_0\alpha_0\alpha_2 e_2 p-12 f_1\alpha_2^3 e_0 e_2^2 p-2 f_3\alpha_2 e_2 p+2\alpha_2 e_2\frac{dq}{dt}-12 f_1\alpha_2^3 e_2 e_1^2 p-12 f_1\alpha_0\alpha_1^2 e_2 p$$
$$-24 f_1\alpha_2\alpha_1^2 e_2 e_1 p=0,$$

$$2 f_4\alpha_2 e_2 p+2\alpha_2 e_2\frac{dp}{dt}=0,$$

$$\frac{d\alpha_1}{dt}+f_4 A\alpha_1=0,$$

$$12 f_1\alpha_2^3 e_0^3 p=0,$$

$$24 f_1\alpha_1^2 e_0^2\alpha_2 p+24 f_1\alpha_2^3 e_0^2 e_1 p+24 f_1\alpha_0\alpha_2^2 e_0^2 p+12 f_0\alpha_2^2 e_0^2 p-24 f_2\alpha_2 e_0^2 p^3=0,$$

$$-2\alpha_2 e_0\frac{dp}{dt}-2 f_4\alpha_2 e_0 p=0,$$

$$12 f_1\alpha_2^3 e_1^2 e_0 p + 12 f_1\alpha_2^3 e_0^2 e_2 p + 6 f_0\alpha_1^2 e_0 p + 12 f_1\alpha_0\alpha_1^2 e_0 p + 12 f_0\alpha_2^2 e_1 e_0 p + 2 f_3\alpha_2 e_0 p$$
$$-8 f_2\alpha_2 e_0 e_1 p^3 + 24 f_1\alpha_1^2 e_1\alpha_2 e_0 p + 12 f_0\alpha_0\alpha_2 e_0 p - 2\alpha_2 e_0\frac{dq}{dt} + 24 f_1\alpha_0\alpha_2^2 e_0 e_1 p \tag{3.3}$$
$$+12 f_1\alpha_2\alpha_0^2 e_0 p = 0.$$

We can solve the above system (3.3) with the aid of Maple or Mathematica to obtain the following sets of solutions:

Case 1.

$$\alpha_2(t) = A_1 e^{-\int f_4(t)dt}, \qquad \alpha_1(t) = 0, \qquad \alpha_0(t) = A_3 e^{-A\int f_4(t)dt},$$
$$f_2(t) = \frac{f_0(t) A_2 e^{(2-A)\int f_4(t)dt}}{2A_1^2}, \qquad p(t) = A_1 e^{-\int f_4(t)dt}, \tag{3.4}$$
$$q(t) = \int\left[A_1 f_3(t) e^{-\int f_4(t)dt} + 6(\frac{2A_2 e_1}{3} + A_3) A_1 f_0(t) e^{-2\int f_4(t)dt}\right]dt, \qquad f_1(t) = 0,$$

where $f_0(t), f_3(t), f_4(t)$ are arbitrary functions of t and e_1, A_1, A_2, A_3 are arbitrary constants.

Case 2.

$$\alpha_1(t) = A_1 e^{-A\int f_4(t)dt}, \qquad \alpha_2(t) = 0, \qquad \alpha_0(t) = A_3 A_1 e^{-A\int f_4(t)dt},$$
$$f_2(t) = \frac{f_1(t) A_1{}^{2(A-1)/A}}{2A_2^2} e^{-2(A-1)\int f_4(t)dt}, \qquad p(t) = A_2 A_1{}^{1/A} e^{-\int f_4(t)dt},$$
$$f_0(t) = -2 f_1(t) A_3 A_1 e^{-A\int f_4(t)dt}, \tag{3.5}$$
$$q(t) = \int\left[A_2 A_1{}^{1/A} f_3(t) e^{-\int f_4(t)dt} - 6(-\frac{e_1}{3} + A_3^2) A_2 A_1{}^{(2A+1)/A} f_1(t) e^{-(2A+1)\int f_4(t)dt}\right]dt,$$

where $f_1(t), f_3(t), f_4(t)$ are arbitrary functions of t and e_1, A_1, A_2, A_3 are arbitrary constants.

