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On the Positive and Negative Solutions of p -Laplacian BVP with Neumann Boundary Conditions

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Abstract. In this paper, we consider the following Neumann boundary value problem

$$\begin{cases} \varphi_p(u'(x))' = |u(x)|^p u(x) \quad \lambda, \quad x\epsilon(0,1), \\ u'(0) = 0 = u'(1) \end{cases}$$

Where $\lambda \in \mathbb{R}$ and p>1 are parameters. We study the positive and negative solutions of this problem with respect to a parameter r (i.e. $u(0) = r$) in all $\mathbb{R}^*$. By using a quadrature method, we obtain our results. Also we provide some details about the solutions that are obtained.
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1. Introduction

Consider the nonlinear two point boundary value problem

$$-\varphi_p(u'(x))' = |u(x)|^p u(x) - \lambda, x \in (0,1) \qquad (1)$$

$$u'(0) = 0 = u'(1) \qquad (2)$$

where $\lambda \in \mathbb{R}$ and p >1 are parameters and $\varphi_p(x) := |x|^{p-2}x$ for all $x \neq 0$ and $\varphi_p(0) = 0$ where $\varphi_p(u')'$ is the one dimensional p -Laplacian operator. We study the positive and negative solution of this problem with respect to a parameter r (that is the value of the solutions at zero, i.e. $u(0) = r$). Also by using a quadrature method, we obtain our results. In [9] problem (1) with Dirichlet boundary value conditions have been studied by Ramaswamy for the case Laplacian and in [1] the same problem with the same boundary value conditions have been extended by Addou to the general quasilinear case p -Laplacian with p >1, i.e. $-\varphi_p(u'(x))' = |u|^p u - \lambda$. In [2] and [7] for semipositon problems with p –Laplacian operator, existence and multiplicity results have been established with Neumann boundary value conditions and Dirichlet boundary value conditions, respectively. In [5], for semipositon and positon problems have been studied by Anuradha, Maya and Shivaji by using a quadrature method with Neumann-Robin boundary conditions and Laplacian operator. In [8] for semipositone problems, existence and multiplicity results have been established with Laplacian operator and Neumann boundary value conditions. Also, in [3] and [6] for semipositon problems with Laplacian operator have been studied for solution curves with Dirichlet boundary value conditions.

This paper is organized as follows. In Section 2, we first state some remarks and then our main results and finally in Section 3, we provide the proof of our main results that contains several lemmas.

2. Main Results

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By a solution of (1)- (2) we mean a function $u \in C^1([0,1])$ for which $\varphi_p(u'(x)) \in C^1([0,1])$ and both the equation and the boundary value conditions are satisfied.

Remark 1 If u is a solution to (1)- (2) at λ then $-u$ is a solution to (1)- (2) at $-\lambda$.

Remark 2 Let u is a solution to (1)- (2) at λ then

$$\int_0^1 |u(x)|^p u(x)dx = \lambda.$$

Remark 3 (cf. [[8], Lemma 2.2]) Every solution u of (1)-(2) is symmetric about any interior critical points such that for any point $x_0 \in (0,1)$ where $u'(x_0)=0$ we have $u(x_0 - z) = u(x_0 + z)$ for all $z \in [0, \min\{x_0, 1 - x_0\}]$.

In fact, if one define $w_1(z) = u(x_0 - z)$ and $w_2(z) = u(x_0 + z)$, then it is clear that both w_1 and w_2 satisfy the IVP

$$\begin{cases} -\varphi_p(w'(x))' = |w(x)|^p w(x) - \lambda, \\ w(0) = u(x_0), \\ w'(x) = 0. \end{cases}$$

Hence, by uniqueness theorem for ODE, one can conclude result.

Remark 4 If u is a solution to (1)-(2), then $u(1-x)$ is also a solution to (1)-(2).

Remark 5 For any $\lambda \geq 0$ the problem(1)-(2) has always a trivial solution $u \equiv \lambda^{\frac{1}{p+1}}$ and for any $\lambda < 0$, the problem (1)-(2) has always a trivial solution $u \equiv -(-\lambda)^{\frac{1}{p+1}}$.

