ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 6, No. 1, 2011, pp. 015-022

Some Improvements in Preconditioned Modified Accelerated Overrelaxation (PMAOR) Method for Solving Linear Systems

Hashem Saberi Najafi, Seyyed Ahmad Edalatpanah+

Department of Mathematics, Faculty of Sciences, Islamic Azad University of Lahijan, P. Code 1616,Lahijan, Iran

Department of Mathematics, Faculty of Sciences, The University of Guilan, Rasht, Iran, P.O.box 41335-1914

(*Received November 1, 2010, accepted December 20, 2010*)

Abstract. In this article a new preconditioner from class of (I+S)-type based on the Modified Accelerated Overrelaxation(MAOR) iterative method has been introduced and convergence properties of the proposed method have been analyzed and compared with the some other preconditioners. Moreover, comparisons between different splittings are derived. Numerical example is also given to illustrate our results.

Keywords: preconditioning; Comparison theorems; MAOR method; Splitting; M-matrix

1. Introduction

Science history indicates that substantial improvements and huge jumps in science and technology require interaction between mathematicians with different scientists. Meanwhile, solving linear equation system play the role of a catalyst for further connection of this interaction between mathematics and sciences.

Consider the following linear system

$$AX=b \qquad (1.1)$$

Where $b, X \in R^n$ and $A \in R^{n\times n}$ is an nonsingular matrix of the following block form

$$A=\begin{bmatrix} D_1 & H \\ K & D_2 \end{bmatrix} \qquad (1.2)$$

Also D_1,D_2 are nonsingular diagonal matrices of orders n_1 and n_2 respectively and $H \in R^{n_1\times n_2}$, $K \in R^{n_2\times n_1}$. For any splitting, A=M-N with det(M) ≠0, the basic iterative methods for solving(1.1) is

$$X^{(t+1)} = M^{-1}NX^{(t)} + M^{-1}b \qquad t=1,2,\ldots \qquad (1.3)$$

This iterative process converges to the unique solution $X = A^{-1}b$ for any initial vector value $X_0 \in R^n$ if and only if the spectral radius $\rho(M^{-1}N) < 1$,where $M^{-1}N$ is called the iterative matrix. There are some specifically iterative methods for solving a linear system (1.1) based on (1.3).see[1].

Modified Overrelaxation methods are also above model .These methods have been discussed and used by many researchers; see [1-7]. Let the matrix **A** have the splitting **A**=D -C_L-C_u=D(I-L-U),where L = $D^{-1}C_L$, U = $D^{-1}C_u$, D = diag(A), C_L and C_u are strictly lower and upper triangular matrices of **A,** respectively. The modified accelerated Overrelaxation (MAOR) method defined by [4] is

$$X^{(t+1)} = \mu_{\Omega,\Gamma} X^{(t)} + \Psi \qquad t=0,1,\cdots \qquad (1.4)$$

With iterative matrix

+ Corresponding author. *E-mail address*: saedala@yahoo.com & saedalatpanah@gmail.com

$$\mu_{\Omega,\Gamma} = M^{-1}N = \underbrace{(D-\Gamma C_L)^{-1}}_{M^{-1}}\underbrace{[(I-\Omega)D+(\Omega-\Gamma)C_L+\Omega C_U]}_{N} = \underbrace{(I-\Gamma L)^{-1}}_{M^{-1}}\underbrace{[(I-\Omega)+(\Omega-\Gamma)L+\Omega U]}_{N} \quad (1.5)$$

And

$$\Psi = (D-\Gamma C_L)^{-1}\Omega b = (I-\Gamma L)^{-1}D^{-1}\Omega b \quad (1.6)$$

With

$$\Omega = diag(w_1 I_1, w_2 I_2), \Gamma = diag(\gamma_1 I_1, \gamma_2 I_2) \quad (1.7)$$

