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Existence of Multiple Solutions for a Class of Nonlinear Elliptic Problems Involving the P-Laplacian

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Abstract. We prove the existence of nontrivial nonnegative solutions to the following nonlinear elliptic problem:

$$\begin{cases} -\Delta_p u + m(x)u^{p-1} = \lambda a(x)u^{\alpha-1} + b(x)u^{\beta-1}, x \in \Omega \\ u = 0, x \in \partial\Omega \end{cases}$$

where Δ_p denotes the p-Laplacian operator defined by $\Delta_p z = div(|\nabla z|^{p-2}\nabla z)$, $1 < p < 2$, $\Omega \subset R^n$ is a bounded domain with smooth boundary , $1 < p < 2 < \beta < \alpha < p^*$ ($p^* = \frac{pn}{n-p}$ if $n > p$, $p^* = \infty$ if $n \le p$) , $\lambda \in R \setminus \{0\}$ is a real parameter , the weight $m(x)$ is a bounded function with $\| m \|_\infty > 0$ and *a*(*x*), *b*(*x*) are continuous functions which change sign in $\overline{\Omega}$.

1. Introduction

We are concerned with the existence and multiplicity of nontrivial nonnegative solutions to the nonlinear elliptic problem:

$$\begin{cases} -\Delta_p u + m(x)u^{p-1} = \lambda a(x)u^{\alpha-1} + b(x)u^{\beta-1}, x \in \Omega \\ u = 0, x \in \partial\Omega \end{cases} \tag{1}$$

where Δ_p denotes the p-Laplacian operator defined by $\Delta_p z = div(|\nabla z|^{p-2}\nabla z)$, $1 < p < 2$, $\Omega \subset R^n$ is a bounded domain with smooth boundary, $1 < p < 2 < \beta < \alpha < p^*$, ($p^* = \frac{pn}{n-p}$ if $n > p$, $p^* = \infty$ if $n = p$), $\lambda \in R \setminus \{0\}$, the weight $m(x)$ is a bounded function with $\| m \|_\infty > 0$ and $a(x), b(x) \in C(\overline{\Omega})$ are satisfying $a^\pm = \max\{\pm a, 0\} \not\equiv 0$ and $b^\pm = \max\{\pm b, \infty\} \not\equiv 0$.

Problems involving the "p-Laplacian" arise from many branches of pure mathematics as in the theory of quasiregular and quasiconformal mapping (see[8,13]) as well as from various problems in mathematical physics notably the flow of non-Newtonian fluids: pseudo-plastic fluids correspond to $p \in (1,2)$ while dilatant fluids correspond to $p > 2$. The case $p = 2$ expresses Newtonian fluids [5].

We are motivated by the paper of Wu [14], in which problem (1) was discussed when $m \equiv 1$, $b \equiv 1$, $p = 2$, and $1 < \alpha < 2 < \beta < 2^*$. The authors proved that, there exists $\lambda_0 > 0$ such that if the parameter λ satisfy $0 < \lambda < \lambda_0$, then problem (1) for $m \equiv 1, b \equiv 1, p = 2$ and $1 < \alpha < 2 < \beta < 2^*$, has at least two positive solutions. Using the technique of Brown and Wu [7], in [15] the author discussed problem (1) with $m \not\equiv 1, b \not\equiv 1$, $p > 2$,and $2 < \beta < p < \alpha < p^*$.They obtained at least two positive solutions. In this paper, we discuss the problem (1) with $m \not\equiv 1, b \not\equiv 1$, $1 < p < 2$ and $1 < p < 2 < \beta < \alpha < p^*$. The change in

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α completely changes the nature of the solution set of (1). In fact, we shall prove that the problem (1) has at least two solutions u_0^+ and u_0^- such that $u_0^\pm \geq 0$ in Ω and $u_0^\pm \neq 0$ when the parameter λ belongs to a certain subset of R.