Also it is well-known that the initial value problem

$$\begin{cases} -\varphi_p(u'(x))' = |u(x)|^p u(x) - \lambda, \\ u(0) = r, \\ u'(0) = 0, \end{cases} \qquad (3)$$

has a local solution beginning at zero (by applying the Schaude r fixed point theorem) which either becomes infinite or exists on all of [0,1] and since $f(u) = |u|^p u - \lambda$ is locally Lipschitz, one can conclude from the classical theory for ODE the solution is locally unique. On the other hand for any given r and λ, there exists a real number $r_0 = r_0(r,\lambda)$ (see Lemma 1(e)), such that $u(1) \in \{r, r_0\} \Leftrightarrow u'(1) = 0$ (due to (4)). Thus if $u(1) \notin \{r, r_0\}$, u (as a unique solution to IVP (3)) do not satisfy the BVP (1)-(2). Also it is clear that every solution to the BVP (1)-(2) at λ with $u(0) = r$ is a solutionto IVP (3). Now, we state the existence of positive and negative solutions to the problem (1)-(2) as described below:

Theorem 1 Let $r \in \mathbb{R}^*$ and $p > 0$, then,

(a) If $r > 0$, the problem (1)-(2) has exactly one positive solution u with $u(0) = r$ at any $\lambda \in S_r$ where $S_r = (\frac{r^{p+1}}{p+2}, r^{p+1}) \cup (r^{p+1}, \infty)$ for which if $\lambda \in (\frac{r^{p+1}}{p+2}, r^{p+1})$ then $||u||_\infty = r$ and if $\lambda \in (r^{p+1}, \infty)$ then $\min_{x \in [0,1]} u(x) = r$ and the problem (1)-(2) has no positive solution with $u(0) = r$ at any $\lambda \in S_r^c$.

(b) if $r < 0$, the problem (1)-(2) has exactly one negative solution u with $u(0) = r$ at any $\lambda \in S_r$ where $S_r = (-\infty, r|r|^p) \cup (r|r|^p, \frac{r|r|^p}{p+2})$ for which if $\lambda \in (r|r|^p, \frac{r|r|^p}{p+2})$ then $\min_{x \in [0,1]} u(x) = r$ and if $\lambda \in (-\infty, r|r|^p)$ then $||u||_\infty = r$ and the problem (1)-(2) has no negative solution with $u(0) = r$ at any $\lambda \in S_r^c$.

3. Proof

Let u be a positive solution to $(1) - (2)$ at λ with $u(0) = r > 0$. Now multiplying (1) throughout by

u' and integrating over $(0,x)$, we obtain

$$|u'|^p = p'\{-\frac{u|u|^{p+1}}{p+2} + \lambda u + c\}$$

where C is a constant. Applying the conditions $u(0) = r$ and $u'(0) = 0$, we have

$$|u'|^p = p'\{\frac{r|r|^{p+1}}{p+2} - \frac{u|u|^{p+1}}{p+2} + \lambda(u-r)\}, x \in (0,1) \tag{4}$$

Now, we define the function,

$$s \mapsto M(p,r,\lambda,s) := \frac{r|r|^{p+1}}{p+2} - \frac{s|s|^{p+1}}{p+2} + \lambda(s-r), onR \tag{5}$$

where $\lambda > 0$ and $r \in \mathbb{R}^*$ are two parameters. The following lemma collects the variations of this function that follows immediately and we omit its proof.

Lemma 1 For all $\lambda \in \mathbb{R}^+$ and $r \in \mathbb{R}^*$,

(a) $\lim_{s\to\pm\infty} M(p,r,\lambda,s) = -\infty$

(b) The function $s \mapsto M(p,r,\lambda,s)$ is concave on $\mathbb{R}$.

(c) The function $s \mapsto M(p,r,\lambda,s)$ is increasing on $\left(-\infty, \lambda^{\frac{1}{p+1}}\right)$ and decreasing on $(\lambda^{\frac{1}{p+1}}, \infty)$, and

$$\max_{s\in\mathbb{R}} M(p,r,\lambda,s) = M\left(p,r,\lambda,\lambda^{\frac{1}{p+1}}\right)\begin{cases} = 0, & if\ \lambda = r|r|^p \\ > 0, & if\ \lambda \neq r|r|^p \end{cases}$$

(d) The y —intercept of the graph of $M(p,r,\lambda,.)$, i.e.

$$M(p,r,\lambda,0)\begin{cases} > 0, & if\ \ 0 < \lambda < \dfrac{r|r|^p}{p+2}, \\ = 0, & if\ \ \lambda = \dfrac{r|r|^p}{p+2}, \\ < 0, & if\ \ \dfrac{r|r|^p}{p+2} < \lambda. \end{cases}$$

(e) The function $M(p,r,\lambda,\cdot)$ has two zeros r and r_0 such that

$$\begin{cases} r_0 < 0 < r, & if & 0 < \lambda < \dfrac{r|r|^p}{p+2} \\ r_0 = 0, & if & \lambda = \dfrac{r|r|^p}{p+2} \\ 0 < r_0 < r, & if & \dfrac{r|r|^p}{p+2} < \lambda < r|r|^p \\ r = r_0 & if & \lambda = r|r|^p \\ r < r_0 & if & \lambda > r|r|^p. \end{cases}$$

