ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 6, No. 2, 2011, pp. 129-142

Exact Solutions for Some Nonlinear Partial Differential Equations in Mathematical Physics

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(*Received December 20, 2010, accepted December 28, 2010*)

Abstract. In this article, by introducing a new general ansatze, the improved (G'/G)- expansion - method is proposed to construct exact solutions of some nonlinear partial differential equations in mathematical physics via the generalized Zakharov equations, the coupled Maccaris equations, the (2+1)-dimensional Wu-Zhang equations and the (1+1) dimensional Fornberg – Whitham equation in terms of the hyperbolic functions , trigonometric functions and rational function, where G satisfies a second order linear ordinary differential equation. When the parameters are taken special values, the solitary wave are derived from the traveling waves. This method is reliable, simple and gives many new exact solutions.

Keywords: The improved (G'/G)- expansion method, Traveling wave solutions, The generalized Zakharov equations, The coupled Maccaris equations, The (1+1) dimensional Fornberg – Whitham equation , The (2+1)-dimensional Wu-Zhang equations.

1. Introduction

Nonlinear partial differential equations are known to describe a wide variety of phenomena not only in physics, where applications extend over magneto fluid dynamics, water surface gravity waves, electromagnetic radiation reactions, and ion acoustic waves in plasma, just to name a few, but also in biology and chemistry, and several other fields. It is one of the important tasks in the study of the nonlinear partial differential equations to seek exact and explicit solutions. In the past several decades both mathematicians and physicists have made many attempts in this direction. Various methods for obtaining exact solutions to nonlinear partial differential equations had been proposed. Among these are the inverse scattering method [1], Hirota's bilinear method [2], Backlund transformation [3,4], Painlevé expansion [5], sine–cosine method [6], homogenous balance method [7], homotopy perturbation method [8–11], variation method [12,13], Adomian decomposition method [14,15], tanh - function method [16–18], Jacobi elliptic function expansion method [19–22], F-expansion method [23–25] and Exp-function method [26–28].

Wang etal [29] proposed a new method called the (G'/G) expansion method to look for the traveling wave solutions for nonlinear partial differential equations (NPDEs) . By using the (G'/G) expansion method, Zayed etal [30,31] and the modified (G'/G) expansion method, Shehata [32] have successfully obtained more traveling wave solutions for some important NPDEs. Recently Guo etal [33] had developed the (G'/G) expansion method for solving the NPDEs. In this paper we use the improvement (G'/G) expansion method to find the traveling wave solutions for the generalized Zakharov equations, the coupled Maccaris equations, the (2+1)-dimensional Wu-Zhang equations and the (1+1) dimensional Fornberg – Whitham equation.

2. Description of the improvement (G'/G) expansion method for NPDEs

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In this section, we give the detailed description of our method. Suppose that a nonlinear evolution equation, say in two independent variables x and t is given by

$$P(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, ..) = 0, \tag{1}$$

where $u = u(x,t)$ is an unknown function, P is a polynomial in $u = u(x,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

To determine u explicitly, we take the following five steps [33]:

Step 1: We use the following travelling wave transformation:

$$u = U(\xi), \qquad \xi = x - kt, \tag{2}$$

where k is a constant to be determined latter. The NPDE (1) is reduced to a nonlinear ordinary differential equation (NODE) in $U(\xi)$:

$$P(U, U', U'', \ldots) = 0 \; . \tag{3}$$

Step 2. We suppose the following series expansion as a solution of Eq. (3):

$$U(\xi) = \sum_{i=-m}^{m} \frac{\alpha_i \left(\frac{G'(\xi_n)}{G(\xi_n)}\right)^i}{\left[1 + \sigma\left(\frac{G'(\xi_n)}{G(\xi_n)}\right)\right]^i}, \tag{4}$$

where $\alpha_i (i = 0, \pm 1, \ldots, \pm m)$, σ are constants to be determined later, m is a positive integer and $G(\xi)$ satisfies a second order linear ordinary differential equation

$$G''(\xi) + \mu G(\xi) = 0, \tag{5}$$

where μ is a real constants. The general solutions of Eq. (5), can be listed as follows. When $\mu < 0$, we obtain the hyperbolic function solution of Eq.(5)

$$G(\xi) = C_1 \cosh(\sqrt{-\mu}\xi) + C_2 \sinh(\sqrt{-\mu}\xi). \tag{6}$$

When $\mu > 0$, we obtain the trigonometric function solution of Eq.(5)

$$G(\xi) = C_1 \sin(\sqrt{\mu}\xi) + C_2 \cos(\sqrt{\mu}\xi) \; . \tag{7}$$

When $\mu = 0$, we obtain the rational function solution of Eq.(5)

$$G(\xi) = C_1\xi + C_2 \, . \tag{8}$$

where C_1 and C_2 are arbitrary constants.

Step 3. Determine the positive integer m by balancing the highest order nonlinear term(s) and the highest order derivative in Eqs. (1) or (3).

Step 4. Substituting Eq. (4) along with (5) into (3) , cleaning the denominator and then setting all the coefficients of $(G'(\xi)/G(\xi))^i, i = 0, \pm 1, \pm 2, ..$ to be zero, yield a set of algebraic equations for which the constants $\alpha_i (i = 0, \pm 1, \ldots, \pm m)$, k and σ .

Step 5. Assuming that the constants $\alpha_i (i = 0, \pm 1, \ldots, \pm m)$, k and σ can be obtained by solving the algebraic equations in Step 4, then substituting these constants and the known general solutions of Eq. (5) into (4), we can obtain the explicit solutions of Eq. (1) immediately.

3. Applications of the improved (G'/G) expansion method for NPDEs

In this section, we apply the improved (G'/G)- expansion method to construct the traveling wave solutions for some nonlinear PDEs vie the generalized Zakharov equations, the coupled Maccaris equations, the (2+1)-dimensional Wu-Zhang equations and the (1+1) dimensional Fornberg – Whitham equation which are very important in the mathematical physics and have been paid attention by many researchers.