In the case when $p=2$, similar problems (with Dirichlet or Neuman boundary condition) have been studied by Binding et al. [6], Ambrosetti et al. [3], and Tehrani [11,12] , by using variational methods and by Amman and Lopez-Gomez [4] used global bifurcation theory to study the problem. Similar problem in the ODE case (semilinear or quasilinear) have been studied in [1,9]. We refer to [2,10] for additional results on elliptic problems involving the *p*-Laplacian.

2. Variational setting

Let $W_0^{1,s}(\Omega)=W_0^{1,s}, (s>0)$, denote the usual Sobolev space. In the Banach spac $W_0^{1,p}(\Omega)=W$ we introduce the norm

$$\| u \|_W = (\int_\Omega (| \nabla u |^p + m(x) | u |^p) dx)^{\frac{1}{p}}$$

which is equivalent to the standard one. First we give the definition of the weak solution of Eq. (1).

Definition 2.1. We say that $u \in W$ is a weak solution to (1) if for any $v \in W$ we have

$$\int_\Omega (| \nabla u |^{p-2} \nabla u.\nabla v + m(x) | u |^{p-2} uv) dx = \lambda \int_\Omega a(x) | u |^{\alpha-2} uv dx + \int_\Omega b(x) | u |^{\beta-2} uv dx$$

It is clear that Problem (1) has a variational structure. Let $J_\lambda : W \to R$ be the corresponding energy functional of problem (1) is defined by

$$J_\lambda(u) = \frac{1}{p} M(u) - \frac{1}{\alpha} A(u) - \frac{1}{\beta} B(u)$$

where

$$M(u) = \int_\Omega (| \nabla u |^p + m(x) | u |^p) dx \text{ , } A(u) = \lambda \int_\Omega a(x) | u |^\alpha dx$$

and

$$B(u) = \int_\Omega b(x) | u |^\beta dx$$

It is well known that the weak solutions of Eq. (1) are the critical points of the energy functional J_λ. Let *I* be the energy functional associated with an elliptic problem on a Banach space *X*. If *I* is bounded below and *I* has a minimizer on *X*, then this minimizer is a critical point of *I*. So, it is a solution of the corresponding elliptic problem. However, the energy functional J_λ, is not bounded below on the whole space *W*, but is bounded on an appropriate subset, and a minimizer on this set (if it exists) gives rise to solution to Eq. (1).

Consider the Nehari minimization problem for $\lambda \in R \setminus \{0\}$,

$$\gamma_\lambda = \inf \{ J_\lambda(u) : u \in N_\lambda \},$$

where $N_\lambda = \{ u \in W \setminus \{0\} : \langle J'_\lambda(u), (u) \rangle = 0 \}$. It is easy to see that $u \in N_\lambda$ if and only if

$$M(u) - A(u) = B(u). \tag{2}$$

Note that N_λ contains every nonzero solution of problem (1). Define

$$g_\lambda(u) = \langle J'_\lambda(u), u \rangle.$$

Then for $u \in N_\lambda$,

$$< g'_\lambda(u), u >= pM(u) - \alpha A(u) - \beta B(u) \tag{3}$$

$$= (p-\alpha)A(u) - (\beta - p)B(u) \tag{4}$$

$$= (p-\alpha)M(u) - (\beta-\alpha)B(u) \tag{5}$$

$$= (p-\beta)M(u) - (\alpha-\beta)A(u). \tag{6}$$

Now, we split N_λ into three parts:

$$N_\lambda^+ = \{ u \in N_\lambda : \langle g'_\lambda(u), u\rangle > 0\}$$
$$N_\lambda^0 = \{ u \in N_\lambda : \langle g'_\lambda(u), u\rangle = 0\}$$
$$N_\lambda^- = \{ u \in N_\lambda : \langle g'_\lambda(u), u\rangle < 0\}$$

To state our main result, we now present some important properties of N_λ^+, N_λ^0 and N_λ^-.

Lemma 2.2. *There exists* δ_0 *such that for* $0<\delta_0<\lambda \| a \|_\infty$, *we have* $N_\lambda^0 = \varphi$.