Lemma 2 Let u be a nontrivial and positive solution to $(1)-(2)$ at λ with $u(0) = r$ and k be the number of interior critical points of u where $k = 0,1,2,3,4,\dots$ and if $k > 0$, x_0 is the first interior critical point, then

(a) The interior critical points of u are $x_0 = \frac{1}{k+1}$, $x_1 = 2x_0$, $x_2 = 3x_0$, ... , $x_{k-1} = kx_0$ and

$$r_0 = \begin{cases} u(x_0), & if\ k > 0, \\ u(1), & if\ k = 0. \end{cases}$$

(b) $u|_{[0,1]} = [r_0, r]$ or $[r, r_0]$.

(c) If u is decreasing at the beginning of $[0,1]$ then:

$$\|u\|_\infty = r = u(0) = u(x_1) = u(x_3) = \cdots \tag{6}$$

$$\min_{x\in[0,1]} u(x) = r_0 = u(x_0) = u(x_2) = u(x_4) = \cdots \tag{7}$$

and if u is increasing at the beginning of $(0,1)$ then:

$$\|u\|_\infty = r_0 = u(x_0) = u(x_2) = u(x_4) = \cdots \tag{8}$$

$$\min_{x\in[0,1]} u(x) = r = u(0) = u(x_1) = u(x_3) = \cdots \tag{9}$$

Proof of Lemma 2.

(a) Let x_0 be the first interior critical point of u and $k > 0$ be the number of interior critical points of u. Thus the values of u for any $x \in (0, x_0)$ must be between $u(0) = r$ and $u(x_0)$. Now we show that $u(x_0) = r_0$. We know that $u'(x_0) = 0$, hence from (4) and (5), one can conclude that $M(p, r, \lambda, u(x_0)) = 0$, also from the Lemma 1(e), $M(p, r, \lambda, u(0)) = 0$. On the other hand $M(p, r, \lambda, u(x)) > 0$ for any $x \in (0, x_0)$. In fact, if there exists a real number $x_{00} \in (0, x_0)$ such that $M(p, r, \lambda, u(x_{00})) = 0$ then from (4), one can conclude that $u'(x_{00}) = 0$, i.e. $x_{00} \in (0, x_0)$ is an interior critical point of u and this is a contradiction, because x_0 is the first interior critical point of u in the interval $[0,1]$. Now, from the Lemma 1(e), it follows that $u\big|_{[0,x_0]} = [r_0, r] or [r, r_0]$. Hence $u(x_0) = r_0$. But if $k = 0$, then the values of u for any $x \in (0,1)$ must be between $u(0)$ and $u(1)$. Hence by similar argument, one can show that $u(1) = r_0$. It is clear that $x_0 = \frac{1}{k+1}, k > 0$ and also by Remark 3, one can conclude that $2x_0, 3x_0, \ldots, kx_0$ are the rest interior critical points of u. The proof of part (a) follows.

(b) It easily follows from the Remark 3 and the proof of Lemma 2(a).

(c) If $k > 0$, u must be strictly increasing or decreasing on the interval $(0, x_0]$. If u is decreasing on $(0, x_0]$, then $max_{x\in[0,x_0]} u(x) = u(0) = r$ and $min_{x\in[0,x_0]} u(x) = u(x_0)$ and by the Remark 3 and the fact that $u(x_0) = r_0$, one can conclude that

$max_{x\in[0,x_0]} u(x) = r = u(0) = u(x_1) = u(x_3) = \cdots$

and $min_{x\in[0,x_0]} u(x) = r_0 = u(x_0) = u(x_2) = u(x_4) = \cdots$. On the other hand by the Lemma 3(b) and the fact that u attains its maximum and minimum values at $x = 0$ and $x = x_0$, respectively, it follows that $||u||_\infty = u(0) = r$ and $min_{x\in[0,1]} u(x) = r_0 = u(x_0)$. Hence (6) and (7) hold. If u is increasing on $(0, x_0)$, by similar argument, one can conclude that (8) and (9) hold. Also, if $k = 0$, by similar argument, one can conclude that (6)-(9) hold. The proof of part (c) follows. Δ

Lemma 3 Let u be a nontrivial and positive solution of (1)-(2) at $\lambda \in S_r$ with $u(0) = r$ and k be the number of interior critical points of u where $k = 0, 1, 2, \ldots$, then:

(a) $S_r = \left(\frac{r|r|^p}{p+2}, r|r|^p\right) \cup (r|r|^p, \infty)$.