3.1. Example 1. The generalized- Zakharov equations

In this section, the generalized- Zakharov equations for the complex envelope [34] reads:

$$\begin{aligned} & i\psi_t + \psi_{xx} - 2\lambda|\psi|^2\psi + 2\psi v = 0, \\ & v_{tt} - v_{xx} + (|\psi|^2)_{xx} = 0, \end{aligned} \tag{9}$$

where λ is nonzero constant. Let us assume the traveling wave solution of Eqs (9) in the form:

$$\psi(x,t) = e^{i\eta}U(\xi), \quad v(x,t) = V(\xi),\ \eta = \alpha x + \beta t,\ \xi = k(x - 2\alpha t), \tag{10}$$

where $U(\xi)$,$V(\xi)$ are real functions and α, β, k are constants to be determined later. Substituting (10) into Eqs.(9), we have:

$$\begin{aligned} & k^2 U'' + 2UV - (\alpha^2 + \beta)U - 2\lambda U^3 = 0, \\ & k^2(4\alpha^2 - 1)V'' + k^2(U^2)'' = 0. \end{aligned} \tag{11}$$

By balancing the highest order derivative terms and nonlinear terms in Eqs. (11), we suppose that Eqs. (11) own the solutions in the following forms:

$$\begin{aligned} U &= a_0 + \frac{a_1\left(\frac{G'(\xi)}{G(\xi)}\right)}{\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]} + \frac{a_2\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]}{\left(\frac{G'(\xi)}{G(\xi)}\right)}, \\ V &= b_0 + \frac{b_1\left(\frac{G'(\xi)}{G(\xi)}\right)}{\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]} + \frac{b_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2}{\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]^2} + \frac{b_3\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]}{\left(\frac{G'(\xi)}{G(\xi)}\right)} + \frac{b_4\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]^2}{\left(\frac{G'(\xi)}{G(\xi)}\right)^2}, \end{aligned} \tag{12}$$

where $G(\xi)$ satisfies Eq.(5) and σ, λ, $a_0, a_1, a_2, b_0, b_1, b_2, b_3, b_4$ are constants to be determined latter. Substituting Eqs. (12) along with (5) into Eqs. (11) and cleaning the denominator and collecting all terms with the same order of $(G'(\xi)/G(\xi))$ together, the left hand side of Eqs. (11) are converted into polynomials in $(G'(\xi)/G(\xi))$. Setting each coefficient of these polynomials to zero , we derive a set of algebraic equations for σ, $a_0, a_1, a_2, b_0, b_1, b_2, b_3, b_4, \alpha, \beta$ and k . Solving the set of algebraic equations by using Maple or Mathematica , we have

Case 1.

$$\begin{aligned} & a_0 = -a_2\sigma, \qquad b_0 = -\frac{(2\sigma^2 a_2^2 + \beta - 4\beta\alpha^2 + \alpha^2 - 4\alpha^4 - 2\mu k^2 + 8\alpha^2 k^2 \mu)}{2(4\alpha^2 - 1)}, \\ & b_3 = \frac{2\sigma a_2^2}{(4\alpha^2 - 1)}, \qquad b_4 = -\frac{a_2^2}{(4\alpha^2 - 1)}, \\ & \lambda = \frac{k^2\mu^2(4\alpha^2 - 1) - a_2^2}{a_2^2(4\alpha^2 - 1)}, \\ & a_1 = b_1 = b_2 = 0, \end{aligned} \tag{13}$$

where $\alpha \neq \pm\frac{1}{2}$ and $\sigma, a_2, \beta, \alpha, k, \mu$ are arbitrary constants.

Case 2.

$$
\begin{aligned}
& a_1 = -\frac{a_0(\sigma^2\mu+1)}{\sigma\mu} \\
& b_1 = \frac{2(\mu\sigma^2+1)a_0^2}{\mu\sigma(4\alpha^2-1)}, \qquad b_2 = -\frac{(\mu\sigma^2+1)^2 a_0^2}{\mu^2\sigma^2(4\alpha^2-1)}, \\
& \lambda = \frac{\sigma^2 k^2\mu^2(4\alpha^2-1)-a_0^2}{a_0^2(4\alpha^2-1)}, \\
& \beta = \frac{(2a_0^2-2b_0+8b_0\alpha^2+\alpha^2-4\alpha^4-2\mu k^2+8\alpha^2k^2\mu)}{(4\alpha^2-1)}, \\
& a_2 = b_3 = b_4 = 0,
\end{aligned} \tag{14}
$$

where $\alpha \neq \pm\frac{1}{2}$ and $\sigma, a_0, \beta, \alpha, k, \mu$ are arbitrary constants.

Note that, there are other cases which are omitted here. We just list some exact solutions corresponding to cases 1,2 to illustrate the effectiveness of the improved (G'/G) – expansion method.

Using case 1, (12) and the general solutions of Eq.(5), we can find the following traveling wave solutions of the generalized- Zakharov equations (9). When $\mu < 0$, we obtain the hyperbolic function solutions of Eq.(9)

$$
\psi(x,t) = \frac{a_2}{\sqrt{-\mu}} e^{i(\alpha x+\beta t)} \left[\frac{C_1\cosh(\sqrt{-\mu}\xi)+C_2\sinh(\sqrt{-\mu}\xi)}{C_1\sinh(\sqrt{-\mu}\xi)+C_2\cosh(\sqrt{-\mu}\xi)} \right],
$$

and

$$
\begin{aligned}
V = & -\frac{(2\sigma^2a_2^2+\beta-4\beta\alpha^2+\alpha^2-4\alpha^4-2\mu k^2+8\alpha^2k^2\mu)}{2(4\alpha^2-1)} \\
& + \frac{2\sigma a_2^2[C_1\{\cosh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\sinh(\sqrt{-\mu}\xi)\}+C_2\{\sinh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\cosh(\sqrt{-\mu}\xi)\}]}{(4\alpha^2-1)\sqrt{-\mu}[C_1\sinh(\sqrt{-\mu}\xi)+C_2\cosh(\sqrt{-\mu}\xi)]} \\
& + \frac{a_2^2[C_1\{\cosh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\sinh(\sqrt{-\mu}\xi)\}+C_2\{\sinh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\cosh(\sqrt{-\mu}\xi)\}]^2}{(4\alpha^2-1)[C_1\sinh(\sqrt{-\mu}\xi)+C_2\cosh(\sqrt{-\mu}\xi)]^2\mu},
\end{aligned} \tag{15}
$$

where $\xi = k(x-2\alpha t)$.

In particular setting $\sigma = C_1 = 0$, $C_2 \neq 0$ the following solitary wave solutions of generalized- Zakharov equations (9) are discovered

$$
\psi(x,t) = \frac{a_2}{\sqrt{-\mu}} e^{i(\alpha x+\beta t)} \tanh(\sqrt{-\mu}\xi),
$$

and

$$
V = -\frac{(\beta-4\beta\alpha^2+\alpha^2-4\alpha^4-2\mu k^2+8\alpha^2k^2\mu)}{2(4\alpha^2-1)} + \frac{a_2^2}{(4\alpha^2-1)\mu}\tanh^2(\sqrt{-\mu}\xi). \tag{16}
$$

Setting again $\sigma = 0$, $C_1 > 0$, $C_1^2 > C_2^2$, then the solitary wave solutions of generalized- Zakharov equations (9) take the following form:

$$
\psi(x,t) = \frac{a_2}{\sqrt{-\mu}} e^{i(\alpha x+\beta t)} \coth[\sqrt{-\mu}\xi+\xi_0],
$$

and

$$V=-\frac{(\beta-4\beta\alpha^2+\alpha^2-4\alpha^4-2\mu k^2+8\alpha^2k^2\mu)}{2(4\alpha^2-1)}+\frac{a_2^2}{(4\alpha^2-1)\mu}\coth^2[\sqrt{-\mu}\xi+\xi_0]\ , \tag{17}$$

where $\xi_0=\tanh^{-1}(\frac{C_2}{C_1})$. It is easy to see that if C_1, C_2, μ and σ are taken as other special values in a proper way, more solitary wave solutions of Eq. (9) can be obtained, here we omit them for simplicity.

