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The Least Squares Solutions of Bisymmetric Matrix for Inverse Quadratic Eigenvalue Problem

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Abstract. The inverse eigenvalue problem of constructing bisymmetric matrices M , C and K of size n for the quadratic pencil $Q(\Lambda) = MX\Lambda^2 + CX\Lambda + KX$ so that has a prescribed subset of eigenvalues and eigenvectors is discussed. A general expression of solution to the problem is provided. The set of such solutions is denoted by S_L . The optimal approximation problem associated with S_L is posed, that is: to find the nearest triple matrix $[\hat{M}, \hat{C}, \hat{K}]$ from S_L . The existence and uniqueness of the optimal approximation problem is discussed and the expression is provided for the nearest triple matrix.
Keywords: bisymmetric matrix, matrix equation, quadratic eigenvalue, inverse problem, SVD.

1. Introduction

Let $R^{n\times n}$ denote the set of $n\times n$ real matrices. $SR^{n\times n}$ denote the set of $n\times n$ real symmetric matrices. $ASR^{n\times n}$ be the set of $n\times n$ real anti-symmetric matrices, R^n denote the set of n dimensional vector. A^T is the transpose of matrix A . I_n is $n\times n$ unit matrix, $\|\bullet\|$ is Frobenius norm, $\|\bullet\|_2$ is 2-norm. e_i be i-th row of the unit matrix I_n . Let A be a real $m\times n$ matrix and let B be real $p\times q$ matrix. Then the Kronecker product of matrices A and B is defined as

$$A\otimes B := \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}. \tag{1}$$

That is, $A\otimes B$ is the $mp\times nq$ matrix formed all possible pairwise element products of A and B. If we let $vec(X)\in R^{mn}$ be the vector formed by the columns of a permutation matrix $X\in R^{m\times n}$.

Definition[1]**.** Let $A=(a_{ij})_{n\times n}$, $a_1=(a_{11},a_{21},\cdots,a_{n1})$, $a_2=(a_{22},a_{32},\cdots,a_{n2})$, $\cdots$,

$a_{n-1}=(a_{(n-1)(n-1)},a_{n(n-1)})$, $a_n=(a_{nn})$. Then we denote $vec_S(A)$ as follow

$$vec_S(A) := (a_1, a_2, \cdots, a_{n-1}, a_n)^T \in R^{\frac{n(n+1)}{2}}. \tag{2}$$

Definition[2]**.** $A=(a_{ij})\in R^{n\times n}$ is termed bisymmetric matrix, if

$$a_{ij} = a_{ji} = a_{n-j+1,n-i+1},\ i,j=1,2,\cdots,n \tag{3}$$

Let $G = \{[X,Y,Z] \mid X\in BSR^{n\times n}, Y\in BSR^{n\times n}, Z\in BSR^{n\times n}\}$.

In this paper, we discuss the following problems:

Problem I. Given matrices $X\in R^{n\times p}$, $\Lambda\in R^{p\times p}$, find $[\tilde{M},\tilde{C},\tilde{K}]\in G$ such that

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$$\left\|\tilde{M}X\Lambda^2+\tilde{C}X\Lambda+\tilde{K}X\right\|=\min_{[M,C,K]\in G}\left\|MX\Lambda^2+CX\Lambda+KX\right\| \tag{4}$$

where $\|\bullet\|$ is Frobenius norm.

Let $\tilde{G}=\{[M,C,K]\,|\,\left\|MX\Lambda^2+CX\Lambda+KX\right\|=\min,[M,C,K]\in G\}$.

Problem II. Find $[\tilde{M},\tilde{C},\tilde{K}]\in\tilde{G}$, such that

$$\|\tilde{M}\|^2+\|\tilde{C}\|^2+\|\tilde{K}\|^2=\min_{[M,C,K]\in\tilde{G}}(\|M\|^2+\|C\|^2+\|K\|^2) \tag{5}$$

where $\|\bullet\|$ is Frobenius norm.

