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On Order of a Function of Several Complex Variables Analytic in the Unit Polydisc

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Abstract: This paper is concerned with the study of the maximum modulus and the coefficients of the power series expansion of a function of several complex variables analytic in the unit polydisc.

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1. Introduction and Definitions

Let $f(z)=\sum_{n=0}^{\infty} c_n z^n$ be analytic in the unit disc $U=\{z:|z|<1\}$ and $M(r)=M(r,f)$ be the maximum of $|f(z)|$ on $|z|=r$.

In 1968 Sons [8] introduced the following definition of the order ρ and the lower order λ as

$$\begin{matrix}\rho \\ \lambda\end{matrix} = \lim_{r\to 1} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log\log M(r,f)}{-\log(1-r)}.$$

Maclane [6] and Kapoor [5] proved the following results which are the characterization of order and lower order of a function f analytic in U, in terms of the coefficients c_n.

Theorem 1.1 [6] Let $f(z)=\sum_{n=0}^{\infty} c_n z^n$ be analytic in U, having order $\rho\,(0\le\rho\le\infty)$. Then

$$\frac{\rho}{1+\rho} = \limsup_{n\to\infty} \frac{\log^+\log^+|c_n|}{\log n}.$$

Theorem 1.2 [5] Let $f(z)=\sum_{n=0}^{\infty} c_n z^n$ be analytic in U, having lower order $\lambda\,(0\le\lambda\le\infty)$. Then

$$\frac{\lambda}{1+\lambda} \ge \liminf_{n\to\infty} \frac{\log^+\log^+|c_n|}{\log n}.$$

Notation 1.3 [7] $\log^{[0]} x = x$, $\exp^{[0]} x = x$ and for positive integer m, $\log^{[m]} x = \log(\log^{[m-1]} x)$, $\exp^{[m]} x = \exp(\exp^{[m-1]} x)$.

In a paper [4] Juneja and Kapoor introduced the definition of p-th order and lower p-th order and in 2005 Banerjee [1] generalized Theorem 1.1 and Theorem 1.2 for p-th order and lower p-th order respectively.

Definition 1.4 [4] If $f(z)=\sum_{n=0}^{\infty} c_n z^n$ be analytic in U, its p-th order ρ_p and lower p-th order λ_p are defined as

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$$\begin{matrix}\rho_p \\ \lambda_p\end{matrix} = \lim_{r\to 1} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p]} M(r)}{-\log(1-r)}, \; p \geq 2.$$

Theorem 1.5 [1] Let $f(z)=\sum_{n=0}^{\infty} c_n z^n$ be analytic in U and having p-th order ρ_p $(0 \leq \rho_p \leq \infty)$. Then

$$\frac{\rho_p}{1+\rho_p} = \limsup_{n\to\infty} \frac{\log^{+[p]} |c_n|}{\log n}.$$

Theorem 1.6 [1] Let $f(z)=\sum_{n=0}^{\infty} c_n z^n$ be analytic in U and having lower p-th order λ_p $(0 \leq \lambda_p \leq \infty)$. Then

$$\frac{\lambda_p}{1+\lambda_p} \geq \liminf_{n\to\infty} \frac{\log^{+[p]} |c_n|}{\log n}.$$

In 2008 Banerjee and Dutta [2] introduced the following definition.

Definition 1.7 Let $f(z_1,z_2)$ be a non-constant analytic function of two complex variables z_1 and z_2 holomorphic in the closed unit polydisc

$$P:\{(z_1,z_2):|z_j|\leq 1;\; j=1,2\}$$

then order of f is denoted by ρ and is defined by

$$\rho = \inf\left\{\mu>0 : F(r_1,r_2) < \exp\left(\frac{1}{1-r_1}.\frac{1}{1-r_2}\right)^{\mu} ;\text{ for all } 0<r_0(\mu)<r_1,r_2<1\right\}.$$

Equivalent formula for ρ is

$$\rho = \limsup_{r_1,r_2\to 1} \frac{\log\log F(r_1,r_2)}{-\log(1-r_1)(1-r_2)}.$$

In a resent paper [3] Banerjee and Dutta introduce the definition of p-th order and lower p-th order of functions of two complex variables analytic in the unit polydisc and generalized the above results for functions of two complex variables analytic in the unit polydisc.