When $\mu>0$, we get the trigonometric function solutions of Eq.(9)

$$\psi(x,t)=\frac{a_2}{\sqrt{\mu}}e^{i(\alpha x+\beta t)}\left[\frac{C_1\sin(\sqrt{\mu}\xi)+C_2\cos(\sqrt{\mu}\xi)}{C_1\cos(\sqrt{\mu}\xi)-C_2\sin(\sqrt{\mu}\xi)}\right]\ ,$$

and

$$\begin{aligned}V=&-\frac{(2\sigma^2a_2^2+\beta-4\beta\alpha^2+\alpha^2-4\alpha^4-2\mu k^2+8\alpha^2k^2\mu)}{2(4\alpha^2-1)}\\&+\frac{2\sigma a_2^2[C_1\{\sin(\sqrt{\mu}\xi)+\sigma\sqrt{\mu}\cos(\sqrt{\mu}\xi)\}+C_2\{\cos(\sqrt{\mu}\xi)-\sigma\sqrt{\mu}\sin(\sqrt{\mu}\xi)\}]}{(4\alpha^2-1)\sqrt{\mu}[C_1\cos(\sqrt{\mu}\xi)-C_2\sin(\sqrt{\mu}\xi)]}\\&-\frac{a_2^2[C_1\{\sin(\sqrt{\mu}\xi)+\sigma\sqrt{\mu}\cos(\sqrt{\mu}\xi)\}+C_2\{\cos(\sqrt{\mu}\xi)-\sigma\sqrt{\mu}\sin(\sqrt{\mu}\xi)\}]^2}{(4\alpha^2-1)\mu[C_1\cos(\sqrt{\mu}\xi)-C_2\sin(\sqrt{\mu}\xi)]^2},\end{aligned} \tag{18}$$

In particular setting $\sigma=C_1=0$, $C_2\neq 0$, the following solitary wave solutions of generalized- Zakharov equations (9) are discovered

$$\psi(x,t)=-\frac{a_2}{\sqrt{\mu}}e^{i(\alpha x+\beta t)}\cot(\sqrt{\mu}\xi)\ ,$$

and

$$V=-\frac{(\beta-4\beta\alpha^2+\alpha^2-4\alpha^4-2\mu k^2+8\alpha^2k^2\mu)}{2(4\alpha^2-1)}-\frac{a_2^2}{(4\alpha^2-1)\mu}\cot^2(\sqrt{\mu}\xi), \tag{19}$$

Setting again $\sigma=0, C_1>0$, $C_1^2>C_2^2$, then the solitary wave solutions of generalized- Zakharov equations (9) take the following form:

$$\psi(x,t)=\frac{a_2}{\sqrt{\mu}}e^{i(\alpha x+\beta t)}\tan(\sqrt{\mu}\xi+\eta_0)$$

and

$$V=-\frac{(\beta-4\beta\alpha^2+\alpha^2-4\alpha^4-2\mu k^2+8\alpha^2k^2\mu)}{2(4\alpha^2-1)}-\frac{a_2^2}{(4\alpha^2-1)\mu}\tan^2(\sqrt{\mu}\xi+\eta_0), \tag{20}$$

where $\eta_0=\tan^{-1}(\frac{C_2}{C_1})$.

When $\mu=0$, we get the exact wave solutions of Eq.(9) take the following form:

$$\psi(x,t)=\frac{a_2}{C_1}(C_1\xi+C_2)e^{i(\alpha x+\beta t)},$$

and

$$V=-\frac{(2\sigma^2 a_2^2+\beta-4\beta\alpha^2+\alpha^2-4\alpha^4)}{2(4\alpha^2-1)}+\frac{a_2^2}{(4\alpha^2-1)}\left[\sigma^2-\frac{(C_1\xi+C_2)^2}{C_1^2}\right]. \tag{21}$$

In the case 2, (12) and the general solutions of Eq.(5), we can find the following traveling wave solutions of the generalized- Zakharov equations (9). When $\mu<0$, we obtain the hyperbolic function solutions of Eq.(9)

$$\psi(x,t)=\left[a_0+\frac{a_0(\sigma^2\mu+1)\,[C_1\sinh(\sqrt{-\mu}\xi)+C_2\cosh(\sqrt{-\mu}\xi)]}{\sigma\sqrt{-\mu}\,\,[C_1\{\cosh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\sinh(\sqrt{-\mu}\xi)\}+C_2\{\sinh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\cosh(\sqrt{-\mu}\xi)\}]}\right]\times e^{i[\alpha x+\frac{(2a_0^2-2b_0+8b_0\alpha^2+\alpha^2-4\alpha^4-2\mu k^2+8\alpha^2k^2\mu)t}{(4\alpha^2-1)}]},$$

and

$$V=b_0-\frac{2(\mu\sigma^2+1)\,a_0^2[C_1\sinh(\sqrt{-\mu}\xi)+C_2\cosh(\sqrt{-\mu}\xi)]}{\sqrt{-\mu}\,\,\sigma(4\alpha^2-1)[C_1\{\cosh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\sinh(\sqrt{-\mu}\xi)\}+C_2\{\sinh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\cosh(\sqrt{-\mu}\xi)\}]}+\frac{(\mu\sigma^2+1)^2\,a_0^2[C_1\sinh(\sqrt{-\mu}\xi)+C_2\cosh(\sqrt{-\mu}\xi)]^2}{\mu\sigma^2\,(4\alpha^2-1)[C_1\{\cosh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\sinh(\sqrt{-\mu}\xi)\}+C_2\{\sinh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\cosh(\sqrt{-\mu}\xi)\}]^2}, \tag{22}$$