Where $w_1, w_2, \gamma_1, \gamma_2$ are positive real parameters and I_1, I_2 are identity matrix of orders n_1 and n_2 respectively. Darvishi et al. in [8] studied preconditioned MAOR method for linear systems based on preconditioners of class (I+S)-type (For details, we refer to [9-21]). They proposed following splitting of A

$$D = \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix}, C_L = \begin{bmatrix} 0 & 0 \\ -K & 0 \end{bmatrix}, C_U = \begin{bmatrix} D_1^* & -H \\ 0 & D_2^* \end{bmatrix} \quad (1.8)$$

Where

$$D_1^* = I_1 - D_1, D_2^* = I_2 - D_2 \quad (1.9)$$

They assume that

$$H \le 0, \quad K \le 0, \quad 0 \le w_1 \le 1, \quad 0 \le w_2 \le 1, \quad 0 \le \gamma_2 \le \frac{w_2}{w_1} \quad (1.10)$$

Also they presented following preconditioners **P**of**A**,

Where

$$P = (D + \bar{S}) \quad (1.11)$$

With

$$\bar{S} = \begin{bmatrix} 0 & 0 \\ s_i & 0 \end{bmatrix} \qquad i = 1,2,3 \quad (1.12)$$

Where

$$S_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\dfrac{K_{n_2,1}}{\alpha} & 0 & \cdots & 0 \end{bmatrix}, \alpha > 0 \quad (1.13)$$

$$S_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -K_{11} & 0 & \cdots & 0 \\ -K_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -K_{n_2,1} & 0 & \cdots & 0 \end{bmatrix} \quad (1.14)$$

And

$$S_3 = \begin{cases} -K_{i,j} & for \quad i = j, i = 1,2,\ldots,n_2 \\ 0 & for \quad otherwise \end{cases} \quad (1.15)$$

Moreover, they showed that the preconditioned matrix

$$\bar{A} = PA \quad (1.16)$$

Can be decomposed by following splitting

$$\overline{A} = \overline{D} - \overline{C}_{L_i} - \overline{C}_{U_i} \qquad i = 1,2,3 \tag{1.17}$$

Where

$$\begin{cases} \overline{D} = D \\ \overline{C}_{L_i} = \begin{bmatrix} 0 & 0 \\ -K - s_i(I_1 - D_1^*) & 0 \end{bmatrix} \\ \overline{C}_{U_i} = \begin{bmatrix} D_1^* & -H \\ 0 & D_2^* - s_i H \end{bmatrix} \end{cases} \tag{1.18}$$

And similar to (1.4), preconditioned MAOR (PMAOR) iterative method defined as

$$Y^{(t+1)} = \overline{\mu}_{\Omega,\Gamma}^{(i)} Y^{(t)} + V^{(i)} \qquad t = 0,1,\cdots \;\&\; i = 1,2,3 \tag{1.19}$$

Where

$$\begin{cases} \overline{\mu}_{\Omega,\Gamma}^{(i)} = \overline{M}^{-1}\overline{N} = \underbrace{(\overline{D} - \Gamma\overline{C}_{L_i})^{-1}}_{\overline{M}^{-1}} \underbrace{[(I-\Omega)\overline{D} + (\Omega-\Gamma)\overline{C}_{L_i} + \Omega\overline{C}_{U_i}]}_{\overline{N}} \\ \qquad\qquad = \underbrace{(I - \Gamma\overline{L}_i)^{-1}}_{\overline{M}^{-1}} \underbrace{[(I-\Omega) + (\Omega-\Gamma)\overline{L}_i + \Omega\overline{U}_i]}_{\overline{N}} \\ V^{(i)} = (\overline{D} - \Gamma\overline{C}_{L_i})^{-1}\Omega\overline{b} = (I - \Gamma\overline{L}_i)^{-1}\overline{D}^{-1}\Omega\overline{b} \\ \overline{b} = Pb \end{cases} \tag{1.20}$$

In this paper, we present alternative splitting of **A,** $\overline{A}$ and propose a new preconditioner of **A**. also, we prove that our splitting and preconditioner compare with the above splitting and preconditioners works better. Numerical example is also reported to confirm our convergence analysis.

2. Prerequisite

We begin with some basic notation and preliminary results which we refer to later.

Definition 2.1 [22-23].