Definition 1.8 Let $f(z_1,z_2)=\sum_{m,n=0}^{\infty} c_{mn} z_1^m z_2^n$ be a function of two complex variables z_1, z_2 holomorphic in the unit polydisc

$$U=\{(z_1,z_2):|z_j|\leq 1;\; j=1,2\}$$

and

$$F(r_1,r_2)=\max\{|f(z_1,z_2)|:|z_j|\leq r_j;\; j=1,2\},$$

be its maximum modulus. Then the p-th order ρ_p and lower p-th order λ_p are defined as

$$\begin{matrix}\rho_p \\ \lambda_p\end{matrix} = \lim_{r_1,r_2\to 1} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p]} F(r_1,r_2)}{-\log(1-r_1)(1-r_2)}, \; p \geq 2.$$

When $p=2$, Definition 1.8 coincides with Definition 1.7.

Theorem 1.9 Let $f(z_1,z_2)$ be analytic in U and having p-th order ρ_p $(0 \leq \rho_p \leq \infty)$. Then

$$\frac{\rho_p}{1+\rho_p} = \limsup_{m,n\to\infty} \frac{\log^{+[p]} |c_{mn}|}{\log mn}.$$

Theorem 1.10 Let $f(z_1,z_2)$ be analytic in U and having lower p-th order λ_p $(0 \leq \lambda_p \leq \infty)$. Then

$$\frac{\lambda_p}{1+\lambda_p} \geq \liminf_{m,\, n\to\infty} \frac{\log^{+[p]} |c_{mn}|}{\log mn}.$$

When $p=2$ then from Theorem 1.9 and Theorem 1.10 we get this two theorems.

Theorem 1.11 Let $f(z_1, z_2)$ be analytic in U and having order $\rho(0 \leq \rho \leq \infty)$. Then

$$\frac{\rho}{1+\rho} = \limsup_{m,n\to\infty} \frac{\log^+ \log^+ |c_{mn}|}{\log mn}.$$

Theorem 1.12 Let $f(z_1, z_2)$ be analytic in U and having lower order $\lambda(0 \leq \lambda \leq \infty)$.

Then

$$\frac{\lambda}{1+\lambda} \geq \liminf_{m,n\to\infty} \frac{\log^+ \log^+ |c_{mn}|}{\log mn}$$

In this paper we consider a more general situation in the case of analytic functions of several complex variables in the unit polydisc and for which we introduce the following definition.

Definition 1.13 Let $f(z_1, z_2, \ldots\ldots z_n) = \sum_{m_1, m_2, \ldots\ldots m_n = 0}^{\infty} c_{m_1 m_2 \ldots\ldots m_n} z_1^{m_1} z_2^{m_2} \ldots\ldots z_n^{m_n}$ be a function of n complex variables $z_1, z_2, \ldots\ldots z_n$ holomorphic in the unit polydisc

$$U = \{(z_1, z_2, \ldots\ldots z_n) : |z_j| \leq 1;\ j = 1, 2, \ldots\ldots n\}$$

and

$$F(r_1, r_2, \ldots\ldots r_n) = \max\{|f(z_1, z_2, \ldots\ldots z_n)| : |z_j| \leq r_j;\ j = 1, 2, \ldots\ldots n\},$$

be its maximum modulus. Then the order ρ and lower order λ are defined as

$$\begin{matrix}\rho \\ \lambda\end{matrix} = \lim_{r_1, r_2, \ldots\ldots r_n \to 1} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log\log F(r_1, r_2, \ldots\ldots r_n)}{-\log(1-r_1)(1-r_2)\ldots\ldots(1-r_n)}.$$

When $p=2$, Definition 1.13 coincides with Definition 1.7.