Note that, there are other cases which are omitted here. Since the solutions obtained here are so many, we just list some of the exact solutions corresponding to case 1 to illustrate the effectiveness of the generalized $(G'/G)-$ expansion method with the variable coefficients. Substituting (3.4) into (3.2) yields

$$u(\xi) = A_1 e^{-\int f_4(t)dt}\left(\frac{G'(\xi)}{G(\xi)}\right)^2 + A_3 e^{-A\int f_4(t)dt}, \tag{3.6}$$

where

$$\xi = A_1 x e^{-\int f_4(t)dt} + \int\left[A_1 f_3(t) e^{-\int f_4(t)dt} + 6(\frac{2A_2 e_1}{3} + A_3) A_1 f_0(t) e^{-2\int f_4(t)dt}\right]dt. \tag{3.7}$$

According to the appendix A, we have the following families of exact solutions:

Family 1. If $e_0 = 1, e_1 = -(m^2+1), e_2 = m^2$, then we get

$$u(\xi) = A_1 e^{-\int f_4(t)dt} cs^2(\xi) dn^2(\xi) + A_3 e^{-A\int f_4(t)dt}, \tag{3.8}$$

or

$$u(\xi) = A_1 e^{-\int f_4(t)dt} (1-m^2)^2 sc^2(\xi) nd^2(\xi) + A_3 e^{-A\int f_4(t)dt}, \tag{3.9}$$

where $\xi = A_1 x e^{-\int f_4(t)dt} + \int\left[A_1 f_3(t) e^{-\int f_4(t)dt} + 6[-\frac{2}{3}A_2(m^2+1) + A_3] A_1 f_0(t) e^{-2\int f_4(t)dt}\right]dt.$

Family 2. If $e_0 = 1-m^2, e_1 = 2m^2-1, e_2 = -m^2$, then we get

$$u(\xi)=A_1e^{-\int f_4(t)dt}sc^2(\xi)dn^2(\xi)+A_3e^{-A\int f_4(t)dt}, \tag{3.10}$$

where $\xi=A_1xe^{-\int f_4(t)dt}+\int\left[A_1f_3(t)e^{-\int f_4(t)dt}+6[\frac{2}{3}A_2(2m^2-1)+A_3]A_1f_0(t)e^{-2\int f_4(t)dt}\right]dt.$

Family 3. If $e_0=m^2-1, e_1=2-m^2, e_2=-1$, then we get

$$u(\xi)=A_1m^4e^{-\int f_4(t)dt}sd^2(\xi)cn^2(\xi)+A_3e^{-A\int f_4(t)dt}, \tag{3.11}$$

where $\xi=A_1xe^{-\int f_4(t)dt}+\int\left[A_1f_3(t)e^{-\int f_4(t)dt}+6[\frac{2}{3}A_2(2-m^2)+A_3]A_1f_0(t)e^{-2\int f_4(t)dt}\right]dt.$

Family 4. If $e_0=1-m^2, e_1=2-m^2, e_2=1$, then we get

$$u(\xi)=A_1e^{-\int f_4(t)dt}nc^2(\xi)ds^2(\xi)+A_3e^{-A\int f_4(t)dt}, \tag{3.12}$$

where $\xi=A_1xe^{-\int f_4(t)dt}+\int\left[A_1f_3(t)e^{-\int f_4(t)dt}+6[\frac{2}{3}A_2(2-m^2)+A_3]A_1f_0(t)e^{-2\int f_4(t)dt}\right]dt.$