Proof. Suppose otherwise, then for $\delta_0 = \left[\frac{\alpha - p}{(\alpha-\beta)C_2^\beta \| b\|_\infty}\right]^{\frac{\alpha-\beta}{p-\beta}} \left[\frac{\beta-p}{(\alpha-\beta)C_1^\alpha}\right]$, where C_1, C_2 are positive constants and specified laters, there exists λ with $0<\lambda \| a\|_\infty < \delta_0$ such that $N_\lambda^0 \neq \phi$. Then for $u \in N_\lambda^0$ we have

$$0 = \langle g'_\lambda(u), u\rangle = (p-\beta)M(u) + (\beta-\alpha)A(u) \tag{7}$$
$$= (p-\alpha)M(u) + (\alpha-\beta)B(u) \tag{8}$$

By the Sobolev imbedding theorem,

$$A(u) \le \lambda \| \alpha \|_\infty \| u\|_\alpha^\alpha \le \lambda C_1^\alpha \| a \|_\infty \| u \|_W^\alpha \tag{9}$$

and

$$B(u) \le \| b\|_\infty \| u \|_\beta^\beta \le C_2^\beta \| b \|_\infty \| u\|_W^\beta \tag{10}$$

By using (9)–(10) in (7)–(8) we get

$$\| u \|_w \ge \left(\frac{p-\beta}{\alpha-\beta}\right)^{\frac{1}{\alpha-P}} \left(\frac{1}{C_1^\alpha \lambda \| \alpha \|_\infty}\right)^{\frac{1}{\alpha-P}}$$

and

$$\| u \|_w \le \left(\frac{\alpha-\beta}{\alpha-p}\right)^{\frac{1}{P-\beta}} (C_2^\beta \| b \|_\infty)^{\frac{1}{P-\beta}}$$

This implies $\lambda \| a\|_\infty \le \delta_0$, which is a contradiction. Thus, we can conclude that there exists $\delta_\circ > 0$ such that for $0<\delta_0<\lambda\| a \|_\infty$, we have $N_\lambda^0 = \phi$. □

By Lemma 2.2, for $0<\delta_0<\lambda\| a\|_\infty$ we write $N_\lambda = N_\lambda^+ \cup N_\lambda^-$ and define

$$\gamma_\lambda^+ = \inf_{u\in N_\lambda^+} J_\lambda(u) \ ; \gamma_\lambda^- = \inf_{u \in N_\lambda^-} J_\lambda(u) \ ;$$

Lemma 2.3. *We have*

(i) *If* $u \in N_\lambda^+$, *then* $B(u) > 0$;

(ii) *If* $u \in N_\lambda^-$, *then* $A(u) > 0$.

Proof. (i) We consider the following two cases:

Case $(i-a)$: $A(u) = 0$. We have

$$B(u) = M(u) > 0.$$

Case $(i-b)$: $A(u) \neq 0$. Since $u \in N_\lambda^+$, by (5), we have

$$(p-\alpha)M(u) + (\alpha-\beta)B(u) > 0$$

which implies

$$B(u) > \frac{\alpha-p}{\alpha-\beta} M(u) > 0.$$

(ii) We consider the following two cases:

Case $(ii-a)$: $B(u)\le 0$, we have

$$A(u)=M(u)-B(u)>0$$

Case $(ii-b)$: $B(u)>0$. By (4), we have

$$(p-\alpha)A(u)+(P-\beta)B(u)<0,$$

which implies

$$A(u)>\frac{p--\beta}{\alpha-p}B(u)>0\cdot$$

It follows that the conclusion is true. □

Lemma 2.4. *Suppose that* u_0 *is a local minimizer for* J_λ *on* N_λ*. Then, if* $u_0\notin N_\lambda$ *,* u_0 *is a critical point of* J_λ.

Proof. If u_0 is a local minimizer for J_λ on N_λ, then u_0 is a solution of the optimization

Problem minimize $J_\lambda(u)$ subject to $g_\lambda(u)=0$.