In this paper we find a similar analytic expression for ρ and λ in terms of the coefficients $c_{m_1 m_2 \ldots\ldots m_n}$ for several complex variables.

Here $f(z_1, z_2, \ldots\ldots z_n) = \sum_{m_1, m_2, \ldots\ldots m_n = 0}^{\infty} c_{m_1 m_2 \ldots\ldots m_n} z_1^{m_1} z_2^{m_2} \ldots\ldots z_n^{m_n}$ will denote a function analytic in the unit polydisc.

2. Lemmas

The following lemmas will be needed in the sequel.

Lemma 2.1 Let the maximum modulus $F(r_1, r_2, \ldots\ldots r_n)$ of a function $f(z_1, z_2, \ldots\ldots z_n)$ analytic in U, satisfy

$$\log F(r_1, r_2, \ldots\ldots r_n) < A \left\{ \prod_{j=1}^{n} (1 - r_j) \right\}^{-B} \tag{1}$$

$1 < A, B < \infty$ for all r_j such that $r_0(A, B) < r_j < 1;\ j = 1, 2, \ldots\ldots n,$

Then for all $m_j > m_{j_0}(A, B) > 1;\ j = 1, 2, \ldots\ldots n$

$$\log |c_{m_1 m_2 \ldots\ldots m_n}| \leq S(A, B) \left(\prod_{j=1}^{n} m_j \right)^{\frac{B}{B+1}}$$

where

$$S(A,B)=(1+2B)\left(\frac{A}{B^B}\right)^{\frac{1}{1+B}}.$$

Proof. At first define n sequences $\{r_{jm_j}\}$ by

$$(1-r_{jm_j})^{-1}=\left(\frac{m_j}{AB}\right)^{\frac{1}{B+1}};\ j=1,2,......n.$$

Then $r_{jm_j}\to 1$ as $m_j\to\infty$ for all $j=1,2,......n.$

Also,

$$\begin{aligned}
|c_{m_1 m_2 ...... m_n}| &= \frac{1}{\prod_{j=1}^{n}(m_j!)}\left|\frac{\partial^{m_1+m_2+......+m_n} f(0,0,......0)}{\partial z_1^{m_1}\ \partial z_2^{m_2}\ ......\partial z_n^{m_n}}\right| \\
&=\left|\frac{1}{(2\pi i)^n}\int_{|z_1|=r_1}\ \int_{|z_2|=r_2}\ ......\int_{|z_n|=r_n}\frac{f(z_1,z_2\,......\,z_n)dz_1\,dz_2......dz_n}{z_1^{m_1+1}z_2^{m_2+1}\,......\,z_n^{m_n+1}}\right| \\
&\le \frac{F(r_1,r_2\,......r_n)}{r_1^{m_1}\ r_2^{m_2}\ ......r_n^{m_n}} \\
&=\frac{F(r_1,r_2\,......r_n)}{\prod_{j=1}^{n} r_j^{m_j}} \qquad (2)
\end{aligned}$$

From (1) and (2) we have for all $m_j>m_{j_0}(A,B)>1;\ j=1,2,......n$

$$\begin{aligned}
\log|c_{m_1 m_2 ...... m_n}| &\le \log F(r_{1m_1},r_{2m_2}\,......r_{nm_n})-\sum_{j=1}^{n} m_j\log r_{jm_j} \\
&< A\left\{\prod_{j=1}^{n}\left(1-r_{jm_j}\right)\right\}^{-B}+\left[\sum_{j=1}^{n} m_j\left(1-r_{jm_j}\right)\right][1+O(1)] \\
&= A\left(\frac{\prod_{j=1}^{n} m_j}{(AB)^n}\right)^{\frac{B}{B+1}}+\left[\sum_{j=1}^{n} m_j\left(\frac{AB}{m_j}\right)^{\frac{1}{B+1}}\right][1+O(1)] \\
&\le A\left(\frac{\prod_{j=1}^{n} m_j}{AB}\right)^{\frac{B}{B+1}}+\prod_{j=1}^{n} m_j\left(\frac{AB}{\prod_{j=1}^{n} m_j}\right)^{\frac{1}{B+1}}[1+O(1)] \\
&= A\left(\frac{\prod_{j=1}^{n} m_j}{AB}\right)^{\frac{B}{B+1}}+AB\left(\frac{\prod_{j=1}^{n} m_j}{AB}\right)^{\frac{B}{B+1}}[1+O(1)]
\end{aligned}$$