Family 5. If $e_0=1, e_1=2m^2-1, e_2=m^2(m^2-1)$, then we get

$$u(\xi)=A_1e^{-\int f_4(t)dt}ns^2(\xi)cd^2(\xi)+A_3e^{-A\int f_4(t)dt}, \tag{3.13}$$

where $\xi=A_1xe^{-\int f_4(t)dt}+\int\left[A_1f_3(t)e^{-\int f_4(t)dt}+6[\frac{2}{3}A_2(2m^2-1)+A_3]A_1f_0(t)e^{-2\int f_4(t)dt}\right]dt.$

Family 6. If $e_0=\frac{1}{4}, e_1=\frac{1}{2}(1-2m^2), e_2=\frac{1}{4}$, then we get

$$u(\xi)=A_1e^{-\int f_4(t)dt}ds^2(\xi)+A_3e^{-A\int f_4(t)dt}, \tag{3.14}$$

where $\xi=A_1xe^{-\int f_4(t)dt}+\int\left[A_1f_3(t)e^{-\int f_4(t)dt}+6[\frac{2}{3}A_2(1-2m^2)+A_3]A_1f_0(t)e^{-2\int f_4(t)dt}\right]dt.$

Family 7. If $e_0=\frac{1}{4}(1-m^2), e_1=\frac{1}{2}(1+m^2), e_2=\frac{1}{4}(1-m^2)$, then we get

$$u(\xi)=A_1e^{-\int f_4(t)dt}dc^2(\xi)+A_3e^{-A\int f_4(t)dt}, \tag{3.15}$$

where $\xi=A_1xe^{-\int f_4(t)dt}+\int\left[A_1f_3(t)e^{-\int f_4(t)dt}+6[\frac{1}{3}A_2(1+m^2)+A_3]A_1f_0(t)e^{-2\int f_4(t)dt}\right]dt.$

Family 8. If $e_0=\frac{m^2}{4}, e_1=\frac{1}{2}(m^2-2), e_2=\frac{1}{4}$, then we get

$$u(\xi)=A_1m^4e^{-\int f_4(t)dt}cs^2(\xi)+A_3e^{-A\int f_4(t)dt}, \tag{3.16}$$

where $\xi=A_1xe^{-\int f_4(t)dt}+\int\left[A_1f_3(t)e^{-\int f_4(t)dt}+6[\frac{1}{3}A_2(m^2-2)+A_3]A_1f_0(t)e^{-2\int f_4(t)dt}\right]dt.$

Family 9. If $e_0=\frac{m^2}{4}, e_1=\frac{1}{2}(m^2-2), e_2=\frac{m^2}{4}$, then we get

$$u(\xi)=A_1e^{-\int f_4(t)dt}dn^2(\xi)+A_3e^{-A\int f_4(t)dt}, \tag{3.17}$$

where $\xi=A_1xe^{-\int f_4(t)dt}+\int\left[A_1f_3(t)e^{-\int f_4(t)dt}+6[\frac{1}{3}A_2(m^2-2)+A_3]A_1f_0(t)e^{-2\int f_4(t)dt}\right]dt.$

Similarly, we can write down the other families of exact solutions of Eq. (3.1) which are omitted for convenience.

Example 2. The coupled modified KdV equations with non-uniformity terms

In this section, we study the coupled modified KdV equations with non-uniformity terms, which is a case of non- isospectral coupled system [36] in the following form:

$$\begin{aligned} u_t + 6C_{12}uvu_x + u_{xxx} + \beta u + (\alpha + \beta x)u_x = 0, \\ v_t + 6C_{12}uvv_x + v_{xxx} + \beta v + (\alpha + \beta x)v_x = 0, \end{aligned} \tag{3.18}$$

where C12 is a positive constant parameter while α and β are arbitrary constants parameters.