Hence, by the theory of Lagrange multipliers, there exists $\Lambda\in R$ such that

$$J'_\lambda(u_\circ)=\Lambda g'_\lambda(u_\circ)\text{ in }W^{-1}(\Omega)$$

Here $W^{-1}(\Omega)$ is the dual space of the Sobolev space W. Thus,

$$<J'_\lambda(u),u>_W=\Lambda<g'_\lambda(u),u>_W.$$

But $<g'_\lambda(u),u>_W\neq 0$, since $u\notin N_\lambda^0$. Hence $\Lambda=0$. This completes the proof. □

Then we have the following result.

Lemma 2.5. J_λ *is coercive and bounded below on* N_λ .

Proof. If $u\in N_\lambda$, it follows from (2) and the Sobolev embedding theorem

$$\begin{aligned}J_\lambda(u)&=\left(\frac{\beta-p}{\beta p}\right)M(u)-\left(\frac{\beta-\alpha}{\beta\alpha}\right)A(u)\\&\ge\left(\frac{\beta-p}{\beta p}\right)M(u)-\left(\frac{\beta-\alpha}{\beta\alpha}\right)\lambda C_1^\alpha\|a\|_\infty\|u\|_W^\alpha\\&=\left(\frac{\beta-p}{\beta p}\right)M(u)-\left(\frac{\beta-\alpha}{\beta\alpha}\right)\lambda C_1^\alpha\|a\|_\infty(M(u))^{\frac{\alpha}{\mathrm{P}}}\end{aligned}\tag{11}$$

Thus $J_\lambda(u)$ is coercive and bounded below on N_λ. □

Lemma 2.6. *Let* $\delta^*=\left(\frac{\beta}{p}\right)^{\frac{\alpha-\mathrm{P}}{\mathrm{P}-\beta}}\delta_0$. *Then if* $0<\delta^*<\lambda\|a\|_\infty$, *We have*

(i) $\gamma^+>0$

(ii) $\gamma^-\ge k_0$, *for some* $k_0=k_0(\alpha,\beta,C_1,C_2)$.

Proof. (i) Let $u\in N_\lambda^+$. By (6)

$$M(u)>\frac{(\beta-\alpha)}{(p-\alpha)}B(u),$$

and so

$$\begin{aligned}J_\lambda(u) &= \left(\frac{1}{p}-\frac{1}{\alpha}\right)M(u)+\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)B(u)\\ &\le \left(\frac{\alpha-p}{p\alpha}\right)M(u)+\left(\frac{\beta-\alpha}{\alpha\beta}\right)\left[\frac{(p-\alpha)}{(\beta-\alpha)}M(u)\right]\\ &= \left[\frac{\alpha-p}{p\alpha}+\frac{p-\alpha}{\alpha\beta}\right]M(u)\\ &= \frac{(p-\alpha)(p-\beta)}{p\alpha\beta}M(u)>0.\end{aligned}$$

Thus $\gamma_\lambda^+ > 0$.

(ii) Let $u\in N_\lambda^-$, by (6) and (9) we have

$$M(u) < \frac{\beta-\alpha}{p-\alpha}B(u) \le \frac{\beta-\alpha}{p-\alpha}C_2^\beta(\| b\|_\infty)\| u\|_W^\beta$$

This implies

$$\| u\|_w > \left[\frac{p-\alpha}{(\beta-\alpha)C_2^\alpha(\| b\|_\infty)}\right]^{\frac{1}{\beta-p}} \text{for all } u\in N_\lambda^-. \tag{12}$$

By Lemma 2.5, we have

$$J_\lambda(u) \ge \| u\|_w^\alpha\left[\left(\frac{p-\beta}{\beta p}\right)\| u\|_w^{\mathrm{P}-\alpha}-\left(\frac{\beta-\alpha}{\alpha\beta}\right)C_1^\alpha(\lambda\| a\|_\infty)\right]$$

$$> \left(\frac{p-\alpha}{(\beta-\alpha)C_2^\beta(\| b\|_\infty)}\right)^{\frac{\alpha}{\beta-\mathrm{P}}}\left[\left(\frac{p-\beta}{\beta p}\right)\left(\frac{p-\alpha}{(\beta-\alpha)C_2^\beta(\| b\|_\infty)}\right)^{\frac{p-\alpha}{\beta-p}}\right.$$

$$\left.-\left(\frac{\beta-\alpha}{\alpha\beta}\right)C_1^\alpha(\lambda\| a\|_\infty)\right]$$