$$= [1 + B\{1 + O(1)\}]\left\{ A \left(\frac{\prod_{j=1}^{n} m_j}{AB} \right)^{\frac{B}{B+1}} \right\}$$

$$\le \left\{ (1+2B)\left(\frac{A}{B^B} \right)^{\frac{1}{B+1}} \right\} \left(\prod_{j=1}^{n} m_j \right)^{\frac{B}{B+1}}.$$

Therefore

$$\log |c_{m_1 m_2 \ldots\ldots m_n}| \le S(A,B)\left(\prod_{j=1}^{n} m_j \right)^{\frac{B}{B+1}}$$

where

$$S(A,B) = (1+2B)\left(\frac{A}{B^B} \right)^{\frac{1}{1+B}}.$$

This proves the lemma.

Lemma 2.2 Let $f(z_1, z_2, \ldots\ldots z_n)$ be analytic in U and satisfy

$$|c_{m_1 m_2 \ldots\ldots m_n}| < \exp\left\{ \sum_{j=1}^{n} \left(C m_j^{\,D} \right) \right\}, \quad 0 < C < \infty,\ 0 < D < 1$$

for all $m_j > m_{j_0}(C,D);\ j = 1,2,\ldots\ldots n.$ Then for all r_j such that $r_{j0}(C,D) < r_j < 1;\ j = 1,2,\ldots\ldots n$

$$\log F(r_1, r_2, \ldots\ldots r_n) < T(C,D)\left(\prod_{j=1}^{n} \log \frac{1}{r_j} \right)^{\frac{-D}{1-D}},$$

where

$$T(C,D) = C^{\frac{1}{1-D}}\, D^{\frac{D}{1-D}}[1 + o(1)].$$

Proof. For all $m_j > m_{j_0}(C,D);\ j = 1,2,\ldots\ldots n,$

$$|c_{m_1 m_2 \ldots\ldots m_n}| < \exp\left\{ \sum_{j=1}^{n} \left(C m_j^{\,D} \right) \right\}$$

$$= \prod_{j=1}^{n} \exp\left(C m_j^{\,D} \right).$$

Now for $|z_j| = r_j < 1;\ j = 1,2,\ldots\ldots n$

$$F(r_1, r_2, ...r_n) < \sum_{m_1, m_2, ... m_n = 0}^{\infty} |c_{m_1 m_2 ...... m_n}| r_1^{m_1} r_2^{m_2} ...... r_n^{m_n}$$

$$< K\left(m_{1_0}, m_{2_0}, ...m_{n_0}\right) + \sum_{\substack{m_1 = m_{1_0}+1 \\ m_2 = m_{2_0}+1 \\ \vdots \\ m_n = m_{n_0}+1}}^{\infty} \left\{ \prod_{j=1}^{n} \exp\left(C m_j^{D}\right) r_j^{m_j} \right\}$$

$$\leq K\left(m_{1_0}, m_{2_0}, ...m_{n_0}\right) + \prod_{j=1}^{n} \left[\sum_{m_j = m_{j_0}+1}^{\infty} \exp\left(C m_j^{\frac{B}{B+1}} \right) r_j^{m_j} \right],$$

where $B = \dfrac{D}{1-D}$.

Choose

$$M_j = M(r_j) = \left[\left(\frac{2C}{\log \frac{1}{r_j}} \right)^{B+1} \right]; \; j = 1, 2......n,$$

where $[x]$ denotes the greatest integer not greater than x.

Clearly $M(r_j) \to \infty$ as $r_j \to 1$ for all $j = 1, 2, ...n$.