(a) A matrix A = a_{ij} is called a Z-matrix if for any $i \neq j, a_{ij} \leq 0$

(b) A Z-matrix is an L-matrix ,if $a_{ii} > 0$

(c) A Z-matrix is an M-matrix, if A is nonsingular, and $A^{-1} \geq 0$.

Definition 2.2[22-23]. Let *A* be a real matrix. The splitting A=M-N is called

(a) Convergent if ρ ($M^{-1}N$)<1

(b) Regular if $M^{-1} \geq 0$ and $N \geq 0$

(c) Weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$

Clearly, a regular splitting is weak regular.

Lemma 2.1(Varga [22]).let $A=M_1-N_1=M_2-N_2$ be two regular splittings of A , where $A^{-1} \geq 0$.If $N_2 \geq N_1 \geq 0$,then $0 \leq \rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2) < 1$

Lemma2.2 (Berman and Pelemmons [23]).Let A be a Z-matrix. Then A is M-matrix if and only if there is a positive vector X such that AX>0 .

Lemma 2.3 ([24-25]) let A, B are Z-matrix and A is an M-matrix, if A≤B then B is an M-matrix too.

Lemma2.4(Varga [22]) Let $A=M_1-N_1=M_2-N_2$ be two regular splittings of A, where $A^{-1} \geq$ 0.If $M_1^{-1} \geq M_2^{-1}$,Then $\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2) < 1$.

Lemma2.5([25-26]) If $A \geq 0$, then

(1) A has a nonnegative real eigenvalue equal to its spectral radius,

(2) To $\rho(A) > 0$, there corresponds an eigenvector $x \geq 0$,

(3) $\rho(A)$ does not decrease when any entry of A is increased.

Lemma2.6(Berman and Pelemmons [23])

Let $T \geq 0$. If there exist $X > 0$ and a scalar $\alpha > 0$ such that

(1) $TX \leq \alpha X$,then $\rho(T) \leq \alpha$. Moreover, if $TX < \alpha X$, then $\rho(T) < \alpha$.

(2) $TX \geq \alpha X$,then $\rho(T) \geq \alpha$. Moreover, if $TX > \alpha X$, then $\rho(T) > \alpha$.

3. Theoretical Analysis

In the following we will compare standard splitting with splitting of(1.8).

To solve linear system (1.1) we consider the following splitting

$$A = D - C_L - C_U$$
$$D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}, C_L = \begin{bmatrix} 0 & 0 \\ -K & 0 \end{bmatrix}, C_U = \begin{bmatrix} 0 & -H \\ 0 & 0 \end{bmatrix} \tag{3.1}$$

Following Theorem indicate that standard splitting for **A** is the best.

Theorem 3.1. let $\mu^{(1)}_{\Omega,\Gamma}$, $\mu^{(2)}_{\Omega,\Gamma}$ be the iterative matrices of the **MAOR** method by splittings of(1.3)and (1.8) , respectively. If **A** in (1.1) is an M-matrix and conditions of (1.9),(1.10) are satisfied. Then we have $\rho(\mu^{(2)}_{\Omega,\Gamma}) \leq \rho(\mu^{(1)}_{\Omega,\Gamma}) < 1$

Proof. By(1.5) ,(1.9),(1.10)and Definition 2.2,these splittings are regular.

Also let $\mu^{(1)}_{\Omega,\Gamma} = M_1^{-1}N_1$, $\mu^{(2)}_{\Omega,\Gamma} = M_2^{-1}N_2$

Then

$$N_1 = \left\{ \begin{pmatrix} (1-w_1)I_1 & 0 \\ 0 & (1-w_2)I_2 \end{pmatrix} + (\Omega - \Gamma)C_L + \begin{pmatrix} w_1 D^*_1 & -w_1 H \\ 0 & w_2 D^*_2 \end{pmatrix} \right\}$$

$$N_2 = \left\{ \begin{pmatrix} (1-w_1)D_1 & 0 \\ 0 & (1-w_2)D_2 \end{pmatrix} + (\Omega - \Gamma)C_L + \begin{pmatrix} 0 & -w_1 H \\ 0 & 0 \end{pmatrix} \right\}$$

$$\Rightarrow N_1 - N_2 = \begin{pmatrix} D^*_1 & 0 \\ 0 & D^*_2 \end{pmatrix} \geq 0 \to N_1 \geq N_2$$