(b) The corresponding solution is defined by

$$\int_r^{u(x)} \{M(p, r, \lambda, s)\}^{\frac{-1}{p}} ds = \kappa_1 \{p'\}^{1/p} x, \quad x \in (0, x_0).$$

$$\int_{u(x)}^{u(1)} \{M(p,r,\lambda,s)\}^{\frac{-1}{p}} ds = \kappa_2 \{p'\}^{1/p}(1 \quad x), \ x \subset (kx_0,1),$$

such that

$$\kappa_1 = \begin{cases} -, & \text{if} \quad \lambda \in \left(\frac{r|r|^p}{p+2}, r|r|^p\right), \\ +, & \text{if} \quad \lambda \in (r|r|^p, \infty). \end{cases}$$

κ_2 may be $+$ or $-$ for any $\lambda \in S_r$, and if $k > 0$, x_0 is the first interior critical point of u.

Proof of Lemma 3.

By the Lemma 2(b), $(x) \in [r_0, r] or [r, r_0]$ for any $x \in [0,1]$, and it, by the Lemma 1(e), (4) and(5), yields that λ must belong to the set $\left(\frac{r|r|^p}{p+2}, \infty\right)$. Now we show that, $\lambda \neq r|r|^p$. In fact, if $\lambda = r|r|^p$, then $r_0 = r$ (by the Lemma 1(e)), hence by the Lemma 2(b), $u \equiv r$ and this a contradiction, because the solution u is nontrivial. Thus we conclude that $S_r = \left(\frac{r|r|^p}{p+2}, r|r|^p\right) \cup (r|r|^p, \infty)$.

(b) Note that since every solution of (1)-(2) is symmetric about each of its interior critical points, thus it is enough to study solution on $[0, x_0]$ and $[kx_0, 1]$ where x_0 is the first interior critical point. If $\lambda \in \left(\frac{r|r|^p}{p+2}, r|r|^p\right)$, then by the Lemma 1(e), $r_0 < r$ and so, by the Lemma 2(c), $u(x_0) < u(0)$. Therefore u must be decreasing on $[0, x_0]$ and $\min_{x \in [0,1]} u(x) = r_0$. Hence from (4), we have

$$u'(x) = -\{p'\}^{\frac{1}{p}} \{M(p,r,\lambda,u(x))\}^{\frac{1}{p}}, \ x \in (0, x_0). \tag{10}$$

Also if $\lambda \in (r|r|^p, \infty)$, then by the Lemma 1(e), $r_0 > r$ and so, by the Lemma 2(c), $u(x_0) > u(0)$.

Therefore u must be increasing on $[0, x_0]$ and $||u||_\infty = r_0$. Hence from (4), we have

$$u'(x) = +\{p'\}^{\frac{1}{p}} \{M(p,r,\lambda,u(x))\}^{\frac{1}{p}}, \ x \in (0, x_0). \tag{11}$$

Also by (2), u may be increasing or decreasing on the interval $[kx_0, 1]$. Hence from (4), we have

$$u'(x) = \kappa_2 \{p'\}^{\frac{1}{p}} \{M(p,r,\lambda,u(x))\}^{\frac{1}{p}}, \ x \in (kx_0, 1), \tag{12}$$

where $\kappa_2 = + or -$.

Now, integrating (10) and (11) on $(0, x_0)$ where $x \in (0, x_0)$ and (12) on $(x, 1)$ $where$ $x \in (kx_0, 1)$, one can obtain

$$\int_r^{u(x)} \{M(p,r,\lambda,s)\}^{\frac{-1}{p}} ds = \kappa_1 \{p'\}^{1/p} x, \ x \in (0, x_0) \tag{13}$$

$$\int_{u(x)}^{u(1)} \{M(p,r,\lambda,s)\}^{\frac{-1}{p}} ds = \kappa_2 \{p'\}^{1/p}(1 \quad x), \ x \subset (kx_0, 1) \tag{14}$$

where κ_1 and κ_2 have been defined before in the Lemma 4(b).