Thus, if $0<\delta^*<\lambda\| a\|_\infty$, then $J_\lambda > k_0$, for all $u\in N_\lambda^-$, for some $k_0=k_0(\alpha,\beta,C_1,C_2)>0$. This completes the proof. □

For each $u\in W$ with $B(u)>0$, we write

$$t_{\max} = \left(\frac{(p-\alpha)M(u)}{(\beta-\alpha)B(u)}\right)^{\frac{1}{(\beta-p)}} > 0. \tag{13}$$

Then we have the following lemma.

Lemma 2.7. *For each $u\in W$ with $B(u)>0$ and $0<\delta_0<\lambda\| a\|_\infty$, we have*

(i) *if $A(u)\le 0$, then there is a unique $0<t^+<t_{\max}$ such that $t^+u\in N_\lambda^+$ and*

$$J_\lambda(t^+u)=\sup_{0\le t\le t_{\max}} J_\lambda(tu)$$

(ii) *if $A(u)>0$, then there are unique $0<t^+=t^+(u)<t_{\max}<t^-$ such that $t^+u\in N_\lambda^+$, $t^-u\in N_\lambda^-$ and*

$$J_\lambda(t^+u)=\sup_{0\le t\le t_{\max}} J_\lambda(tu),\quad J_\lambda(t^-u)=\inf_{t\ge 0} J_\lambda(tu)$$

Proof. Fix $u\in W$ with $B(u)>0$. Let

$$E(t)=-t^{p-\alpha}M(u)+t^{\beta-\alpha}B(u) \quad \text{for } t>0. \tag{14}$$

Clearly, $E(t)\to-\infty$ as $t\to\overset{+}{0}$. Since

$$E'(t)=-(p-\alpha)t^{p-\alpha-1}M(u)+(\beta-\alpha)t^{\beta-\alpha-1}B(u),$$

we have $E'(t)=0$ at $t=t_{\max}$, $E'(t)>0$ for $t\in[0,t_{\max})$ and $E'(t)<0$ for $t\in(t_{\max},\infty)$. Then $E(t)$ achieves its maximum at $t_{\max}$, increasing for $t\in[0,t_{\max})$ and decreasing for $t\in(t_{\max},\infty)$. Moreover,

$$\begin{aligned} E(t_{\max}) &= \left(\frac{(p-\alpha)M(u)}{(\beta-\alpha)B(u)}\right)^{\frac{p-\alpha}{\beta-\alpha}} M(u) - \left(\frac{(p-\alpha)M(u)}{(\beta-\alpha)A(u)}\right)^{\frac{\beta-\alpha}{\beta-p}} B(u) \\ &= \| u\|_w^{\alpha}\left[\left(\frac{(p-\alpha)}{(\beta-\alpha)}\right)^{\frac{p-\alpha}{\beta-\alpha}} - \left(\frac{p-\alpha}{\beta-\alpha}\right)^{\frac{\beta-\alpha}{\beta-p}}\left(\frac{\| u\|_W^{\alpha}}{B(u)}\right)^{\frac{p-\alpha}{\beta-p}}\right. \\ &\geq \| u\|_w^{\alpha}\left(\frac{1}{\| b\|_\infty}\right)^{\frac{p-\alpha}{\beta-p}}\left(\frac{\beta-p}{\beta-\alpha}\right)\left(\frac{(p-\alpha)}{(\beta-\alpha)C_2^{\alpha}}\right)^{\frac{p-\alpha}{\beta-p}} \end{aligned} \tag{15}$$