Therefore by Lemma2.1 we obtain finishes the proof of theorem.■

Now, we consider following splitting for preconditioned matrix $\overline{A}$ and show that our splitting is better than splitting (1.18).

$$\overline{A} = \overline{\overline{D}} - \overline{\overline{C}}_{L_i} - \overline{\overline{C}}_{U_i} = \begin{pmatrix} D_1 & H \\ K + s_i D_I & D_2 + s_i H \end{pmatrix} \qquad i = 1,2,3$$
$$s_i H = \underbrace{d_1}_{\leq 0} - \underbrace{l_1}_{\geq 0} - \underbrace{u_1}_{\geq 0} \tag{3.2}$$

Where d_1, l_1, u_1 are diagonal, strictly lower and strictly Upper triangular parts of $s_i H$, respectively.

And

$$\begin{cases} \overline{\overline{D}} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 + d_1 \end{pmatrix} \\ \overline{\overline{C}}_{L_i} = \begin{bmatrix} 0 & 0 \\ -K - s_i D_1 & l_1 \end{bmatrix} \\ \overline{\overline{C}}_{U_i} = \begin{bmatrix} 0 & -H \\ 0 & u_1 \end{bmatrix} \end{cases} \tag{3.3}$$

Theorem 3.2. let $\overline{\mu}_{\Omega,\Gamma}$, $\overline{\overline{\mu}}_{\Omega,\Gamma}$ be the iterative matrices of the **PMAOR** method by splittings of(1.17)and (3.2) , respectively. If conditions of Theorem 3.1 are satisfied. Then we have $\rho(\overline{\overline{\mu}}_{\Omega,\Gamma}) \le \rho(\overline{\mu}_{\Omega,\Gamma}) < 1$

Proof. Since A is an M-matrix, by lemma2.2 it is easy to see that $\overline{A}$ also is M-matrix. Therefore entries of its diagonal are positive,i.e $\overline{D} \ge 0, \overline{\overline{D}} \ge 0$

Also let $\overline{\mu}_{\Omega,\Gamma} = \overline{M}^{-1}\overline{N}$, $\overline{\overline{\mu}}_{\Omega,\Gamma} = \overline{\overline{M}}^{-1}\overline{\overline{N}}$

Then $\overline{\overline{M}} - \overline{M} = \begin{pmatrix} D_1 & 0 \\ K + s_i D_1 & D_2 + d_1 - l_1 \end{pmatrix} - \begin{pmatrix} I_1 & 0 \\ K + s_i D_1 & I_2 \end{pmatrix}$

And by (1.9) $D_1 \le I_1$ & $D_2 + d_1 \le I_2$ & $-l_1 \le 0$

$\Rightarrow \overline{\overline{M}} \le \overline{M}$

Since $\overline{A}$ is an M-matrix by Lemma 2.3 $\overline{\overline{M}}, \overline{M}$ are M-matrix too.

Thus $\overline{\overline{M}}^{-1} \ge \overline{M}^{-1}$

Therefore by lemma 2.4 the proof is completed.■

Now, we will propose alternative preconditioner $\hat{P}$ of **A**

$$\hat{P} = (I + S) = \begin{pmatrix} I_1 & 0 \\ -KD_1^{-1} & I_2 \end{pmatrix} \tag{3.4}$$

Then

$$\hat{A} = \hat{P}A = \begin{pmatrix} I_1 & 0 \\ -KD_1^{-1} & I_2 \end{pmatrix}\begin{pmatrix} D_1 & H \\ K & D_2 \end{pmatrix} = \begin{pmatrix} D_1 & H \\ 0 & -KD_1^{-1}H + D_2 \end{pmatrix} \tag{3.5}$$

Let $\Psi = -KD_1^{-1}H = \underbrace{d_2}_{\le 0} - \underbrace{l_2}_{\ge 0} - \underbrace{u_2}_{\ge 0}$

Where d_2, l_2, u_2 are diagonal, strictly lower and strictly upper triangular parts of Ψ ,respectively.