By substituting $x = x_0$ and $x = kx_0$ in (13) and (14), respectively, and using the fact that $u(x_0) = r_0$ And $u(kx_0) = r$ or r_0 (by the Lemma 2(c)), we get

$$\int_{\Omega_1} \{M(p,r,\lambda,s)\}^{\frac{-1}{p}} ds = \{p'\}^{1/p} \ x_0 \tag{15}$$

$$\int_{\Omega_2} \{M(p,r,\lambda,s)\}^{\frac{-1}{p}} ds = \{p'\}^{1/p}(1 - kx_0), \tag{16}$$

where

$$\Omega_1 = \begin{cases} (r_0, r), & \text{if } \lambda \in \left(\frac{r|r|^p}{p+2}, r|r|^p\right), \\ (r, r_0), & \text{if } \lambda \in (r|r|^p, \infty) \end{cases} \quad \text{and} \quad \Omega_2 = (r_0, r) \text{ or } (r, r_0).$$

Note that in (15) and (16) the integrals are convergent. In fact,

Claim 1 The integrals $\int_{\Omega_i} \{M(p,r,\lambda,s)\}^{\frac{-1}{p}} ds \in (0,\infty)$, when $i = 1,2$.

Proof of Claim 1. It is suffice to show that $\int_{r_0}^{r} \{M(p,r,\lambda,s)\}^{\frac{-1}{p}} ds \in (0,\infty)$. For this mean, by (5) and Lemma 1(e), one can conclude that

$$\lim_{s\to r} |s-r|^{\frac{1}{p}} \{M(p,r,\lambda,s)\}^{\frac{-1}{p}} = \frac{1}{|\lambda - r|r|^p|^{\frac{1}{p+1}}} \in (0,\infty),$$

$$\lim_{s\to r_0} |s-r_0|^{\frac{1}{p}} \{M(p,r,\lambda,s)\}^{\frac{-1}{p}} = \frac{1}{|\lambda - r_0|r_0|^p|^{\frac{1}{p+1}}} \in (0,\infty).$$

Also we know that the integrals $\int_{r_0}^{r} |s-r|^{\frac{-1}{p}} ds$ *and* $\int_{r_0}^{r} |s-r_0|^{\frac{-1}{p}} ds$ for $p > 1$ are convergent. Thus one can conclude that the convergence of the integral $\int_{r_0}^{r} \{M(p,r,\lambda,s)\}^{\frac{-1}{p}} ds$ is a consequence of that of the integrals $\int_{r_0}^{r} |s-r|^{\frac{-1}{p}} ds$ and $\int_{r_0}^{r} |s-r_0|^{\frac{-1}{p}} ds$. Δ

Here the proof of Lemma 3 is complete. Δ

If $\lambda \in \left(\frac{r|r|^p}{p+2}, r|r|^p\right)$, by the Lemma 1(e), $r > min_{x\in[0,1]} u(x) = r_0 > 0$. Hence u must be positive solution. Also If $\lambda \in (r|r|^p, \infty)$, by the Lemma 1(e), $min_{x\in[0,1]} u(x) = r > 0$. Hence u must be positive solution. Here the proof of Theorem 1(a) is complete. By the proof of the Theorem 1(a) and Remark 1, the proof of Theorem 1(b) is clear.

4. Refrences

[1] I. Addou. On the number of solutions for boundary-value problems with jumping nonlinearities, Ph.D. Thesis. Université des Sciences et de la Technologie Houari Boumedienne. Algiers, Algeria, 2000.

[2] G.A. Afruzi and M.Khaleghy Moghadaam. Existence and multiplicity results for a class of p-Laplacian problems with Neumann-Robin boundary conditions. *Ghaos, Solitons & Fractals.* 2006, **30**: 967-973.

[3] G. A. Afrouzi and M. Khalehghy Moghaddam. Nonnegative solution Curves of Semipositone Problems With Dirichlet Boundary conditions. *Nonlinear Analysis, Theory, methods, and Applications.* 2005, **61**: 485-489.

[4] F. Ammar-Khodja. Une revue et quelques compléments sur la détermination du nombre des solutions de certains problémes elliptiques semi-linéaires. These Doctorat 3e Cycle, Université Pierre et Marie Curie, Paris VI, 1983.

[5] V. Anuradha, C. Maya and R. Shivaji. Positive solutions for a class of nonlinear boundary value problems with Neumann-Robin boundary conditions. *J. Math. Anal. Appl.* 1999, **236**: 94-124.

[6] A. Castro and R. Shivaji, Nonnegative solutions for a class non-positone problems. *Proc. Roy. Soc. Edinburgh, Sect*. 1988, A 108: 291-302.

[7] M. Guedda and L. Veron. Bifurcation phenomena associated to the p-Laplacian operator. *Trans. Amer. Math. Soc*. 1998, **310**: 419-431.

[8] A. R. Miciano and R. Shivaji. Multiple positive solutions for a class of semipositone Neumann two point boundary value problems. *J. Math. Anal. Appl.* 1993, **178**: 102-115.

[9] M. Ramaswamy. *Thése 3 éme cycle.* Univ. Pierre et Marie Curie, Paris VI, 1983.