2. The Solution Problem I

Lemma 1[1]. Let $A\in R^{m\times n},b\in R^n$, then the sufficiency and necessary condition of the solution exist for linear equation $Ax=b$ as follow

$$AA^+b=b. \tag{6}$$

the general solution for linear equation $Ax=b$ can write as follow

$$x=A^+b+(I-A^+A)\tau, \tag{7}$$

where $\tau\in R^n$.

Lemma 2[1]. Let $A\in R^{m\times n},b\in R^n$, then the least squares solution of incompatibility linear equation $Ax=b$ can write as follow

$$x=A^+b+(I-A^+A)\tau, \tag{8}$$

where $\tau\in R^n$.

For any k of positive integer, let

$$D_{2k}=\frac{1}{\sqrt{2}}\begin{pmatrix}I_k & S_k\\ S_k & -I_k\end{pmatrix},\ D_{2k+1}=\frac{1}{\sqrt{2}}\begin{pmatrix}I_k & O & S_k\\ O & \sqrt{2} & O\\ S_k & O & -I_k\end{pmatrix}, \tag{9}$$

where $S_k=(e_k,e_{k-1},\cdots,e_2,e_1)$.

We easy know, for any positive integer n, have $D_n^TD_n=I_n$, $D_n^T=D_n$, then D_n is symmetric orthogonal matrix.

Lemma 3[2]. For any n is odd number or even number, the sufficiency and necessary condition of $n\times n$ real matrix being bisymmetric matrix is

$$X=D_n\begin{pmatrix}X_1 & O\\ O & X_2\end{pmatrix}D_n, \tag{10}$$

where $X_1\in SR^{(n-k)\times(n-k)}$, $X_2\in SR^{k\times k}$, $k=[\frac{n}{2}]$, D_n as (9).

Lemma 4[2]. Given matrix $X\in R^{n\times n}$, then the sufficiency and necessary condition for $X\in SR^{n\times n}$ as follow

$$vec(X)=\Gamma_n vec_S(X) \tag{11}$$

where $vec_S(X) \in R^{\frac{n(n+1)}{2}}$ by (2) to present,

$$\Gamma_n = \begin{bmatrix} e_1 & e_2 & e_3 & \cdots & e_{n-1} & e_n & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & e_1 & 0 & \cdots & 0 & 0 & e_2 & e_3 & \cdots & e_{n-1} & e_n & \cdots & 0 & 0 & 0 \\ 0 & 0 & e_1 & \cdots & 0 & 0 & 0 & e_2 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e_1 & 0 & 0 & 0 & \cdots & e_2 & 0 & \cdots & e_{n-1} & e_n & 0 \\ 0 & 0 & 0 & \cdots & 0 & e_1 & 0 & 0 & \cdots & 0 & e_2 & \cdots & & e_{n-1} & e_n \end{bmatrix} \quad (12)$$

where e_i be i row of the unit matrix I_n , $\Gamma_n \in R^{n^2 \times \frac{n(n+1)}{2}}$.

Lemma 5[9]**.** Given matrices $X,Y,Z \in R^{n\times m}$, $F \in R^{m\times n}$, then matrix equation $AX+BY+CZ=F$ has solution for $[A,B,C] \in G$ condition as follow

$$NN^+vec(F) = vec(F) . \quad (13)$$

where $N = (P_1,P_2,P_3)$, $P_1 = (X^T \otimes I)\Gamma_n$, $P_2 = (Y^T \otimes I)\Gamma_n$, $P_3 = (Z^T \otimes I)\Gamma_n$.