Suppose that the solutions of Eqs. (3.18) can be expressed by polynomials in (G'/G) as follows

$$u(\xi) = \sum_{i=0}^{n} \alpha_i(t)\left(\frac{G'(\xi)}{G(\xi)}\right)^i, \tag{3.19}$$

and

$$v(\xi) = \sum_{i=0}^{m} \beta_i(t)\left(\frac{G'(\xi)}{G(\xi)}\right)^i, \tag{3.20}$$

where $\xi = p(t)x + q(t)$ and $\alpha_i(t), \beta_i(t)(\mathrm{i} = 0,1,2,..,\mathrm{n}), \mathrm{p(t)}, \mathrm{q(t)}$ are arbitrary functions of t to be determined later. Considering the homogeneous balance between the highest order partial derivatives u_{xxx} or v_{xxx} with the nonlinear terms uvu_x or uvv_x in (3.18) respectively, we deduce that $n + m = 2$. Suppose that $n = m = 1$, hence the exact solutions

to Eqs. (3.18) can be expressed as:

$$u(\xi) = \alpha_1(t)\left(\frac{G'(\xi)}{G(\xi)}\right) + \alpha_0(t), \tag{3.21}$$

and

$$v(\xi) = \beta_1(t)\left(\frac{G'(\xi)}{G(\xi)}\right) + \beta_0(t), \tag{3.22}$$

where $\beta_1(t)$, $\alpha_1(t)$, $\alpha_0(t)$ and $\beta_0(t)$ are arbitrary functions of t. Substituting (3.21) and (3.22) along with Eq. (2.4) into (3.18). Collecting all terms with the same power of $x^i (G'(\xi))^j G^k(\xi)$, $(i, j = 0,1, k = 0,\pm1,\pm2,...)$ and setting each coefficients of this polynomial to zero, we get a system of ordinary differential equations which can be solved by Maple or Mathematica. Thus we obtain the two sets of solutions as:

Case1.

$$\begin{aligned} &\alpha_1(t) = C_3 e^{-\beta t}, \qquad \alpha_0(t) = C_1 e^{-\beta t}, \qquad \beta_1(t) = -\frac{C_2 C_3}{C_1} e^{-\beta t}, \\ &p(t) = \pm\frac{C_3}{C_1}\sqrt{C_{12}C_1C_2}\, e^{-\beta t}, \qquad \beta_0(t) = C_2 e^{-\beta t}, \\ &q(t) = \pm\frac{C_3}{C_1^2}\sqrt{C_{12}C_1C_2}\left[\frac{2C_{12}C_1^2C_2}{\beta}e^{-3\beta t} - \frac{2C_{12}e_1C_2C_3^2}{3\beta}e^{-3\beta t} + \frac{\alpha}{\beta}e^{-\beta t}\right] + C_4. \end{aligned} \tag{3.23}$$

Case 2.

$$\begin{aligned}
&\alpha_1(t) = C_1 e^{-\beta t}, \qquad\qquad \beta_1(t) = C_2 e^{-\beta t},\\
&p(t) = \mp\sqrt{-C_{12}C_1C_2}\, e^{-\beta t},\\
&q(t) = \pm\sqrt{-C_{12}C_1C_2}\left[-\frac{2C_{12}C_1C_2e_1}{3\beta}e^{-3\beta t} - \frac{\alpha}{\beta}e^{-\beta t}\right] + C_3.\\
&\beta_0(t) = \alpha_0(t) = 0,
\end{aligned} \tag{3.24}$$

where C_1, C_2, C_3 and C_4 are arbitrary constants.

Note that, there are other cases which are omitted here. Since the solutions obtained here are so many, we just list some of the exact solutions corresponding to case 1 to illustrate the effectiveness of the generalized $(G'/G)-$ expansion method with the variable coefficients.