Also, for our preconditioned matrix we have the following splitting

$$\hat{A} = \hat{D} - \hat{C}_L - \hat{C}_U$$

Where

$$\hat{D} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 + d_2 \end{pmatrix} \& \hat{C}_L = \begin{pmatrix} 0 & 0 \\ 0 & l_2 \end{pmatrix} \& \hat{C}_U = \begin{pmatrix} 0 & -H \\ 0 & u_2 \end{pmatrix} \tag{3.6}$$

Here ,we prove that our preconditioner compare with the Darvishi et al's preconditioners work better point of view spectral radius.

Theorem 3.3. let $\overline{\mu}^{(i)}_{\Omega,\Gamma}$, $\hat{\mu}_{\Omega,\Gamma}$ (i=1,2,3)be the iterative matrices of the **PMAOR** method by preconditioners (1.11)and (3.4) , respectively. If conditions of Theorem3.1 are satisfied. Then we have $\rho(\hat{\mu}_{\Omega,\Gamma}) \le \rho(\overline{\mu}^{(i)}_{\Omega,\Gamma}) < 1$

Proof. Let $\bar{\mu}^{(i)}_{\Omega,\Gamma} = \bar{M}^{-1}\bar{N}$, $\hat{\mu}_{\Omega,\Gamma} = \hat{M}^{-1}\hat{N}$ since $\hat{\mu}_{\Omega,\Gamma} \geq 0$ then by lemma2.5there exist a positive vector X such that $(\hat{M}^{-1}\hat{N})X = \rho(\hat{M}^{-1}\hat{N})X$.

We have $\hat{N}X \geq 0$ because $\hat{N} \geq 0$. And so

$$\hat{M}X = \frac{1}{\rho(\hat{M}^{-1}\hat{N})}\hat{N}X \geq 0$$

$$\Rightarrow \hat{A}X = \hat{M}(I - \hat{M}^{-1}\hat{N})X = \frac{1-\rho(\hat{M}^{-1}\hat{N})}{\rho(\hat{M}^{-1}\hat{N})}\hat{N}X \geq 0$$

Also we know $\hat{A}X = (I+S)AX \geq 0$ and $(I+S) \geq 0$,therefore $AX \geq 0$.

Now we have.

$$\begin{aligned}
\hat{A}X &= (I+S)AX \\
&= \begin{pmatrix} I_1 & 0 \\ -KD_1^{-1} & I_2 \end{pmatrix} AX + (\bar{S} - \bar{S})AX \\
&= (D+\bar{S})AX + \left\{ \begin{pmatrix} 0 & 0 \\ -KD_1^{-1} & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ s_i & 0 \end{pmatrix} \right\} AX \geq (D+\bar{S})AX \geq 0 \\
&\longrightarrow \hat{A}X \geq \bar{A}X
\end{aligned}$$

Since $\hat{M}^{-1} \geq \bar{M}^{-1}$ we have;

$$\begin{aligned}
\rho(\hat{M}^{-1}\hat{N})X &= \hat{M}^{-1}\hat{N}X = X - \hat{M}^{-1}\hat{A}X \\
&\leq X - \hat{M}^{-1}\bar{A}X \leq X - \bar{M}^{-1}\bar{A}X \\
&= (I - \bar{M}^{-1}\bar{A})X = \bar{M}^{-1}\bar{N}X
\end{aligned}$$

Therefore by lemma 2.6 the proof is completed.■

4. Numerical example

In this section, we give an example to illustrate the results obtained in previous Sections.