(i) $A(u)\leq 0$: There is a unique $0<t^+<t_{\max}$ such that $E(t^+)=-\lambda A(u)$ and $E'(t^+)>0$. Now,

$$\begin{aligned} &-(p-\alpha)M(t^+u)+(\beta-\alpha)B(t^+u)=(t^+)^{1+\alpha} \\ &\left[-(p-\alpha)(t^+)^{p-\alpha-1}M(u)+(\beta-\alpha)(t^+)^{\beta-\alpha-1}B(u)\right], \\ &=(t^+)^{1+\beta}E'(t^+)>0 \end{aligned}$$

and

$$\begin{aligned} &< J_\lambda'(t^+u),t^+u >= (t^+)^p M(u)-(t^+)^{\alpha}A(u)-(t^+)^{\beta}B(u) \\ &=-(t^+)^{\alpha}\left[-(t^+)^{p-\alpha}M(u)+(t^+)^{\beta-\alpha}B(u)+A(u)\right] \\ &=-(t^+)^{\alpha}\left[E(t^+)+A(u)\right]=0 \end{aligned}$$

Thus, $t^+u\in N_\lambda^+$.

Since for $t<t_{\max}$, we have

$$\begin{aligned} &-(p-\beta)M(tu)+(\alpha-\beta)B(tu)>0 \\ &\frac{d^2}{dt^2}J_\lambda(tu)<0 \end{aligned}$$

and

$$\frac{d}{dt}J_\lambda(tu)=t^{p-1}M(u)-t^{\alpha-1}A(u)-t^{\beta-1}B(u)=0 \text{ for } t=t^+.$$

Thus, $J_\lambda(t^+u)=\sup_{0\leq t\leq t_{\max}} J_\lambda(tu)\cdot$

(ii) $A(u)>0$. By (15) and

$$\begin{aligned} &E(-\infty)=0<A(u) \\ &\leq C_1^{\alpha}\left(\lambda\| a\|_\infty\right)\| u\|_w^{\alpha} \\ &<\| u\|_w^{\alpha}\left(\frac{1}{\| b\|_\infty}\right)^{\frac{p-\alpha}{\beta-p}}\left(\frac{\beta-p}{\beta-\alpha}\right)\left(\frac{(p-\alpha)}{(\beta-\alpha)C_2^{\beta}}\right)^{\frac{p-\alpha}{\beta-p}} \\ &\leq E(t_{\max}) \end{aligned}$$

for $0<\delta_0<\lambda\| a\|_\infty$, there are unique t^+ and t^- such that $0<t^+<t_{\max}<t^-$,

$$\begin{aligned} E(t^+)=A(u)=E(t^-) \\ E'(t^+)>0>E'(t^-) \end{aligned}$$

We have $t^+u\in N_\lambda^+, t^-u\in N_\lambda^-$, and $J_\lambda(t^+u)\geq J_\lambda(tu)\geq J_\lambda(t^-u)$ for each $t\in\left[t^+,t^-\right]$ and $J_\lambda(t^+u)\geq J_\lambda(tu)$ for each $0<t<t^+$. Thus,

$$J_\lambda(t^+u)=\sup_{0\leq t\leq t_{\max}} J_\lambda(tu),\quad J_\lambda(t^-u)=\inf_{t\geq 0} J_\lambda(tu)\quad\cdot$$

This completes the proof. □

3. Existence of solutions

Now we can state our main result.

Theorem 3.1. *If the parameter* λ *satisfy* $0<\delta_0<\lambda\|a\|_\infty$, *then problem* (1) *has at least two solutions* u_0^+ *and* u_0^- *such that* $u_0^\pm \geq 0$ *in* Ω *and* $u_0^\pm \neq 0$.

The proof of this Theorem will be a consequence of the next two propositions.

Proposition 3.2. *If* $0<\delta_0<\lambda\|a\|_\infty$, *then the functional* J_λ *has a minimizer* u_0^- *in* N_λ^+ *and it satisfies*

(i) $J_\lambda(u_0^-)=\gamma_\lambda^-$

(ii) u_0^- *is a nontrivial nonnegative solution of problem* (1), *such that* $u_0^- \geq 0$ *in* Ω *and* $u_0^- \neq 0$.