The above estimate of $F(r_1, r_2, ...r_n)$ for r_j; $j = 1, 2, ...n$ sufficiently close to 1 gives,

$$F(r_1, r_2, ...r_n) < K(m_{1_0}, m_{2_0}, ...m_{n_0}) + \prod_{j=1}^{n} \left[M(r_j) H(r_j) + \sum_{m_j = M_j + 1}^{\infty} r_j^{m_j/2} \right] \tag{3}$$

where

$$H(r_j) = \max_{m_j} \left\{ \exp\left(C m_j^{\frac{B}{B+1}} \right) r_j^{m_j} \right\}; \; j = 1, 2, ...n$$

for if $m_j \geq M_j + 1$, then

$$m_j > \left(\frac{2C}{\log \frac{1}{r_j}} \right)^{B+1}$$

$$i.e. \;\; C m_j^{\frac{B}{B+1}} < \frac{m_j}{2} \log \frac{1}{r_j}$$

$$i.e. \;\; \exp\left(C m_j^{\frac{B}{B+1}} \right) r_j^{m_j} < r_j^{\frac{m_j}{2}}$$

for all $j = 1, 2, ...n$.

Therefore the infinite series $\sum_{m_j = M_j + 1}^{\infty} r_j^{\frac{m_j}{2}}$ in (3) is bounded by $r_j^{\frac{M_j+1}{2}} \left(\dfrac{1}{1 - r_j^{\frac{1}{2}}} \right)$ for all $j = 1, 2, ...n$.

Since $B > 0$, we have

$$-\frac{M_j+1}{2}\log\frac{1}{r_j}-\log\left(1-r_j^{\frac{1}{2}}\right)<-\frac{1}{2}\left(\frac{2C}{\log\frac{1}{r_j}}\right)^{B+1}\log\frac{1}{r_j}-\log(1-r_j)+\log\left(1+r_j^{\frac{1}{2}}\right)$$

$$\to -\infty \ as\, r_j \to 1.$$

Thus for r_j sufficiently close to 1,

$$\sum_{m_j=M_j+1}^{\infty} r_j^{\frac{m_j}{2}} = o(1)$$

for all $j=1,2,......n.$

The maximum of $\exp\left(Cm_j^{\frac{B}{B+1}}\right)r_j^{m_j}$ assume at the point $m_j=\left(\frac{BC}{(B+1)\log\frac{1}{r_j}}\right)^{B+1}$ and $H(r_j)$ is given by

$$\log H(r_j)=Cm_j^{\frac{B}{B+1}}+m_j\log r_j$$
$$=\left\{\frac{C.B^B.C^B}{(B+1)^B\left(\log\frac{1}{r_j}\right)^B}\right\}-\left(\frac{(BC)^{B+1}\log\frac{1}{r_j}}{(B+1)^{B+1}\left(\log\frac{1}{r_j}\right)^{B+1}}\right) \tag{4}$$
$$\le\left\{\frac{C^{B+1}.B^B}{(B+1)^B\left(\log\frac{1}{r_j}\right)^B}\right\}.$$

Thus for $r_j;\, j=1,2,......n$ sufficiently close to 1, from (3)

$$F(r_1,r_2,......r_n)<\prod_{j=1}^{n}\left[M(r_j)H(r_j)+o(1)\right]\left[1+\frac{K\left(m_{1_0},m_{2_0},......m_{n_0}\right)}{\prod_{j=1}^{n}\left[M(r_j)H(r_j)+o(1)\right]}\right]$$
$$=\prod_{j=1}^{n}\left[M(r_j)H(r_j)+o(1)\right][1+O(1)].$$

$$\therefore \log F(r_1, r_2, ...r_n) < \sum_{j=1}^{n} \left\{ \log M(r_j) + \log H(r_j) \right\} + O(1)$$

$$\leq \sum_{j=1}^{n} \left\{ -(B+1)\log^{[2]} \frac{1}{r_j} + \frac{C^{B+1}.B^B}{(B+1)^B \left(\log \frac{1}{r_j} \right)^B} \right\} + O(1) \ [\textit{from } (4)]$$