Example.[see(8,Example1)]The coefficient matrix A of is given by

$$A = \begin{bmatrix} D_1 & H \\ K & D_2 \end{bmatrix}$$

$$D_1 = \frac{1}{2}I_{6\times6} \ \& \ D_2 = \frac{1}{2}I_{5\times5} \ \& \ H = \begin{bmatrix} 0 & \frac{-1}{8} & 0 & \frac{-1}{8} & \frac{-1}{8} \\ 0 & \frac{-1}{8} & 0 & 0 & 0 \\ \frac{-1}{8} & \frac{-1}{8} & \frac{-1}{8} & \frac{-1}{8} & 0 \\ \frac{-1}{8} & \frac{-1}{8} & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{8} & \frac{-1}{8} & 0 \\ 0 & 0 & 0 & \frac{-1}{8} & \frac{-1}{8} \end{bmatrix} \ \& \ K = \begin{bmatrix} 0 & 0 & \frac{-1}{8} & \frac{-1}{8} & 0 & 0 \\ \frac{-1}{8} & \frac{-1}{8} & \frac{-1}{8} & \frac{-1}{8} & 0 & 0 \\ 0 & 0 & \frac{-1}{8} & 0 & \frac{-1}{8} & 0 \\ \frac{-1}{8} & 0 & \frac{-1}{8} & 0 & \frac{-1}{8} & \frac{-1}{8} \\ \frac{-1}{8} & 0 & 0 & 0 & 0 & \frac{-1}{8} \end{bmatrix}$$

If we apply all the last methods for A and compute the spectral radius in each case ,we have the following results.

In the Table1, we reported the spectral radius of the **MAOR** method with **different splittings**.

Also ρ^1, ρ^2 are, spectral radius of iteration matrix with splitting (1.3),(1.8) ,respectively.

In the Table2, we reported the spectral radius of the **PMAOR** method with **different splittings**. also $\overline{\rho}, \overline{\overline{\rho}}$ are ,spectral radius of iteration matrix with splitting(1.17),(3.2) ,respectively.

In the Table3, we reported the spectral radius of the **PMAOR** method with **different preconditioners and our splitting**. Also ρ^1 , ρ^2 , ρ^3 and ρ are ,spectral radius of iteration matrix with preconditioners (1.13),(1.14),(1.15)and(3.4) ,respectively.($\alpha = 2$)

From the tables, we can see that our splittings are superior to the Darvishi et al.'s splittings and our preconditioner is better than other preconditioners.

5. Conclusions

In this paper, we have proposed a new preconditioner from class of (I+S)-type based on the MAOR iterative method. We have studied how the iterative method is affected if the system is preconditioned by our model. Also we let the coefficient matrix of linear system be Z-matrices, M- matrices that often occur in a wide variety of sciences. Finally, from theoretical speaking and numerical example, it is may be concluded that the convergence rate of our proposed methods are superior to the basic iterative methods and better than the some preconditioner of (I+S)-type.

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Table1 The spectral radius of the MAOR method with two different splittings

MAOR	w_1	w_2	γ_2	ρ^1	ρ^2
	0.8913	0.9273	0.8842	0.8501	0.6143
	0.9213	0.9773	0.3442	0.8606	0.7011
	0.9462	0.9751	0.8649	0.8422	0.5969

Table2 The spectral radius of the PMAOR method with two different splittings

PMAOR Preconditioner	α	w_1	w_2	γ_2	$\overline{\rho}$	$\overline{\overline{\rho}}$
S_1	3/2	0.8913	0.9273	0.8842	0.8494	0.6112
	3/2	0.9213	0.9773	0.3442	0.8597	0.6989
	2	0.9462	0.9751	0.8649	0.8416	0.5945

Table3 The spectral radius of the PMAOR method with two different splittings and $\alpha = 2$

w_1	w_2	γ_2	ρ^1	ρ^2	ρ^3	ρ
0.7032	0.8720	0.8722	0.6892	0.6738	0.6832	0.3331
0.7408	0.9856	0.4976	0.7219	0.7065	0.7159	0.3356
0.8913	0.9273	0.8842	0.6120	0.5946	0.6050	0.2869
0.9213	0.9773	0.3442	0.6994	0.6835	0.6931	0.3664
0.9462	0.9751	0.8649	0.5945	0.5762	0.5872	0.2567