When $\mu>0$, we get the trigonometric function solutions of Eq.(9)

$$\psi(x,t)=\left[a_0-\frac{a_0(\sigma^2\mu+1)\,\,[C_1\cos(\sqrt{\mu}\xi)-C_2\sin(\sqrt{\mu}\xi)]}{\sigma\sqrt{\mu}\,\,[C_1\{\sin(\sqrt{\mu}\xi)+\sigma\sqrt{\mu}\cos(\sqrt{\mu}\xi)\}+C_2\{\cos(\sqrt{\mu}\xi)-\sigma\sqrt{\mu}\sin(\sqrt{\mu}\xi)\}]}\right]\times e^{i[\alpha x+\frac{(2a_0^2-2b_0+8b_0\alpha^2+\alpha^2-4\alpha^4-2\mu k^2+8\alpha^2k^2\mu)t}{(4\alpha^2-1)}]},$$

and

$$V=b_0+\frac{2(\mu\sigma^2+1)\,a_0^2[C_1\cos(\sqrt{\mu}\xi)-C_2\sin(\sqrt{\mu}\xi)]}{\sqrt{\mu}\,\,\sigma(4\alpha^2-1)[C_1\{\sin(\sqrt{\mu}\xi)+\sigma\sqrt{\mu}\cos(\sqrt{\mu}\xi)\}+C_2\{\cos(\sqrt{\mu}\xi)-\sigma\sqrt{\mu}\sin(\sqrt{\mu}\xi)\}]}-\frac{(\mu\sigma^2+1)^2\,a_0^2[C_1\cos(\sqrt{\mu}\xi)-C_2\sin(\sqrt{\mu}\xi)]^2}{\mu\sigma^2\,(4\alpha^2-1)[C_1\{\sin(\sqrt{\mu}\xi)+\sigma\sqrt{\mu}\cos(\sqrt{\mu}\xi)\}+C_2\{\cos(\sqrt{\mu}\xi)-\sigma\sqrt{\mu}\sin(\sqrt{\mu}\xi)\}]^2}. \tag{23}$$

where $\xi=k(x-2\alpha t)$. It is easy to see that if C_1, C_2, μ and σ are taken as other special values in a proper way, more solitary wave solutions of Eq. (9) can be obtained, here we omit them for simplicity.

3.2. Example 2. The coupled Maccaris equations

In this subsection we study the coupled Maccaris equations [34].

$$\begin{aligned} &iQ_t+Q_{xx}+QR=0,\\ &R_t+R_y+(|Q|^2)_x=0. \end{aligned} \tag{24}$$

In order to seek the exact solutions of Eqs.(24), we suppose

$$Q(x,y,t)=u(x,y,t)e^{[i(kx+\alpha y+\lambda t+l)]}, \tag{25}$$

where k,α,λ and l are constants to be determined later. Substituting Eq.(25) into Eqs.(24), we have :

$$\begin{aligned} &i(u_t+2ku_x)+u_{xx}-(\lambda+k^2)u+uR=0,\\ &R_t+R_y+2uu_x=0. \end{aligned} \tag{26}$$

We use the following traveling wave transformations

$$u(x,y,t)=U(\xi),\qquad R(x,y,t)=V(\xi),\quad \xi=w(x+\beta y-2kt+x_0)\ , \tag{27}$$

where w and β are constants to be determined later, x_0 is an constant , Eqs. (26) become the following NODEs:

$$\begin{aligned} &w^2U''-(\lambda+k^2)U+UV=0,\\ &(\beta-2k)V'+2UU'=0, \end{aligned} \tag{28}$$

By balancing the highest order derivative terms and nonlinear terms in Eqs. (28), we suppose that Eqs. (28) own the solutions (12) . Substituting Eqs. (12) along with (5) into Eqs. (28) and cleaning the denominator and collecting all terms with the same order of $(G'(\xi)/G(\xi))$ together, the left hand side of Eqs. (28) are converted into polynomials in $(G'(\xi)/G(\xi))$. Setting each coefficient of these polynomials to zero , we derive a set of algebraic equations for σ, a_0, a_1, a_2, b_0, b_1, b_2, b_3, b_4, α,β , w,λ,l and k . Solving the set of algebraic equations by using Maple or Mathematica , we have

Case 1.

$$\begin{aligned} &a_1=-\frac{a_0(\mu\sigma^2+1)}{\mu\sigma}, \qquad \beta=\frac{a_0^2+4w^2\mu^2\sigma^2k}{2w^2\mu^2\sigma^2},\\ &b_0=\lambda+k^2-2w^2\mu^2\sigma^2-2w^2\mu, \qquad b_1=4w^2\mu\sigma(\mu\sigma^2+1),\\ &b_2=-2w^2(\mu\sigma^2+1)^2,\\ &a_2=b_3=b_4=0, \end{aligned} \tag{29}$$

where a_0,μ,σ,λ,k and w are arbitrary constants.

Case 2.

$$\begin{aligned} &a_0=-a_2\sigma, \qquad \beta=\frac{4w^2\mu^2k+a_2^2}{2w^2\mu^2},\\ &b_0=\lambda+k^2-2w^2\mu^2\sigma^2-2w^2\mu, \qquad b_3=4w^2\mu^2\sigma,\\ &b_4=-2w^2\mu^2,\\ &a_1=b_1=b_2=0, \end{aligned} \tag{30}$$

where a_2,μ,σ,λ,k and w are arbitrary constants.

Using case 1, (29) and the general solutions of Eq.(5), we can find the following traveling wave solutions of the coupled Maccaris equations (24). When $\mu<0$, we obtain the hyperbolic function solutions of Eq.(24)

$$Q(x,y,t)=\left[a_0+\frac{a_0(\mu\sigma^2+1)[C_1\sinh(\sqrt{-\mu}\xi)+C_2\cosh(\sqrt{-\mu}\xi)]}{\sqrt{-\mu}\ \sigma[C_1\{\cosh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\sinh(\sqrt{-\mu}\xi)\}+C_2\{\sinh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\cosh(\sqrt{-\mu}\xi)\}]}\right]\times e^{[i(kx+\alpha y+\lambda t+l)]},\qquad(31)$$

and

$$\begin{aligned}R(x,y,t)&=\lambda+k^2-2w^2\mu^2\sigma^2-2w^2\mu+\\&+\frac{4w^2\mu\sigma(\mu\sigma^2+1)\sqrt{-\mu}\ [C_1\sinh(\sqrt{-\mu}\xi)+C_2\cosh(\sqrt{-\mu}\xi)]}{[C_1\{\cosh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\sinh(\sqrt{-\mu}\xi)\}+C_2\{\sinh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\cosh(\sqrt{-\mu}\xi)\}]}+\\&+\frac{2w^2(\mu\sigma^2+1)^2\mu[C_1\sinh(\sqrt{-\mu}\xi)+C_2\cosh(\sqrt{-\mu}\xi)]^2}{[C_1\{\cosh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\sinh(\sqrt{-\mu}\xi)\}+C_2\{\sinh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\cosh(\sqrt{-\mu}\xi)\}]^2},\end{aligned}\qquad(32)$$

where $\xi=w[x+(\frac{a_0^2+4w^2\mu^2\sigma^2k}{2w^2\mu^2\sigma^2})y-2kt+x_0]$.