If matrix equation of $AX+BY+CZ=F$ has solution for $[A,B,C] \in G$, the set of solutions can write as follow

$$S_L = \left\{ [X,Y,Z] | \begin{pmatrix} vec(X) \\ vec(Y) \\ vec(Z) \end{pmatrix} = \Omega[N^+vec(F) + (I - N^+N)\tau] \right\}, \quad (14)$$

where $\Omega = diag(\Gamma_n,\Gamma_n,\Gamma_n)$, $N = (P_1,P_2,P_3)$, $P_1 = (X^T \otimes I)\Gamma_n$, $P_2 = (Y^T \otimes I)\Gamma_n$, $P_3 = (Z^T \otimes I)\Gamma_n$, $\tau \in R^{3n^2}$.

When $F=0$, the matrix equation $AX+BY+CZ=0$ always has solution for $[A,B,C] \in G$, and the solution can write as follow

$$S'_L = \left\{ [X,Y,Z] | \begin{pmatrix} vec(X) \\ vec(Y) \\ vec(Z) \end{pmatrix} = M(I - N^+N)\tau \right\} \quad (15)$$

where

$$\tau = [I - [M(I - N^+N)]^+[M(I - N^+N)]]\varsigma , \; \varsigma \in R^{n^2} . \quad (16)$$

Theorem 1. Given matrices $X \in R^{n\times p}$, $\Lambda \in R^{p\times p}$, let

$$D_n^T X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} , \quad (17)$$

where $X_1 \in C^{k\times p}$, $X_2 \in C^{(n-k)\times p}$. Then, the solution for matrix equation (1) can write as follow:

$$M = D_n \begin{bmatrix} M_1 & O \\ O & M_2 \end{bmatrix} D_n, C = D_n \begin{bmatrix} C_1 & O \\ O & C_2 \end{bmatrix} D_n, K = D_n \begin{bmatrix} K_1 & O \\ O & K_2 \end{bmatrix} D_n, \quad (18)$$

where

$$\begin{bmatrix} vec(M_1) \\ vec(C_1) \\ vec(K_1) \end{bmatrix} = \Omega_1 (I_1 - N_1^+ N_1)\tau_1, \begin{bmatrix} vec(M_2) \\ vec(C_2) \\ vec(K_2) \end{bmatrix} = \Omega_2 (I_2 - N_2^+ N_2)\tau_2, \tag{19}$$

$\Omega_1 = diag(\Gamma_{n-k}, \Gamma_{n-k}, \Gamma_{n-k})$, $\Omega_2 = diag(\Gamma_k, \Gamma_k, \Gamma_k)$, $N_1 = (P_1, P_2, P_3)$, $N_2 = (Q_1, Q_2, Q_3)$, $P_1 = ((X_1\Lambda^2)^T \otimes I_k)\Gamma_k$, $P_2 = ((X_1\Lambda)^T \otimes I_{n-k})\Gamma_{n-k}$, $P_3 = (X_1^T \otimes I_{n-k})\Gamma_{n-k}$, $Q_1 = ((X_2\Lambda^2)^T \otimes I_k)\Gamma_k$, $Q_2 = ((X_2\Lambda)^T \otimes I_k)\Gamma_k$, $Q_3 = (X_2^T \otimes I_k)\Gamma_k$, $\tau_1 \in R^{3(n-k)^2}$ and $\tau_2 \in R^{3k^2}$ are any vector. I_k is $k \times k$ unit matrix, I_{n-k} is $(n-k)\times(n-k)$ unit matrix.

Proof. By Lemma 1 and (9), we have easily matrix equation (1) equivalence with

$$D_n \begin{bmatrix} M_1 & O \\ O & M_2 \end{bmatrix} \begin{bmatrix} X_1\Lambda^2 \\ X_2\Lambda^2 \end{bmatrix} + D_n \begin{bmatrix} C_1 & O \\ O & C_2 \end{bmatrix} \begin{bmatrix} X_1\Lambda \\ X_2\Lambda \end{bmatrix} + D_n \begin{bmatrix} K_1 & O \\ O & K_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0 \cdot \tag{20}$$

where $M_1, C_1, K_1 \in R^{(n-k)\times(n-k)}, M_2, C_2, K_2 \in\in R^{k\times k}$.