Substituting (3.23) into (3.21) and (3.22) yield

$$u(\xi) = C_3 e^{-\beta t}\left(\frac{G'(\xi)}{G(\xi)}\right) + C_1 e^{-\beta t}, \tag{3.25}$$

and

$$v(\xi) = -\frac{C_2C_3}{C_1} e^{-\beta t}\left(\frac{G'(\xi)}{G(\xi)}\right) + C_2 e^{-\beta t}, \tag{3.26}$$

where

$$\xi = \pm\frac{C_3}{C_1}\sqrt{C_{12}C_1C_2}\, e^{-\beta t} x \pm \frac{C_3}{C_1^2}\sqrt{C_{12}C_1C_2}\left[\frac{6C_{12}C_1^2C_2 - 2C_{12}e_1C_2C_3^2}{3\beta}e^{-3\beta t} + \frac{\alpha}{\beta}e^{-\beta t}\right] + C_4 \tag{3.27}$$

According to the appendix A, we have the following families of exact solutions:

Family 1. If $e_0 = 1, e_1 = -(m^2+1), e_2 = m^2$, then we get

$$u(\xi) = C_3 e^{-\beta t} cs(\xi)\, dn(\xi) + C_1 e^{-\beta t},$$

and

$$v(\xi) = -\frac{C_2C_3}{C_1} e^{-\beta t} cs(\xi)\; dn(\xi) + C_2 e^{-\beta t}, \tag{3.28}$$

or

$$u(\xi) = -(1-m^2)C_3 e^{-\beta t} sc(\xi)\, nd(\xi) + C_1 e^{-\beta t},$$

and

$$v(\xi) = \frac{(1-m^2)C_2C_3}{C_1} e^{-\beta t} sc(\xi)\, nd(\xi) + C_2 e^{-\beta t}, \tag{3.29}$$

where

$$\xi = \pm\frac{C_3}{C_1}\sqrt{C_{12}C_1C_2}\, e^{-\beta t} x \pm \frac{C_3}{C_1^2}\sqrt{C_{12}C_1C_2}\left[\frac{6C_{12}C_1^2C_2 + 2C_{12}C_2C_3^2(m^2+1)}{3\beta}e^{-3\beta t} + \frac{\alpha}{\beta}e^{-\beta t}\right] + C_4. \tag{3.30}$$

Family 2. If $e_0 = 1-m^2, e_1 = 2m^2-1, e_2 = -m^2$, then we get

$$u(\xi) = -C_3 e^{-\beta t} sc(\xi)\, dn(\xi) + C_1 e^{-\beta t},$$

and

$$v(\xi) = \frac{C_2C_3}{C_1} e^{-\beta t} sc(\xi)\; dn(\xi) + C_2 e^{-\beta t}, \tag{3.31}$$

Where

$$\xi=\pm\frac{C_3}{C_1}\sqrt{C_{12}C_1C_2}e^{-\beta t}x\pm\frac{C_3}{C_1^2}\sqrt{C_{12}C_1C_2}\left[\frac{6C_{12}C_1^2C_2-2C_{12}C_2C_3^2(2m^2-1)}{3\beta}e^{-3\beta t}+\frac{\alpha}{\beta}e^{-\beta t}\right]+C_4. \quad (3.32)$$

Family 3. If $e_0=m^2-1, e_1=2-m^2,\ e_2=-1$, then we get

$$u(\xi)=-m^2C_3e^{-\beta t}sd(\xi)\,cn(\xi)+C_1e^{-\beta t},$$

and

$$v(\xi)=\frac{C_2C_3}{C_1}e^{-\beta t}sd(\xi)\,cn(\xi)+C_2e^{-\beta t}, \quad (3.33)$$

Where

$$\xi=\pm\frac{C_3}{C_1}\sqrt{C_{12}C_1C_2}e^{-\beta t}x\pm\frac{C_3}{C_1^2}\sqrt{C_{12}C_1C_2}\left[\frac{6C_{12}C_1^2C_2-2C_{12}C_2C_3^2(2-m^2)}{3\beta}e^{-3\beta t}+\frac{\alpha}{\beta}e^{-\beta t}\right]+C_4. \quad (3.34)$$