In particular setting $C_1=0$, $C_2\neq 0$ the following solitary wave solutions of the coupled Maccaris equations are discovered

$$Q(x,y,t)=\left[a_0+\frac{a_0(\mu\sigma^2+1)[\coth(\sqrt{-\mu}\xi)]}{\sqrt{-\mu}\ \sigma[1+\sigma\sqrt{-\mu}\coth(\sqrt{-\mu}\xi)\}]}\right]\ e^{[i(kx+\alpha y+\lambda t+l)]}\ ,\qquad(33)$$

and

$$\begin{aligned}R(x,y,t)&=\lambda+k^2-2w^2\mu^2\sigma^2-2w^2\mu+\\&+\frac{4w^2\mu\sigma(\mu\sigma^2+1)\sqrt{-\mu}\coth(\sqrt{-\mu}\xi)}{1+\sigma\sqrt{-\mu}\coth(\sqrt{-\mu}\xi)}+\frac{2w^2(\mu\sigma^2+1)^2\mu\coth^2(\sqrt{-\mu}\xi)}{[1+\sigma\sqrt{-\mu}\coth(\sqrt{-\mu}\xi)]^2},\end{aligned}\qquad(34)$$

When $\mu>0$, we obtain the hyperbolic function solutions of Eq.(24)

$$Q(x,y,t)=\left[a_0-\frac{a_0(\mu\sigma^2+1)[C_1\cos(\sqrt{\mu}\xi)-C_2\sin(\sqrt{\mu}\xi)]}{\sqrt{\mu}\ \sigma[C_1\{\sin(\sqrt{\mu}\xi)+\sigma\sqrt{\mu}\cos(\sqrt{\mu}\xi)\}+C_2\{\cos(\sqrt{\mu}\xi)-\sigma\sqrt{\mu}\sin(\sqrt{\mu}\xi)\}]}\right]e^{[i(kx+\alpha y+\lambda t+l)]},\qquad(35)$$

and

$$\begin{aligned}R(x,y,t)&=\lambda+k^2-2w^2\mu^2\sigma^2-2w^2\mu+\\&+\frac{4w^2\mu\sigma(\mu\sigma^2+1)\sqrt{\mu}\ [C_1\cos(\sqrt{\mu}\xi)-C_2\sin(\sqrt{\mu}\xi))]}{[C_1\{\sin(\sqrt{\mu}\xi)+\sigma\sqrt{\mu}\cos(\sqrt{\mu}\xi)\}+C_2\{\cos(\sqrt{\mu}\xi)-\sigma\sqrt{\mu}\sin(\sqrt{\mu}\xi)\}]}+\\&-\frac{2w^2(\mu\sigma^2+1)^2\mu[C_1\cos(\sqrt{\mu}\xi)-C_2\sin(\sqrt{\mu}\xi))]^2}{[C_1\{\sin(\sqrt{\mu}\xi)+\sigma\sqrt{\mu}\cos(\sqrt{\mu}\xi)\}+C_2\{\cos(\sqrt{\mu}\xi)-\sigma\sqrt{\mu}\sin(\sqrt{\mu}\xi)\}]^2},\end{aligned}\qquad(36)$$

In particular setting $C_1=0$, $C_2\neq 0$ the following solitary wave solutions of the coupled Maccaris equations are discovered

$$Q(x,y,t)=\left[a_0+\frac{a_0(\mu\sigma^2+1)\tan(\sqrt{\mu}\xi)]}{\sqrt{\mu}\ \sigma[1-\sigma\sqrt{\mu}\tan(\sqrt{\mu}\xi)]}\right]e^{[i(kx+\alpha y+\lambda t+l)]},\qquad(37)$$

and

$$R(x,y,t)=\lambda+k^2-2w^2\mu^2\sigma^2-2w^2\mu+ \\ -\frac{4w^2\mu\sigma(\mu\sigma^2+1)\sqrt{\mu}\tan(\sqrt{\mu}\xi)}{[1-\sigma\sqrt{\mu}\tan(\sqrt{\mu}\xi)]}-\frac{2w^2(\mu\sigma^2+1)^2\mu\tan^2(\sqrt{\mu}\xi)}{[1-\sigma\sqrt{\mu}\tan(\sqrt{\mu}\xi)]^2}, \tag{38}$$

where $\xi=w[x+(\frac{a_0^2+4w^2\mu^2\sigma^2k}{2w^2\mu^2\sigma^2})y-2kt+x_0]$.

3.3. Example 3. The (2+1)-dimensional Wu-Zhang equations

In this subsection, we study the (2+1)-dimensional Wu-Zhang equations [35,36].

$$\begin{aligned} &u_t+uu_x+vu_y+w_x=0,\\ &v_t+uv_x+vv_y+w_y=0,\\ &w_t+(uv)_x+(uw)_y+\frac{1}{3}(u_{xxx}+u_{xyy}+v_{xxy}+v_{yyy})=0. \end{aligned} \tag{39}$$

Let us assume the traveling wave solutions of Eqs (39) in the following forms:

$$u(x,y,t)=U(\xi),\quad v(x,y,t)=V(\xi),\quad w(x,y,t)=W(\xi),\quad \xi=x+y-kt, \tag{40}$$

where k is an arbitrary constant. Substituting (40) into Eqs. (39), we have:

$$\begin{aligned} &-kU'+UU'+VU'+W'=0,\\ &-kV'+UV'+VV'+W'=0,\\ &-kW+UV+UW+\frac{2}{3}(U''+V'')+L=0, \end{aligned} \tag{41}$$

where L is the integration constant. By balancing the highest order derivative terms and nonlinear terms in Eqs. (41), we suppose that Eqs. (41) own the solutions in the following forms:

$$\begin{aligned} U&=a_0+\frac{a_1\left(\frac{G'(\xi)}{G(\xi)}\right)}{\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]}+\frac{a_2\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]}{\left(\frac{G'(\xi)}{G(\xi)}\right)},\\ V&=b_0+\frac{b_1\left(\frac{G'(\xi)}{G(\xi)}\right)}{\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]}+\frac{b_2\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]}{\left(\frac{G'(\xi)}{G(\xi)}\right)},\\ W&=c_0+\frac{c_1\left(\frac{G'(\xi)}{G(\xi)}\right)}{\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]}+\frac{c_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2}{\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]^2}+\frac{c_3\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]}{\left(\frac{G'(\xi)}{G(\xi)}\right)}+\frac{c_4\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]^2}{\left(\frac{G'(\xi)}{G(\xi)}\right)^2}. \end{aligned} \tag{42}$$

where $a_0,a_1,a_2,b_0,b_1,b_2,c_0,c_1,c_2,c_3$ and c_4 are constants to be determined later. Substituting Eqs. (42) along with (5) into Eqs. (41) and cleaning the denominator and collecting all terms with the same order of $(G'(\xi)/G(\xi))$ together, the left hand side of Eqs. (41) are converted into polynomials in $(G'(\xi)/G(\xi))$. Setting each coefficient of these polynomials to zero , we derive a set of algebraic equations for $a_0,a_1,a_2,b_0,b_1,b_2,c_0,c_1,c_2,c_3,c_4,k,L$ and σ. Solving the set of algebraic equations by using Maple or Mathematica , we have