By matrix computation, matrix equation (20) equivalence with

$$M_1 X_1 \Lambda^2 + C_1 X_1 \Lambda + K_1 X_1 = 0 \text{, and } M_2 X_2 \Lambda^2 + C_2 X_2 \Lambda + K_2 X_2 = 0 . \tag{21}$$

Then, matrix equation (21) equivalence with

$$I_{n-k} M_1 X_1 \Lambda^2 + I_{n-k} C_1 X_1 \Lambda + I_{n-k} K_1 X_1 = 0 \text{, and } I_k M_2 X_2 \Lambda^2 + I_k C_2 X_2 \Lambda + I_k K_2 X_2 = 0 . \tag{22}$$

By the Kronecker product of matrices , (22) equivalence with

$$[(X_1\Lambda^2)^T \otimes I_{n-k}]vec(M_1) + [(X_1\Lambda)^T \otimes I_{n-k}]vec(C_1) + (X_1^T \otimes I_{n-k})vec(K_1) = 0 , \tag{23}$$

and

$$[(X_2\Lambda^2)^T \otimes I_k]vec(M_2) + [(X_2\Lambda)^T \otimes I_k]vec(C_2) + (X_2^T \otimes I_k)vec(K_2) = 0 . \tag{24}$$

Let

$\Omega_1 = diag(\Gamma_{n-k}, \Gamma_{n-k}, \Gamma_{n-k})$, $\Omega_2 = diag(\Gamma_k, \Gamma_k, \Gamma_k)$, $N_1 = (P_1, P_2, P_3)$, $N_2 = (Q_1, Q_2, Q_3)$, $P_1 = ((X_1\Lambda^2)^T \otimes I_{n-k})\Gamma_{n-k}$, $P_2 = ((X_1\Lambda)^T \otimes I_{n-k})\Gamma_{n-k}$, $P_3 = (X_1^T \otimes I_{n-k})\Gamma_{n-k}$, $Q_1 = ((X_2\Lambda^2)^T \otimes I_k)\Gamma_k$, $Q_2 = ((X_2\Lambda)^T \otimes I_k)\Gamma_k$, $Q_3 = (X_2^T \otimes I_k)\Gamma_k$.

Therefore, by Lemma 4, the solution of matrix equation (23) and (24) is having, and the general solution can write as (18) and (19).

3. The Solution Problem II

Theorem 2. Given matrices $X \in R^{n\times p}$, $\Lambda \in R^{p\times p}$, the solution exist of Problem II and can write as follow

$$M = D_n \begin{bmatrix} M_1 & O \\ O & M_2 \end{bmatrix} D_n, C = D_n \begin{bmatrix} C_1 & O \\ O & C_2 \end{bmatrix} D_n, K = D_n \begin{bmatrix} K_1 & O \\ O & K_2 \end{bmatrix} D_n, \tag{25}$$

where

$$\begin{bmatrix} vec(M_1) \\ vec(C_1) \\ vec(K_1) \end{bmatrix} = \Omega_1 (I_1 - N_1^+ N_1)\tau_1, \begin{bmatrix} vec(M_2) \\ vec(C_2) \\ vec(K_2) \end{bmatrix} = \Omega_2 (I_2 - N_2^+ N_2)\tau_2, \tag{26}$$

$\tilde{\tau}_1 = \left(I - (\Omega_1(I - N_1^+ N_1))^+ (\Omega_1(I - N_1^+ N_1))\right)\mu_1$, $\tilde{\tau}_2 = \left(I - (\Omega_2(I - N_2^+ N_2))^+ (\Omega_2(I - N_2^+ N_2))\right)\mu_2$, $\mu_1 \in R^{(n\text{-}k)^2}$ and $\mu_2 \in R^{k^2}$ are any vector.