$$\leq \sum_{j=1}^{n} \left\{ \frac{C^{B+1}.B^B}{(B+1)^B \left(\log \frac{1}{r_j} \right)^B} \right\} + O(1)$$

$$= \frac{C^{B+1}.B^B}{(B+1)^B} \sum_{j=1}^{n} \left(\log \frac{1}{r_j} \right)^{-B} + O(1)$$

$$\leq \left[C^{B+1} \left(\frac{B}{1+B} \right)^B \prod_{j=1}^{n} \left(\log \frac{1}{r_j} \right)^{-B} \right] [1+O(1)]$$

$$= C^{\frac{1}{1-D}} D^{\frac{D}{1-D}} [1+o(1)] \left(\prod_{j=1}^{n} \log \frac{1}{r_j} \right)^{\frac{-D}{1-D}}.$$

Therefore

$$\log F(r_1, r_2, ...r_n) < T(C, D) \left(\prod_{j=1}^{n} \log \frac{1}{r_j} \right)^{\frac{-D}{1-D}}$$

where

$$T(C, D) = C^{\frac{1}{1-D}} D^{\frac{D}{1-D}} [1+o(1)].$$

This proves the lemma.

3. Theorems

We prove the following theorems.

Theorem 3.1 Let $f(z_1, z_2, ...z_n)$ be analytic in U and having order $\rho (0 \leq \rho \leq \infty)$. Then

$$\frac{\rho}{1+\rho} = \limsup_{m_1 m_2 ...... m_n \to \infty} \frac{\log^+ \log^+ |c_{m_1 m_2 ...... m_n}|}{\log \left(\prod_{j=1}^{n} m_j \right)}. \quad (5)$$

Proof. If $|c_{m_1 m_2 ...... m_n}|$ is bounded by K for all $m_j; j = 1, 2......n$ then $\sum_{m_1, m_2, ... m_n = 0}^{\infty} c_{m_1 m_2 ...... m_n} z_1^{m_1} z_2^{m_2} ...... z_n^{m_n}$ is bounded by $\frac{K}{\prod_{j=1}^{n}(1-r_j)}$.

Therefore

$$F(r_1, r_2 ......r_n) \le \sum_{m_1, m_2, ... m_n = 0}^{\infty} | c_{m_1 m_2 ...... m_n} | r_1^{m_1} r_2^{m_2} ...... r_n^{m_n}$$

$$\le \frac{K}{\prod_{j=1}^{n} (1 - r_j)}$$

$$< \exp\left(\frac{1}{\prod_{j=1}^{n} (1 - r_j)} \right)^{\varepsilon}$$

for any $0 < \varepsilon < 1$ and for all $r_j; j = 1, 2, ... n$ sufficiently close to 1.

Therefore

$$\rho = \limsup_{r_1, r_2, ... r_n \to 1} \frac{\log \log F(r_1, r_2 ...... r_n)}{-\log\left(\prod_{j=1}^{n} (1 - r_j) \right)} \le \varepsilon$$

since $0 < \varepsilon < 1$ arbitrary, $\rho = 0$ and so (5) is satisfied.

Thus we need to consider only the case

$$\limsup_{m_1 m_2 ...... m_n \to \infty} | c_{m_1 m_2 ...... m_n} | = \infty.$$

In this case all the $\log^+$ in (5) may be replaced by log. First let $0 < \rho < \infty$ and $\rho' > \rho.$

Then for all $r_j; j = 1, 2, ... n$ sufficiently close to 1,

$$\log F(r_1, r_2 ...... r_n) < \left\{ \prod_{j=1}^{n} (1 - r_j) \right\}^{-\rho'}.$$

Using Lemma 2.1 with $A = 1, B = \rho'$ it follows from the above inequality that for $m_j > m_{j_0}(\rho'); j = 1, 2 ...... n$

$$\log | c_{m_1 m_2 ...... m_n} | \le (1 + 2\rho') \left(\frac{1}{\rho'^{\rho'}} \right)^{\frac{1}{1+\rho'}} \left(\prod_{j=1}^{n} m_j \right)^{\frac{\rho'}{\rho'+1}}.$$