Proof. By Lemma 2.5, J_λ is coercive and bounded below on N_λ. Let $\{u_n\}$ be a minimizing sequence for J_λ on N_λ^-, i.e., $\lim_{n\to\infty} J_\lambda(u_n)=\inf_{u\in N_\lambda^-} J_\lambda(tu)$. Then by Lemma 2.5 and the Rellich–Kondrachov theorem, there exist a subsequence $\{u_n\}$ and $u_0^- \in W$ such that u_0^- is a solution of problem (1) and

$$u_n \to u_0^- \text{ weakly in } W,$$

$$u_n \to u_0^- \text{ strongly in } L^\alpha(\Omega) \text{ and in } L^\beta(\Omega).$$

This implies

$$B(u_n)\to B(u_0^-) \qquad \text{as } n\to+\infty$$

$$A(u_n)\to A(u_0^-) \qquad \text{as } n\to+\infty$$

Let $B(u_0)>0$. In particular $u_0^-\neq 0$. Now we prove that $u_n \to u_0^-$ strongly in *W*. Suppose otherwise, then

$$\|u_0^-\|_W < \liminf_{n\to\infty}\|u_n\|_W \tag{16}$$

Fix $u\in W$ with $B(u)>0$. Let

$$k_u(t)=E(t)+A(u),$$

where $E(t)$ is as in (14). Clearly, $k_u(t)\to-\infty$ as $t\to 0^+$, and $k_u(t)\to A(u)$ as $t\to\infty$.(Since $k_u'(t)=E'(t)$, By similar argument as in the proof of Lemma 2.7, we have $k_u(t)$ achieves its maximum at $\bar{t}_{\max}$, $k_u(t)$ is increasing for $t\in(0,\bar{t}_{\max})$ and decreasing for $t\in(\bar{t}_{\max},\infty)$, where $t\in(0,\bar{t}_{\max})$

$$\bar{t}_{\max}=\left(\frac{(p-\alpha)M(u)}{(\beta-\alpha)B(u)}\right)^{\frac{1}{(\beta-p)}}>0,$$

is as in (13), since $k_u'(t)=E'(t)$. Since $B(u_0^-)>0$, by Lemma 2.7, there is unique $t_0^->\bar{t}_{\max}$ such that $t_0^-u_0^-\in N_\lambda^-$ and

$$J_\lambda(t_0^-u_0^-)=\inf_{t\geq 0} J_\lambda(tu_0^-)$$

Then

$$\begin{aligned} K_{u_0^-}(t_0^-) &= -(t_0^-)^{P-\alpha}M(u_0^-)+(t_0^-)^{\beta-\alpha}B(u_0^-)+A(u_0^-) \\ &= -(t_0^-)^{-\alpha}((M(t_0^-u_0^-)-B(t_0^-u_0^-)-A(t_0^-u_0^-))=0 \end{aligned} \tag{17}$$

By (16) and (17) we obtain $k_{u_n}(t_0^-)>0$ for n sufficiently large. Since $u_n\in N_\lambda^-$, we have $\bar{t}_{\max}(u_n)<1$. Moreover,

$$k_{u_n}(1)=-M(u_n)+B(u_n)+A(u_n)=0,$$

and $k_{u_n}(t)$ is decreasing for $t\in(\bar{t}_{\max},t^-)$. This implies $k_{u_n}(t)<0$ for all $t\in[1,\infty)$ and n sufficiently large.