Family 4. If $e_0=1-m^2, e_1=2-m^2,\ e_2=1$, then we get

$$u(\xi)=-C_3e^{-\beta t}nc(\xi)\,ds(\xi)+C_1e^{-\beta t},$$

and

$$v(\xi)=\frac{C_2C_3}{C_1}e^{-\beta t}nc(\xi)\,ds(\xi)+C_2e^{-\beta t}, \quad (3.35)$$

Where

$$\xi=\pm\frac{C_3}{C_1}\sqrt{C_{12}C_1C_2}e^{-\beta t}x\pm\frac{C_3}{C_1^2}\sqrt{C_{12}C_1C_2}\left[\frac{6C_{12}C_1^2C_2-2C_{12}C_2C_3^2(2-m^2)}{3\beta}e^{-3\beta t}+\frac{\alpha}{\beta}e^{-\beta t}\right]+C_4. \quad (3.36)$$

Family 5. If $e_0=1, e_1=2m^2-1,\ e_2=m^2(m^2-1)$, then we get

$$u(\xi)=C_3e^{-\beta t}ns(\xi)\,cd(\xi)+C_1e^{-\beta t},$$

and

$$v(\xi)=-\frac{C_2C_3}{C_1}e^{-\beta t}ns(\xi)\,cd(\xi)+C_2e^{-\beta t}, \quad (3.37)$$

Where

$$\xi=\pm\frac{C_3}{C_1}\sqrt{C_{12}C_1C_2}e^{-\beta t}x\pm\frac{C_3}{C_1^2}\sqrt{C_{12}C_1C_2}\left[\frac{6C_{12}C_1^2C_2-2C_{12}C_2C_3^2(2m^2-1)}{3\beta}e^{-3\beta t}+\frac{\alpha}{\beta}e^{-\beta t}\right]+C_4. \quad (3.38)$$

Family 6. If $e_0=\frac{1}{4}, e_1=\frac{1}{2}(1-2m^2),\ e_2=\frac{1}{4}$, then we get

$$u(\xi)=\mp C_3e^{-\beta t}ds(\xi)+C_1e^{-\beta t},$$

and

$$v(\xi)=\pm\frac{C_2C_3}{C_1}e^{-\beta t}ds(\xi)+C_2e^{-\beta t}, \quad (3.39)$$

where

$$\xi=\pm\frac{C_3}{C_1}\sqrt{C_{12}C_1C_2}e^{-\beta t}x\pm\frac{C_3}{C_1^2}\sqrt{C_{12}C_1C_2}\left[\frac{6C_{12}C_1^2C_2-C_{12}C_2C_3^2(1-2m^2)}{3\beta}e^{-3\beta t}+\frac{\alpha}{\beta}e^{-\beta t}\right]+C_4. \quad (3.40)$$

Family 7. If $e_0=\frac{1}{4}(1-m^2), e_1=\frac{1}{2}(1+m^2), e_2=\frac{1}{4}(1-m^2)$, then we get

$$u(\xi)=\pm C_3e^{-\beta t}dc(\xi)+C_1e^{-\beta t},$$

and

$$v(\xi)=\mp\frac{C_2C_3}{C_1}e^{-\beta t}dc(\xi)+C_2e^{-\beta t}, \quad (3.41)$$

Where

$$\xi=\pm\frac{C_3}{C_1}\sqrt{C_{12}C_1C_2}e^{-\beta t}x\pm\frac{C_3}{C_1^2}\sqrt{C_{12}C_1C_2}\left[\frac{6C_{12}C_1^2C_2-C_{12}C_2C_3^2(1+m^2)}{3\beta}e^{-3\beta t}+\frac{\alpha}{\beta}e^{-\beta t}\right]+C_4. \quad (3.42)$$