Proof. By (18) and (19), we have

$$\min_{[M,C,K]\in S} (\| M \|^2 + \| C \|^2 + \| K \|^2)$$

$$= \min_{[M,C,K]\in S} (\| D_n \begin{bmatrix} M_1 & O \\ O & M_2 \end{bmatrix} D_n \|^2 + \| D_n \begin{bmatrix} C_1 & O \\ O & C_2 \end{bmatrix} D_n \|^2 + \| D_n \begin{bmatrix} K_1 & O \\ O & K_2 \end{bmatrix} D_n \|^2)$$

$$= \min_{[M,C,K]\in S} ((\| M_1 \|^2 + \| C_1 \|^2 + \| K_1 \|^2) + (\| M_2 \|^2 + \| C_2 \|^2 + \| K_2 \|^2))$$

$$= \min_{[M,C,K]\in S} (\| M_1 \|^2 + \| C_1 \|^2 + \| K_1 \|^2) + \min_{[M,C,K]\in S} (\| M_2 \|^2 + \| C_2 \|^2 + \| K_2 \|^2)$$

$$= \min_{[M_1, C_1, K_1]\in \tilde{G}_1} \left\| \begin{pmatrix} vec(M_1) \\ vec(C_1) \\ vec(K_1) \end{pmatrix} \right\|_2^2 + \min_{[M_2, C_2, K_2]\in \tilde{G}_2} \left\| \begin{pmatrix} vec(M_2) \\ vec(C_2) \\ vec(K_2) \end{pmatrix} \right\|_2^2$$

$$= \min_{\tau_1} \left\| \Omega_1 (I - N_1^+ N_1) \tau_1 \right\|_2^2 + \min_{\tau_2} \left\| \Omega_2 (I - N_2^+ N_2) \tau_2 \right\|_2^2$$

Thereby, $\min_{[M,C,K]\in S} (\| M \|^2 + \| C \|^2 + \| K \|^2)$ equivalence with

$$\min_{\tau_1} \left\| \Omega_1 (I - N_1^+ N_1) \tau_1 \right\|_2^2 \text{ and } \min \left\| \Omega_2 (I - N_2^+ N_2) \tau_2 \right\|_2^2 \tag{27}$$

By Lemma 1 and Lemma 2, we have the solution of

$$\min_{\tau_1} \left\| \Omega_1 (I - N_1^+ N_1) \tau_1 \right\|_2^2 \tag{28}$$

as follow

$$\tilde{\tau}_1 = \left(I - (\Omega_1(I - N_1^+ N_1))^+ (\Omega_1(I - N_1^+ N_1))\right)\mu_1, \tag{29}$$

where $\mu_1 \in R^{(n\text{-}k)^2}$ is any vector.

The solution of

$$\min_{\tau_2} \left\| \Omega_2 (I - N_2^+ N_2) \tau_2 \right\|_2^2 \tag{30}$$

as follow

$$\tilde{\tau}_2 = \left(I - (\Omega_2(I - N_2^+ N_2))^+ (\Omega_2(I - N_2^+ N_2))\right)\mu_2, \tag{31}$$

where $\mu_2 \in R^{k^2}$ is any vector.

By (18), (19), (29) and (31), we easy know the solution of Problem II present by (25) and (26).

Theorem 3. Given matrices $X \in R^{n\times p}$, $\Lambda \in R^{p\times p}$, $M^* \in R^{n\times n}$, $C^* \in R^{n\times n}$, $K^* \in R^{n\times n}$. Let

$$D_nM^*D_n=\begin{bmatrix} M_{11}^* & M_{12}^* \\ M_{21}^* & M_{22}^* \end{bmatrix},\ D_nC^*D_n=\begin{bmatrix} C_{11}^* & C_{12}^* \\ C_{21}^* & C_{22}^* \end{bmatrix},\ D_nK^*D_n=\begin{bmatrix} K_{11}^* & K_{12}^* \\ K_{21}^* & K_{22}^* \end{bmatrix}. \tag{32}$$