Therefore

$$\limsup_{m_1, m_2, ... m_n \to \infty} \frac{\log \log | c_{m_1 m_2 ...... m_n} |}{\log\left(\prod_{j=1}^{n} m_j \right)} \le \frac{\rho'}{1 + \rho'}.$$

Since $\rho' > \rho$ is arbitrary, it follows that

$$\limsup_{m_1, m_2, ... m_n \to \infty} \frac{\log \log | c_{m_1 m_2 ...... m_n} |}{\log\left(\prod_{j=1}^{n} m_j \right)} \le \frac{\rho}{1 + \rho}. \tag{6}$$

Since $f(z_1, z_2, ... z_n)$ is analytic in U, the above inequality is trivially true if $\rho = \infty$ and the right hand side is interpreted as 1 in this case.

Conversely, if

$$\theta = \limsup_{m_1, m_2, \ldots\ldots m_n \to \infty} \frac{\log\log |c_{m_1 m_2 \ldots\ldots m_n}|}{\log\left(\prod_{j=1}^{n} m_j\right)}$$

then $0 \le \theta \le 1$.

First let $\theta < 1$ and choose $\theta < \theta' < 1$.

Then for all sufficiently large $m_j;\ j = 1, 2, \ldots\ldots n$,

$$\log |c_{m_1 m_2 \ldots\ldots m_n}| < \left(\prod_{j=1}^{n} m_j\right)^{\theta'}.$$

Using Lemma 2.2 with $C = 1, D = \theta'$ it follows from the above inequality that for all r_j such that $r_{j_0}(\theta') < r_j < 1;\ j = 1, 2, \ldots\ldots n$,

$$\log F(r_1, r_2, \ldots\ldots r_n) < \theta'^{\frac{\theta'}{1-\theta'}} \left(\prod_{j=1}^{n} \log \frac{1}{r_j}\right)^{\frac{-\theta'}{1-\theta'}} [1 + o(1)].$$

$$\therefore\ \log\log F(r_1, r_2, \ldots\ldots r_n) < \frac{\theta'}{1-\theta'} \log(\theta') + \frac{-\theta'}{1-\theta'} \log\left(\prod_{j=1}^{n} \log \frac{1}{r_j}\right) + \log[1 + o(1)]$$

that is

$$\limsup_{r_1, r_2, \ldots\ldots r_n \to 1} \frac{\log\log F(r_1, r_2, \ldots\ldots r_n)}{-\log\left(\prod_{j=1}^{n} (1 - r_j)\right)} \le -\frac{\theta'}{1-\theta'} \limsup_{r_1, r_2, \ldots\ldots r_n \to 1} \frac{\log\left(\prod_{j=1}^{n} \log \frac{1}{r_j}\right)}{-\log\left(\prod_{j=1}^{n} (1 - r_j)\right)}.$$

$$\therefore\quad \rho < \frac{\theta'}{1-\theta'}.$$

Since $\theta' > 0$ is arbitrary, it follows that

$$\frac{\rho}{1+\rho} \le \theta = \limsup_{m_1, m_2, \ldots\ldots m_n \to \infty} \frac{\log\log |c_{m_1 m_2 \ldots\ldots m_n}|}{\log\left(\prod_{j=1}^{n} m_j\right)}. \tag{7}$$

If $\theta = 1$, the above inequality is obviously true.

Inequality (6) and (7) together gives (5) when $\limsup_{m_1 m_2 \ldots\ldots m_n \to \infty} |c_{m_1 m_2 \ldots\ldots m_n}| = \infty$.

This proves the theorem.