Family 8. If $e_0=\frac{m^2}{4}, e_1=\frac{1}{2}(m^2-2), e_2=\frac{1}{4}$, then we get

$$u(\xi)=\mp C_3e^{-\beta t}cs(\xi)+C_1e^{-\beta t},$$

and

$$v(\xi)=\pm\frac{C_2C_3}{C_1}e^{-\beta t}cs(\xi)+C_2e^{-\beta t}, \quad (3.43)$$

Where

$$\xi=\pm\frac{C_3}{C_1}\sqrt{C_{12}C_1C_2}e^{-\beta t}x\pm\frac{C_3}{C_1^2}\sqrt{C_{12}C_1C_2}\left[\frac{6C_{12}C_1^2C_2-C_{12}C_2C_3^2(m^2-2)}{3\beta}e^{-3\beta t}+\frac{\alpha}{\beta}e^{-\beta t}\right]+C_4. \quad (3.44)$$

Family 9. If $e_0=\frac{m^2}{4}, e_1=\frac{1}{2}(m^2-2), e_2=\frac{m^2}{4}$, then we get

$$u(\xi)=\mp i\,C_3e^{-\beta t}dn(\xi)+C_1e^{-\beta t},$$

and

$$v(\xi)=\pm i\,\frac{C_2C_3}{C_1}e^{-\beta t}dn(\xi)+C_2e^{-\beta t}, \quad (3.45)$$

Where

$$\xi=\pm\frac{C_3}{C_1}\sqrt{C_{12}C_1C_2}e^{-\beta t}x\pm\frac{C_3}{C_1^2}\sqrt{C_{12}C_1C_2}\left[\frac{6C_{12}C_1^2C_2-C_{12}C_2C_3^2(m^2-2)}{3\beta}e^{-3\beta t}+\frac{\alpha}{\beta}e^{-\beta t}\right]+C_4. \quad (3.46)$$

Appendix A

The general solutions to the Jacobi elliptic equation (2.3) and its derivatives (see for example [3,9,13,29]) are listed as follows:

e_0	e_1	e_2	$G(\xi)$	$G'(\xi)$
1	$-(1+m^2)$	m^2	$sn(\xi)$ or $cd(\xi)$	$cn(\xi)dn(\xi)$ or $-(1-m^2)sd(\xi)nd(\xi)$
$1-m^2$	$2m^2-1$	$-m^2$	$cn(\xi)$	$-sn(\xi)dn(\xi)$
m^2-1	$2-m^2$	-1	$dn(\xi)$	$-m^2 sn(\xi)cn(\xi)$
m^2	$-(1+m^2)$	1	$ns(\xi)$ or $dc(\xi)$	$-ds(\xi)cs(\xi)$ or $(1-m^2)nc(\xi)sc(\xi)$
$-m^2$	$2m^2-1$	$1-m^2$	$nc(\xi)$	$sc(\xi)dc(\xi)$
-1	$2-m^2$	m^2-1	$nd(\xi)$	$m^2 sd(\xi)cd(\xi)$
$1-m^2$	$2-m^2$	1	$cs(\xi)$	$-ns(\xi)ds(\xi)$
1	$2-m^2$	$1-m^2$	$sc(\xi)$	$nc(\xi)dc(\xi)$
1	$2m^2-1$	$m^2(m^2-1)$	$sd(\xi)$	$nd(\xi)cd(\xi)$
$m^2(m^2-1)$	$2m^2-1$	1	$ds(\xi)$	$-cs(\xi)ns(\xi)$
$\frac{1}{4}$	$\frac{1}{2}(1-2m^2)$	$\frac{1}{4}$	$ns(\xi)\pm cs(\xi)$	$-ds(\xi)cs(\xi)\mp ns(\xi)ds(\xi)$
$\frac{1}{4}(1-m^2)$	$\frac{1}{4}(1+m^2)$	$\frac{1}{4}(1-m^2)$	$nc(\xi)\pm sc(\xi)$	$sc(\xi)dc(\xi)\pm nc(\xi)dc(\xi)$
$\frac{m^2}{4}$	$\frac{1}{2}(m^2-2)$	$\frac{1}{4}$	$ns(\xi)\pm ds(\xi)$	$-ds(\xi)cs(\xi)\mp cs(\xi)ns(\xi)$
$\frac{m^2}{4}$	$\frac{1}{2}(m^2-2)$	$\frac{m^2}{4}$	$sn(\xi)\pm i\,cn(\xi)$	$cn(\xi)dn(\xi)\mp i\,sn(\xi)dn(\xi)$