Case 1.

$$a_1 = 2\sqrt{\frac{2}{3}}(\mu\sigma^2+1), \qquad a_2 = 2\mu\sqrt{\frac{2}{3}},$$

$$b_0 = -6\mu\sigma\sqrt{\frac{2}{3}}+1+2k-2a_0, \qquad b_1 = 2\sqrt{\frac{2}{3}}(\mu\sigma^2+1),$$

$$b_2 = 2\mu\sqrt{\frac{2}{3}}, \qquad c_2 = -\frac{8}{3}(\mu\sigma^2+1)^2,$$

$$c_0 = -1-\frac{16}{3}\mu^2\sigma^2+2a_0-3k+2ka_0-a_0^2-k^2+\sqrt{\frac{2}{3}}(-6a_0\mu\sigma+6k\mu\sigma+6\mu\sigma),$$

$$c_1 = \sqrt{\frac{2}{3}}(-2\mu\sigma^2k-2k+2a_0\mu\sigma^2+2a_0-2\mu\sigma^2-2)+8\mu^2\sigma^3+8\mu\sigma,$$

$$c_3 = \sqrt{\frac{2}{3}}(-2\mu k+2a_0\mu-2\mu)+8\mu^2\sigma, \qquad c_4 = \frac{-8\mu^2}{3},$$

$$L = \sqrt{\frac{2}{3}}(6k^2\mu\sigma+6a_0^2\sigma\mu+6k\mu\sigma-\frac{64}{3}\mu^2\sigma-\frac{64}{3}\mu^3\sigma^3-12k\sigma a_0\mu)-k-3k^2$$
$$-k^3+3ka_0+3k^2a_0-3ka_0^2+a_0^3+\frac{16}{3}k\mu-\frac{16}{3}a_0\mu, \tag{43}$$

where a_0, μ, σ and k are arbitrary constants.

Case 2.

$$a_2 = 2\mu\sqrt{\frac{2}{3}}, \qquad b_0 = -6\mu\sigma\sqrt{\frac{2}{3}}+1+2k-2a_0, \qquad b_2 = 2\mu\sqrt{\frac{2}{3}},$$

$$c_3 = \sqrt{\frac{2}{3}}(-2\mu k+2a_0\mu-2\mu)+8\mu^2\sigma, \qquad c_4 = \frac{-8\mu^2}{3},$$

$$c_0 = -1-8\mu^2\sigma^2+2a_0-3k+2ka_0-a_0^2-k^2-\frac{8\mu}{3}+\sqrt{\frac{2}{3}}(-6a_0\mu\sigma+6k\mu\sigma+6\mu\sigma),$$

$$L = \sqrt{\frac{2}{3}}(6k^2\mu\sigma+6a_0^2\sigma\mu+6k\mu\sigma-\frac{64}{3}\mu^2\sigma-\frac{64}{3}\mu^3\sigma^3-12k\sigma a_0\mu)-k-3k^2 \tag{44}$$
$$-k^3+3ka_0+3k^2a_0-3ka_0^2+a_0^3+\frac{16}{3}k\mu-\frac{16}{3}a_0\mu,$$

$$c_1 = c_2 = b_1 = a_1 = 0,$$

Using case 1, (42) and the general solutions of Eq.(5), we can find the following traveling wave solutions of the (2+1)-dimensional Wu-Zhang equations (39). When $\mu < 0$, we obtain the hyperbolic function solutions of Eqs.(39)

$$U = a_0 + \frac{\sqrt{-8\mu}(\mu\sigma^2+1)\,[C_1\sinh(\sqrt{-\mu}\xi)+C_2\cosh(\sqrt{-\mu}\xi)]}{\sqrt{3}[C_1\{\cosh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\sinh(\sqrt{-\mu}\xi)\}+C_2\{\sinh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\cosh(\sqrt{-\mu}\xi)\}]}$$
$$-\frac{\sqrt{-8\mu}[C_1\{\cosh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\sinh(\sqrt{-\mu}\xi)\}+C_2\{\sinh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\cosh(\sqrt{-\mu}\xi)\}]}{\sqrt{3}[C_1\sinh(\sqrt{-\mu}\xi)+C_2\cosh(\sqrt{-\mu}\xi)]}, \tag{45}$$

$$V = -6\mu\sigma\sqrt{\frac{2}{3}}+1+2k-2a_0+\frac{\sqrt{-8\mu}\,(\mu\sigma^2+1)[C_1\sinh(\sqrt{-\mu}\xi)+C_2\cosh(\sqrt{-\mu}\xi)]}{\sqrt{3}[C_1\{\cosh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\sinh(\sqrt{-\mu}\xi)\}+C_2\{\sinh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\cosh(\sqrt{-\mu}\xi)\}]}$$
$$-\sqrt{\frac{-8\mu}{3}}\frac{[C_1\{\cosh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\sinh(\sqrt{-\mu}\xi)\}+C_2\{\sinh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\cosh(\sqrt{-\mu}\xi)\}]}{[C_1\sinh(\sqrt{-\mu}\xi)+C_2\cosh(\sqrt{-\mu}\xi)]}, \tag{46}$$

and

$$W=-1-\frac{16}{3}\mu^2\sigma^2+2a_0-3k+2ka_0-a_0^2-k^2+\sqrt{\frac{2}{3}}(-6a_0\mu\sigma+6k\mu\sigma+6\mu\sigma)$$
$$+\frac{\sqrt{-\mu}[\sqrt{2}(-2\mu\sigma^2k-2k+2a_0\mu\sigma^2+2a_0-2\mu\sigma^2-2)+\sqrt{3}(8\mu^2\sigma^3+8\mu\sigma)][C_1\sinh(\sqrt{-\mu}\xi)+C_2\cosh(\sqrt{-\mu}\xi)]}{\sqrt{3}\,[C_1\{\cosh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\sinh(\sqrt{-\mu}\xi)\}+C_2\{\sinh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\cosh(\sqrt{-\mu}\xi)\}]}+$$
$$+\frac{8(\mu\sigma^2+1)^2\mu[C_1\sinh(\sqrt{-\mu}\xi)+C_2\cosh(\sqrt{-\mu}\xi)]^2}{3[C_1\{\cosh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\sinh(\sqrt{-\mu}\xi)\}+C_2\{\sinh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\cosh(\sqrt{-\mu}\xi)\}]^2}+$$
$$+\frac{(\sqrt{2}\,(-2\mu k+2a_0\mu-2\mu)+8\sqrt{3}\,\mu^2\sigma)[C_1\{\cosh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\sinh(\sqrt{-\mu}\xi)\}+C_2\{\sinh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\cosh(\sqrt{-\mu}\xi)\}]}{\sqrt{-3\mu}[C_1\sinh(\sqrt{-\mu}\xi)+C_2\cosh(\sqrt{-\mu}\xi)]}$$
$$+\frac{8\mu[C_1\{\cosh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\sinh(\sqrt{-\mu}\xi)\}+C_2\{\sinh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}\cosh(\sqrt{-\mu}\xi)\}]^2}{3[C_1\sinh(\sqrt{-\mu}\xi)+C_2\cosh(\sqrt{-\mu}\xi)]^2}, \quad (47)$$