Theorem 3.2 Let $f(z_1, z_2, \ldots\ldots z_n)$ be analytic in U and having lower order $\lambda\,(0 \le \lambda \le \infty)$. Then

$$\frac{\lambda}{1+\lambda} \ge \liminf_{m_1, m_2, \ldots\ldots m_n \to \infty} \frac{\log^{+}\log^{+} |c_{m_1 m_2 \ldots\ldots m_n}|}{\log\left(\prod_{j=1}^{n} m_j\right)}.$$

Proof. Let

$$\liminf_{m_1, m_2, \ldots\ldots m_n \to \infty} \frac{\log^{+}\log^{+} |c_{m_1 m_2 \ldots\ldots m_n}|}{\log\left(\prod_{j=1}^{n} m_j\right)} = A. \tag{8}$$

First suppose that $0 < A < 1$.

From (8), for $0 < \varepsilon < A < 1,$

$$\log |c_{m_1 m_2 \ldots\ldots m_n}| > \left(\prod_{j=1}^{n} m_j \right)^{A-\varepsilon}$$

for $m_j > M_j = M_j(\varepsilon);\ j = 1, 2, \ldots\ldots n.$

Also

$$|c_{m_1 m_2 \ldots\ldots m_n}| \le \frac{F(r_1, r_2, \ldots\ldots r_n)}{\prod_{j=1}^{n} r_j^{\,m_j}}.$$

$$\therefore\ \log F(r_1, r_2, \ldots\ldots r_n) \ge \log |c_{m_1 m_2 \ldots\ldots m_n}| + \sum_{j=1}^{n} m_j \log r_j\,. \tag{9}$$

Choose

$$m_j = \left(\log \frac{1}{r_j} \right)^{\frac{1}{A-\varepsilon-1}} \ where\ \ j = 1, 2 \ldots\ldots n.$$

Then from (9)

$$\log F(r_1, r_2, \ldots\ldots r_n) > \left(\prod_{j=1}^{n} \log \frac{1}{r_j} \right)^{\frac{A-\varepsilon}{A-\varepsilon-1}} - \sum_{j=1}^{n} \left(\log \frac{1}{r_j} \right)^{\frac{1}{A-\varepsilon-1}} \log \frac{1}{r_j}$$

$$= \left(\prod_{j=1}^{n} \log \frac{1}{r_j} \right)^{\frac{A-\varepsilon}{A-\varepsilon-1}} - \sum_{j=1}^{n} \left(\log \frac{1}{r_j} \right)^{\frac{A-\varepsilon}{A-\varepsilon-1}}$$

$$> \frac{1}{k} \left(\prod_{j=1}^{n} \log \frac{1}{r_j} \right)^{\frac{A-\varepsilon}{A-\varepsilon-1}}$$

where *k* is a suitable constant.

$$\therefore \qquad \log\log F(r_1, r_2, \ldots\ldots r_n) > \frac{A-\varepsilon}{A-\varepsilon-1} \log \left(\prod_{j=1}^{n} \log \frac{1}{r_j} \right) + O(1).$$

$$\therefore \quad \frac{\log\log F(r_1, r_2, \ldots\ldots r_n)}{-\log \left(\prod_{j=1}^{n} (1-r_j) \right)} > \frac{A-\varepsilon}{A-\varepsilon-1} \frac{\log \left(\prod_{j=1}^{n} \log \frac{1}{r_j} \right)}{-\log \left(\prod_{j=1}^{n} (1-r_j) \right)} + O(1).$$

$$\therefore \qquad \lambda = \liminf_{r_1, r_2, \ldots\ldots r_n \to 1} \frac{\log\log F(r_1, r_2, \ldots\ldots r_n)}{-\log \left(\prod_{j=1}^{n} (1-r_j) \right)}$$

$$\ge \frac{A-\varepsilon}{1-A+\varepsilon}.$$

Since $0 < \varepsilon < A < 1$ is arbitrary,

$$\lambda \ge \frac{A}{1-A}.$$

This implies

$$\frac{\lambda}{1+\lambda} \geq A.$$

This inequality holds obviously when $A = 0$. For $A = 1$ the above arguments with a number K arbitrarily near to 1 in place of $A - \varepsilon$, give

$$\frac{\lambda}{1+\lambda} = 1.$$

This proves the theorem.

4. Reference

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