Then, the solution for

$$\| \widehat{M}-M^* \|^2+\| \widehat{C}-C^* \|^2+\| \widehat{K}-K^* \|^2=\min_{[M,C,K]\in S}(\| M-M^* \|^2+\| C-C^* \|^2+\| K-K^* \|^2) \tag{33}$$

can write as follow

$$\widehat{M}=D_n\begin{bmatrix} \widehat{M}_1 & O \\ O & \widehat{M}_2 \end{bmatrix}D_n, \widehat{C}=D_n\begin{bmatrix} \widehat{C}_1 & O \\ O & \widehat{C}_2 \end{bmatrix}D_n, \widehat{K}=D_n\begin{bmatrix} \widehat{K}_1 & O \\ O & \widehat{K}_2 \end{bmatrix}D_n, \tag{34}$$

where

$$\begin{pmatrix} vec(\tilde{M}_1) \\ vec(\tilde{C}_1) \\ vec(\tilde{K}_1) \end{pmatrix}=\Omega_1(I-N_1^+N_1)\tilde{\tau}_1,\ \begin{pmatrix} vec(\tilde{M}_2) \\ vec(\tilde{C}_2) \\ vec(\tilde{K}_2) \end{pmatrix}=\Omega_2(I-N_2^+N_2)\tilde{\tau}_2, \tag{35}$$

$\tilde{\tau}_1=\tilde{\tau}_0+\left(I-(\Omega_1(I-N_1^+N_1))^+(\Omega_1(I-N_1^+N_1))\right)\mu_1$, $\tilde{\tau}_0=(\Omega_1(I-N_1^+N_1))^+y_0$, $y_0=\begin{pmatrix} vec(M_{11}^*) \\ vec(C_{11}^*) \\ vec(K_{11}^*) \end{pmatrix}$, $\mu_1\in R^{(n\text{-}k)^2}$ is any vector,

$\tilde{\tau}_2=\tilde{\tau}_{00}+\left(I-(\Omega_2(I-N_2^+N_2))^+(\Omega_2(I-N_2^+N_2))\right)\mu_2$, $\tilde{\tau}_{00}=(\Omega_2(I-N_2^+N_2))^+y_{00}$, $y_{00}=\begin{pmatrix} vec(M_{22}^*) \\ vec(C_{22}^*) \\ vec(K_{22}^*) \end{pmatrix}$, $\mu_2\in R^{k^2}$ is any vector.

Proof. By (24) and (31), we have

$$\begin{aligned}
&\| M-M^* \|^2+\| C-C^* \|^2+\| K-K^* \|^2 \\
&=\| D_n\begin{bmatrix} M_1 & O \\ O & M_2 \end{bmatrix}D_n-D_nM^*D_n \|^2+\| \begin{bmatrix} C_1 & O \\ O & C_2 \end{bmatrix}-D_nC^*D_n \|^2+\| \begin{bmatrix} K_1 & O \\ O & K_2 \end{bmatrix}-D_nK^*D_n \|^2 \\
&=\| \begin{bmatrix} M_1 & O \\ O & M_2 \end{bmatrix}-\begin{bmatrix} M_{11}^* & M_{12}^* \\ M_{21}^* & M_{22}^* \end{bmatrix} \|^2+\| \begin{bmatrix} C_1 & O \\ O & C_2 \end{bmatrix}-\begin{bmatrix} C_{11}^* & C_{12}^* \\ C_{21}^* & C_{22}^* \end{bmatrix} \|^2+\| \begin{bmatrix} K_1 & O \\ O & K_2 \end{bmatrix}-\begin{bmatrix} K_{11}^* & K_{12}^* \\ K_{21}^* & K_{22}^* \end{bmatrix} \|^2 \\
&=\| \begin{bmatrix} M_1-M_{11}^* & -M_{12}^* \\ -M_{21}^* & M_2-M_{22}^* \end{bmatrix} \|^2+\| \begin{bmatrix} C_1-C_{11}^* & -C_{12}^* \\ -C_{21}^* & C_2-C_{22}^* \end{bmatrix} \|^2+\| \begin{bmatrix} K_1-K_{11}^* & -K_{12}^* \\ -K_{21}^* & K_2-K_{22}^* \end{bmatrix} \|^2 \\
&=(\| M_1-M_{11}^* \|^2+\| C_1-C_{11}^* \|^2+\| K_1-K_{11}^* \|^2)+(\| M_2-M_{22}^* \|^2+\| C_2-C_{22}^* \|^2+\| K_2-K_{22}^* \|^2)+ \\
&\quad(\| M_{12}^* \|^2+\| C_{12}^* \|^2+\| K_{12}^* \|^2)+(\| M_{21}^* \|^2+\| C_{21}^* \|^2+\| K_{21}^* \|^2)
\end{aligned}$$