where $\xi=x+y-kt$. In particular setting $C_1=0$, $C_2\neq 0$ the following solitary wave solutions of the (2+1)-dimensional Wu-Zhang equations are discovered

$$U=a_0+2\sqrt{\frac{-2\mu}{3}}\frac{(\mu\sigma^2+1)\coth(\sqrt{-\mu}\xi)]}{1+\sigma\sqrt{-\mu}\coth(\sqrt{-\mu}\xi)}-2\sqrt{\frac{-2\mu}{3}}[\tanh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}], \quad (48)$$

$$V=-6\mu\sigma\sqrt{\frac{2}{3}}+1+2k-2a_0+\frac{\sqrt{-8\mu}\,(\mu\sigma^2+1)\coth(\sqrt{-\mu}\xi)}{\sqrt{3}[1+\sigma\sqrt{-\mu}\coth(\sqrt{-\mu}\xi)\}]}-\sqrt{\frac{-8\mu}{3}}[\tanh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}], \quad (49)$$

and

$$W=-1-\frac{16}{3}\mu^2\sigma^2+2a_0-3k+2ka_0-a_0^2-k^2+\sqrt{\frac{2}{3}}(-6a_0\mu\sigma+6k\mu\sigma+6\mu\sigma)$$
$$+\frac{\sqrt{-\mu}[\sqrt{2}(-2\mu\sigma^2k-2k+2a_0\mu\sigma^2+2a_0-2\mu\sigma^2-2)+\sqrt{3}(8\mu^2\sigma^3+8\mu\sigma)]\coth(\sqrt{-\mu}\xi)}{\sqrt{3}[1+\sigma\sqrt{-\mu}\coth(\sqrt{-\mu}\xi)\}]}+$$
$$+\frac{8(\mu\sigma^2+1)^2\mu\coth^2(\sqrt{-\mu}\xi)}{3[1+\sigma\sqrt{-\mu}\coth(\sqrt{-\mu}\xi)\}]^2}+\frac{(\sqrt{2}\,(-2\mu k+2a_0\mu-2\mu)+8\sqrt{3}\,\mu^2\sigma)}{\sqrt{-3\mu}}[\tanh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}]$$
$$+\frac{8\mu}{3}[\tanh(\sqrt{-\mu}\xi)+\sigma\sqrt{-\mu}]^2, \quad (50)$$

When $\mu>0$, we get the trigonometric function solutions of Eqs.(39)

$$U=a_0+\frac{\sqrt{8\mu}(\mu\sigma^2+1)[C_1\cos(\sqrt{\mu}\xi)-C_2\sin(\sqrt{\mu}\xi)]}{\sqrt{3}[C_1\{\sin(\sqrt{\mu}\xi)+\sigma\sqrt{\mu}\cos(\sqrt{\mu}\xi)\}+C_2\{\cos(\sqrt{\mu}\xi)-\sigma\sqrt{\mu}\sin(\sqrt{\mu}\xi)\}]}$$
$$+\frac{\sqrt{8\mu}[C_1\{\sin(\sqrt{\mu}\xi)+\sigma\sqrt{\mu}\cos(\sqrt{\mu}\xi)\}+C_2\{\cos(\sqrt{\mu}\xi)-\sigma\sqrt{\mu}\sin(\sqrt{\mu}\xi)\}]}{\sqrt{3}[C_1\cos(\sqrt{\mu}\xi)-C_2\sin(\sqrt{\mu}\xi)]}, \quad (51)$$

$$V=-6\mu\sigma\sqrt{\frac{2}{3}}+1+2k-2a_0+\frac{\sqrt{8\mu}\,(\mu\sigma^2+1)[C_1\cos(\sqrt{\mu}\xi)-C_2\sin(\sqrt{\mu}\xi)]}{\sqrt{3}[C_1\{\sin(\sqrt{\mu}\xi)+\sigma\sqrt{\mu}\cos(\sqrt{\mu}\xi)\}+C_2\{\cos(\sqrt{\mu}\xi)-\sigma\sqrt{\mu}\sin(\sqrt{\mu}\xi)\}]}$$
$$+\frac{\sqrt{8\mu}\,[C_1\{\sin(\sqrt{\mu}\xi)+\sigma\sqrt{\mu}\cos(\sqrt{\mu}\xi)\}+C_2\{\cos(\sqrt{\mu}\xi)-\sigma\sqrt{\mu}\sin(\sqrt{\mu}\xi)\}]}{\sqrt{3}[C_1\cos(\sqrt{\mu}\xi)-C_2\sin(\sqrt{\mu}\xi)]}, \quad (52)$$