where $0<m<1$ is the modulus of the Jacobi elliptic functions and $i=\sqrt{-1}$.

Appendix B

The Jacobi elliptic functions $\mathrm{sn}(\xi), \mathrm{cn}(\xi), \mathrm{dn}(\xi), \mathrm{ns}(\xi), \mathrm{cs}(\xi), \mathrm{ds}(\xi), \mathrm{sc}(\xi), \mathrm{sd}(\xi)$ generate into hyperbolic functions when $m \to 1$ as follows:

$\mathrm{sn}(\xi)\to\tanh(\xi)$	$\mathrm{cn}(\xi)\to sech(\xi)$	$\mathrm{dn}(\xi)\to sech(\xi)$	$\mathrm{ns}(\xi)\to\coth(\xi)$
$\mathrm{cs}(\xi)\to\mathrm{csc}h(\xi)$	$\mathrm{ds}(\xi)\to\mathrm{csc}h(\xi)$	$\mathrm{sc}(\xi)\to\sinh(\xi)$	$\mathrm{sd}(\xi)\to\sinh(\xi)$

and into trigonometric functions when $m \to 0$ as follows:

$\mathrm{sn}(\xi)\to sin(\xi)$	$\mathrm{cn}(\xi)\to\cos(\xi)$	$\mathrm{dn}(\xi)\to 1$	$\mathrm{ns}(\xi)\to\csc(\xi)$
$\mathrm{cs}(\xi)\to\cot(\xi)$	$\mathrm{ds}(\xi)\to\csc(\xi)$	$\mathrm{sc}(\xi)\to\tan(\xi)$	$\mathrm{sd}(\xi)\to\sin(\xi)$

Appendix C

$\mathrm{cd}(\xi)=\frac{\mathrm{cn}(\xi)}{\mathrm{dn}(\xi)}$	$\mathrm{dc}(\xi)=\frac{\mathrm{dn}(\xi)}{\mathrm{cn}(\xi)}$	$\mathrm{nc}(\xi)=\frac{1}{cn(\xi)}$	$\mathrm{nd}(\xi)=\frac{1}{dn(\xi)}$
$\mathrm{cs}(\xi)=\frac{cn(\xi)}{\mathrm{sn}(\xi)}$	$\mathrm{sc}(\xi)=\frac{sn(\xi)}{cn(\xi)}$	$\mathrm{sd}(\xi)=\frac{sn(\xi)}{\mathrm{dn}(\xi)}$	$\mathrm{ds}(\xi)=\frac{dn(\xi)}{\mathrm{sn}(\xi)}$

4. Conclusions

The main idea of the generalized (G'/G)- expansion method with variable coefficients is that the exact solutions of nonlinear partial differential equations can be expressed as a polynomial in (G'/G), where $\mathrm{G}(\xi)$ satisfies the Jacobi elliptic equation (2.3) instead of the standard technique used by Wang et al. [22] to some nonlinear PDEs in mathematical physics via the generalized time- dependent variable coefficients KdV- mKdV equation and the coupled modified KdV equations with non-uniformity terms. We have obtained families of exact solutions of these equations in terms of Jacobi elliptic functions. Finally, we conclude according to the appendix B that our results in terms of Jacobi elliptic functions generate into hyperbolic functions when $m \to 1$ and generate into trigonometric functions when $m \to 0$.

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