Then, $\min_{[M,C,K]\in\tilde{G}}(\| M-M^* \|^2+\| C-C^* \|^2+\| K-K^* \|^2)$ equivalence with

$$\min_{[M_1,C_1,K_1]\in\tilde{G}_1}(\| M_1-M_{11}^* \|^2+\| C_1-C_{11}^* \|^2+\| K_1-K_{11}^* \|^2) \tag{36}$$

and

$$\min_{[M_2,C_2,K_2]\in\tilde{G}_2}(\|M_2-M_{22}^*\|^2+\|C_2-C_{22}^*\|^2+\|K_2-K_{22}^*\|^2) \tag{37}$$

For (35), we have

$$\min_{[M_1,C_1,K_1]\in\tilde{G}_1}(\|M_1-M_{11}^*\|^2+\|C_1-C_{11}^*\|^2+\|K_1-K_{11}^*\|^2)$$

$$=\min_{[M_1,C_1,K_1]\in\tilde{G}_1}\left\|\begin{pmatrix}vec(M_1)\\vec(C_1)\\vec(K_1)\end{pmatrix}-\begin{pmatrix}vec(M_{11}^*)\\vec(C_{11}^*)\\vec(K_{11}^*)\end{pmatrix}\right\|_2^2=\min_{\tau_1}\left\|\Omega_1(I-N_1^+N_1)\tau_1-\begin{pmatrix}vec(M_{11}^*)\\vec(C_{11}^*)\\vec(K_{11}^*)\end{pmatrix}\right\|_2^2$$

By Lemma 1 and Lemma 2, we have the solution of

$$\min_{[M_1,C_1,K_1]\in\tilde{G}_1}(\|M_1-M_{11}^*\|^2+\|C_1-C_{11}^*\|^2+\|K_1-K_{11}^*\|^2) \tag{38}$$

as follow

$$\tilde{\tau}_1=\tilde{\tau}_0+\left(I-(\Omega_1(I-N_1^+N_1))^+(\Omega_1(I-N_1^+N_1))\right)\mu_1, \tag{39}$$

where

$\tilde{\tau}_0=(\Omega_1(I-N_1^+N_1))^+y_0$, $y_0=\begin{pmatrix}vec(M_{11}^*)\\vec(C_{11}^*)\\vec(K_{11}^*)\end{pmatrix}$, $\mu_1\in R^{(n-k)^2}$ is any vector.

Analogously, the solution of (36) can write as follow

$$\tilde{\tau}_2=\tilde{\tau}_{00}+\left(I-(\Omega_2(I-N_2^+N_2))^+(\Omega_2(I-N_2^+N_2))\right)\mu_2, \tag{40}$$

where $\tilde{\tau}_{00}=(\Omega_2(I-N_2^+N_2))^+y_{00}$, $y_{00}=\begin{pmatrix}vec(M_{22}^*)\\vec(C_{22}^*)\\vec(K_{22}^*)\end{pmatrix}$, $\mu_2\in R^{k^2}$ is any vector.

By (25), (26), (37), (38), (39) and (40), we easy know the solution of (33) present by (34) and (35).

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