and

$$\begin{aligned}W=&-1-\frac{16}{3}\mu^2\sigma^2+2a_0-3k+2ka_0-a_0^2-k^2+\sqrt{\frac{2}{3}}(-6a_0\mu\sigma+6k\mu\sigma+6\mu\sigma)\\&+\frac{\sqrt{\mu}[\sqrt{2}(-2\mu\sigma^2k-2k+2a_0\mu\sigma^2+2a_0-2\mu\sigma^2-2)+\sqrt{3}(8\mu^2\sigma^3+8\mu\sigma)][C_1\cos(\sqrt{\mu}\xi)-C_2\sin(\sqrt{\mu}\xi)]}{\sqrt{3}[C_1\{\sin(\sqrt{\mu}\xi)+\sigma\sqrt{\mu}\cos(\sqrt{\mu}\xi)\}+C_2\{\cos(\sqrt{\mu}\xi)-\sigma\sqrt{\mu}\sin(\sqrt{\mu}\xi)\}]}\\&-\frac{8(\mu\sigma^2+1)^2\mu[C_1\cos(\sqrt{\mu}\xi)-C_2\sin(\sqrt{\mu}\xi)]^2}{3[C_1\{\sin(\sqrt{\mu}\xi)+\sigma\sqrt{\mu}\cos(\sqrt{\mu}\xi)\}+C_2\{\cos(\sqrt{\mu}\xi)-\sigma\sqrt{\mu}\sin(\sqrt{\mu}\xi)\}]^2}\\&+\frac{[\sqrt{2}(-2\mu k+2a_0\mu-2\mu)+8\sqrt{3}\mu^2\sigma][C_1\{\sin(\sqrt{\mu}\xi)+\sigma\sqrt{\mu}\cos(\sqrt{\mu}\xi)\}+C_2\{\cos(\sqrt{\mu}\xi)-\sigma\sqrt{\mu}\sin(\sqrt{\mu}\xi)\}]}{\sqrt{3\mu}[C_1\cos(\sqrt{\mu}\xi)-C_2\sin(\sqrt{\mu}\xi)]}\\&-\frac{8\mu[C_1\{\sin(\sqrt{\mu}\xi)+\sigma\sqrt{\mu}\cos(\sqrt{\mu}\xi)\}+C_2\{\cos(\sqrt{\mu}\xi)-\sigma\sqrt{\mu}\sin(\sqrt{\mu}\xi)\}]^2}{3[C_1\cos(\sqrt{\mu}\xi)-C_2\sin(\sqrt{\mu}\xi)]^2}.\end{aligned} \tag{53}$$

In particular setting $C_1=0$, $C_2\neq 0$ the following solitary wave solutions of the (2+1)-dimensional Wu-Zhang equations are discovered

$$U=a_0-\frac{\sqrt{8\mu}(\mu\sigma^2+1)\tan(\sqrt{\mu}\xi)]}{\sqrt{3}[1-\sigma\sqrt{\mu}\tan(\sqrt{\mu}\xi)\}]}-\sqrt{\frac{8\mu}{3}}[\cot(\sqrt{\mu}\xi)-\sigma\sqrt{\mu}], \tag{54}$$

$$V=-6\mu\sigma\sqrt{\frac{2}{3}}+1+2k-2a_0-\frac{\sqrt{8\mu}\ (\mu\sigma^2+1)\tan(\sqrt{\mu}\xi)}{\sqrt{3}[1-\sigma\sqrt{\mu}\tan(\sqrt{\mu}\xi)\}]}-\sqrt{\frac{8\mu}{3}}\ [\cot(\sqrt{\mu}\xi)-\sigma\sqrt{\mu}], \tag{55}$$

and

$$\begin{aligned}W=&-1-\frac{16}{3}\mu^2\sigma^2+2a_0-3k+2ka_0-a_0^2-k^2+\sqrt{\frac{2}{3}}(-6a_0\mu\sigma+6k\mu\sigma+6\mu\sigma)\\&-\frac{\sqrt{\mu}[\sqrt{2}(-2\mu\sigma^2k-2k+2a_0\mu\sigma^2+2a_0-2\mu\sigma^2-2)+\sqrt{3}(8\mu^2\sigma^3+8\mu\sigma)]\tan(\sqrt{\mu}\xi)}{\sqrt{3}[1-\sigma\sqrt{\mu}\tan(\sqrt{\mu}\xi)]}\\&-\frac{8(\mu\sigma^2+1)^2\mu\tan^2(\sqrt{\mu}\xi)}{3[1-\sigma\sqrt{\mu}\tan(\sqrt{\mu}\xi)]^2}-\frac{[\sqrt{2}(-2\mu k+2a_0\mu-2\mu)+8\sqrt{3}\mu^2\sigma][\cot(\sqrt{\mu}\xi)-\sigma\sqrt{\mu}]}{\sqrt{3\mu}}\\&-\frac{8\mu}{3}[\cot(\sqrt{\mu}\xi)-\sigma\sqrt{\mu}]^2.\end{aligned} \tag{56}$$

By the similar manner, we can obtain the exact solutions for the case 2. We omitted the results of case 2. for convenience.

3.4. Example 4 The (1+1) dimensional Fornberg – Whitham equation

In this subsection, we study the (1+1) dimensional Fornberg – Whitham equation [37].

$$u_t-u_{xxt}+u_x-uu_{xxx}+uu_x-3u_xu_{xx}=0. \tag{57}$$

The traveling wave transformation (2) permits us converting Eq.(57) to the following ODE:

$$-kU'+kU'''+U'-UU'''+UU'-3U'U''=0. \tag{58}$$

By balancing the highest order derivative terms and nonlinear terms in Eqs. (58), we get

$$U=a_0+\frac{a_1\left(\frac{G'(\xi)}{G(\xi)}\right)}{\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]}+\frac{a_2\left[1+\sigma\left(\frac{G'(\xi)}{G(\xi)}\right)\right]}{\left(\frac{G'(\xi)}{G(\xi)}\right)}, \tag{59}$$

On substituting Eq.(59) along with (5) into Eqs. (58) and cleaning the denominator and collecting all terms with the same order of $(G'(\xi)/G(\xi))$ together, the left hand side of Eq. (58) are converted into polynomial in ($G'(\xi)/G(\xi)$). Setting each coefficient of this polynomial to zero, we derive a set of algebraic equations for a_0, a_1, a_2, k and σ. Solving the set of algebraic equations by using Maple or Mathematica , we have

$$a_1 = \pm 8k \mp \frac{32}{3} \mp 8a_0, \qquad \sigma = \pm 4, \qquad \mu = -\frac{1}{16}, \qquad a_2 = 0 \tag{60}$$

where k and a_0 are arbitrary constants. Consequently, the traveling wave solution takes the following form:

$$U = a_0 + \frac{[\pm 2k \mp \frac{8}{3} \mp 2a_0]\{A\sinh(\frac{1}{4}\xi) + B\cosh(\frac{1}{4}\xi)\}}{\left[(B \pm A)\cosh(\frac{1}{4}\xi) + (A \pm B)\sinh(\frac{1}{4}\xi)\right]}, \tag{61}$$

where A, B are arbitrary constants and $\xi = x - kt$.

4. Conclusion

The proposed method in this paper is more general than the ansätz in (G'/G)-expansion method [30,31] and modified (G'/G)- expansion method [32]. If we set the parameters in (2.4) and (2.5) to special values, the above two methods can be recovered by our proposed method. Therefore, the new method is more powerful than the (G'/G)-expansion method and modified (G'/G)- expansion method and some new types of travelling wave solutions and solitary wave solutions would be expected for some PDEs. This method is concise, effective and can be applied to other nonlinear evolution equations in mathematical physics.

5. Acknowledgement

This work is partially supported by Taif University, Saudi Arabia under the Grant No.1-431-752. The authors wish to thank Taif University for this support.

6. References

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