We obtain $\bar{t}_{\max}(u_0) < t^- < 1$. But $t_0^- u_0^- \in N_\lambda^-$ and

$$J_\lambda(t_0^- u_0^-) = \inf_{t \ge 0} J_\lambda(t u_0^-)$$

This implies

$$J_\lambda(t_0^- u_0^-) < J_\lambda(u_0^-) < \lim_{n\to\infty} J_\lambda(u_n) = \gamma_\lambda^-$$

which is a contradiction. Hence

$$u_n \to u_0^- \text{ strongly in } W.$$

This implies

$$J_\lambda(u_n) \to J_\lambda(u_0^-) = \gamma_\lambda^-$$

Thus u_0^- is a minimizer for J_λ on N_λ^-. Since $J_\lambda(u_0^-) = J_\lambda(|u_0^-|)$ and $|u_0^-| \in N_\lambda^-$,by Lemma 2.4 we may assume that u_0^- is a nontrivial nonnegative solution of Eq. (1). □

Next, we establish the existence of a local minimum for J_λ on N_λ^-.

Proposition 3.3. *If* $0 < \delta_0 < \lambda \| a \|_\infty$*, then the functional* J_λ *has a minimizer* u_0^+ *and it satisfies*

(i) $J_\lambda(u_0^+) = \gamma_\lambda^+$

(ii) u_0^+ *is a nontrivial nonnegative solution of problem* (1)*, such that* $u_0^+ \ge 0$ *in* Ω *and* $u_0^+ \ne 0$.

Proof. Let $\{u_n\}$ be a minimizing sequence for J_λ on N_λ^+, i.e $\lim_{n\to\infty} J_\lambda(u_n) = \inf_{u \in N_\lambda^+} J_\lambda(u)$. Then by Lemma 2.5 and the Rellich–Kondrachov theorem, there exist a subsequence $\{u_n\}$ and $u_0^+ \in W$ such that u_0^+ is a solution of problem (1) and

$$u_n \to u_0^+ \text{ weakly in } W,$$

$$u_n \to u_0^+ \text{ strongly in } L^\alpha(\Omega) \text{ and in } L^\beta(\Omega).$$

This implies

$$A(u_n) \to A(u_0^+) \qquad \text{as } n \to +\infty$$

$$B(u_n) \to B(u_0^+) \qquad as\ n \to +\infty\ .$$

Moreover, by (6) we obtain

$$B(u_n) > \frac{(p-\alpha)}{(\beta-\alpha)} M(u_n); \tag{18}$$

By (12) and (18) there exists a positive number η_0 such that

$$B(u_n) > \eta_0 .$$

This implies

$$B(u_0^+) \ge \eta_0 \ . \tag{19}$$

Now we prove that $u_n \to u_0^+$ strongly in *W*. Suppose otherwise, then

$$\| u_0^+ \|_W < \liminf_{n\to\infty} \| u_n \|_W \ .$$

By Lemma 2.7, there is unique $t \ge 0$ such that $t_0^+ u_0^+ \in N_\lambda^+$. Since $\{u_n\} \in N_\lambda^+$, $J_\lambda(u_n) \ge J_\lambda(t u_n)$ for all $t \ge 0$, we have

$$J_\lambda(t_0^+ u_0^+) < \lim_{n\to\infty} J_\lambda(t_0^+ u_n) \le \lim_{n\to\infty} J_\lambda(u_n) = \gamma_\lambda^+$$

and this is a contradiction. Hence $u_n \to u_0^+$ strongly in W .This implies

$$J_\lambda(u_n) \to J_\lambda(u_0^-) = \gamma_\lambda^+ \quad as \quad n \to \infty$$

Since $J_\lambda(u_0^+) = J_\lambda(|u_0^+|)$ and $|u_0^+| \in N_\lambda^+$, by Lemma 2.4 and (19) we may assume that u_0^+ is a nontrivial □

nonnegative solution of Eq. (1).

Proof of Theorem 3.1. By Propositions 3.2 and 3.3, we obtain Eq. (1) has two nontrivial nonnegative solutions u_0^+ and u_0^- such that $u_0^+ \in N_\lambda^+$ and $u_0^- \in N_\lambda^-$. It remains to show that the solutions found in Propositions 3.2 and 3.3 are distinct. Since $N_\lambda^+ \cap N_\lambda^- = \phi$, this implies that u_0^+ and u_0^- are distinct. This concludes the proof.

4. References

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