

Synchronization Criteria between Two Identical or Different Fractional Order Chaotic Systems

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Abstract. This paper discusses chaos synchronization between two identical or different fractional order chaotic systems. Base on linear matrix inequality, two new synchronization criteria are constructed by which it is proved that two identical (Lü system) or different (Lü and Chen systems) fractional order chaotic systems are synchronized using the simple linear feedback control laws. Finally, simulations results show the method is effective.

Keywords: Synchronization, Fractional order, Linear matrix inequality

1. Introduction

Since Pecora and Carroll established a chaos synchronization scheme for two identical chaotic systems with different initial conditions[1], variety of method and techniques have been proposed for the control and synchronization of chaotic systems such as linear and nonlinear feedback synchronization[2-6], impulsive synchronization [7-8], adaptive synchronization [9-12], observer based control method[13-15], and etc.

Fractional calculus deals with derivatives and integration of arbitrary order [16–18] and has deep and natural connections with many fields of applied mathematics, engineering and physics. Fractional calculus has wide range of applications in control theory [19], Furthermore, recently, study of chaos synchronization in fractional order dynamical systems and related phenomena is receiving growing attention , some synchronization-based strategies have been devised to synchronize fractional chaotic systems [20-25]. In Ref. [26], the synchronization of fractional-order chaotic systems has been presented. In Refs. [27-28], nonlinear control are employed to synchronize two fractional-order chaotic systems. In Ref. [29], Synchronization of N-coupled fractional-order chaotic systems with ring connection has been reported. In Refs. [30-32], Synchronization of different fractional order chaotic systems using active control has been discussed. In Ref. [33], the stability of the fractional order unified chaotic system has been studied. In this paper, base on linear matrix inequality, two new synchronization criteria are constructed by which it is proved that two identical (Lü system) or different (Lü and Chen systems) fractional order chaotic systems are synchronized.

This work is presented as follows: Section 2 describes mathematical preliminaries and model. Chaos synchronization between two identical fractional order Lü systems in Section 3. Section 4 handles chaos synchronization between Lü and Chen systems of fractional order. Section 5 gives the conclusion of the paper.

2. Mathematical preliminaries and model

In this section, we give some useful mathematical preliminaries.

The mathematical definition of fractional derivatives and integrals has been the subject of several different approaches. The definition of fractional integrals by Grunwald–Letnikov and Riemann–Liouville

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is as follows [34]:

$$I_{\mu}^{\lambda} x(t) = \frac{1}{\Gamma(\lambda)} \int_{\mu}^t (t-\tau)^{\lambda-1} x(\tau) d\tau, \quad (1)$$

where $(\mu, t) \in R^2$, $\mu < t$, $0 < \lambda < 1$, $\Gamma(\cdot)$ is Gamma function, $\Gamma(\lambda) = \int_0^{\infty} \vartheta^{\lambda-1} e^{-\vartheta} d\vartheta$, and $\Gamma(z+1) = z\Gamma(z)$.

The definition of fractional derivatives is as follows:

$$D_{\mu}^{\lambda} x(t) = \frac{d}{dt} [I_{\mu}^{\lambda} x(t)] = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dt} \int_{\mu}^t (t-\tau)^{-\lambda} x(\tau) d\tau. \quad (2)$$

The fractional order Lü system and Chen system [35-36] are described by (see Fig.1-2)

$$\begin{cases} \frac{d^{\alpha_1} x_1}{dt^{\alpha_1}} = 36(x_2 - x_1), \\ \frac{d^{\alpha_2} x_2}{dt^{\alpha_2}} = -x_1 x_3 + 20x_2, \\ \frac{d^{\alpha_3} x_3}{dt^{\alpha_3}} = x_1 x_2 - 3x_3, \end{cases} \quad (3)$$

and

$$\begin{cases} \frac{d^{\alpha_1} y_1}{dt^{\alpha_1}} = 35(y_2 - y_1), \\ \frac{d^{\alpha_2} y_2}{dt^{\alpha_2}} = (28 - 35)y_1 + 28y_2 - y_1 y_3, \\ \frac{d^{\alpha_3} y_3}{dt^{\alpha_3}} = y_1 y_2 - 3y_3. \end{cases} \quad (4)$$

3. Chaos synchronization between two identical fractional order Lü systems

In this section we study the synchronization between two identical fractional order Lü systems, we define the drive (master) and response (slave) systems as follows:

$$\begin{cases} \frac{d^{\alpha_1} x_1}{dt^{\alpha_1}} = 36(x_2 - x_1), \\ \frac{d^{\alpha_2} x_2}{dt^{\alpha_2}} = -x_1 x_3 + 20x_2, \\ \frac{d^{\alpha_3} x_3}{dt^{\alpha_3}} = x_1 x_2 - 3x_3, \end{cases} \quad (5)$$

and

$$\begin{cases} \frac{d^{\alpha_1} y_1}{dt^{\alpha_1}} = 36(y_2 - y_1) + u_1, \\ \frac{d^{\alpha_2} y_2}{dt^{\alpha_2}} = -y_1 y_3 + 20y_2 + u_2, \\ \frac{d^{\alpha_3} y_3}{dt^{\alpha_3}} = y_1 y_2 - 3y_3 + u_3. \end{cases} \quad (6)$$

We define control functions u_i as

$$u_1 = -k_1 e_1, u_2 = -k_2 e_2, u_3 = -k_3 e_3, \quad (7)$$

where $k_1 > 0, k_2 > 0, k_3 > 0$.

The error functions as

$$e_1 = y_1 - x_1, e_2 = y_2 - x_2, e_3 = y_3 - x_3. \quad (8)$$

Eq. (8) together with (5) and (6) yields the error system

$$\begin{cases} \frac{d^{\alpha_1} e_1}{dt^{\alpha_1}} = 36(e_2 - e_1) + u_1, \\ \frac{d^{\alpha_2} e_2}{dt^{\alpha_2}} = 20e_2 - x_3 e_1 - y_1 e_3 + u_2, \\ \frac{d^{\alpha_3} e_3}{dt^{\alpha_3}} = y_1 e_2 - 3e_3 + x_2 e_1 + u_3, \end{cases} \quad (9)$$

Theorem 1. The fractional order systems (5) and (6) can realize synchronization using the following linear matrix inequality:

$$\begin{pmatrix} -a(36+k_1) & (36a+b|X_3|)/2 & c|X_2|/2 \\ \Delta & -b(k_2-20) & (b+c)|Y_1|/2 \\ \Delta & \Delta & -c(3+k_3) \end{pmatrix} < 0, \quad (10)$$

where $|X_2|$, $|X_3|$ and $|Y_1|$ are the upper bounds of the absolute values of the states x_2 , x_3 and y_1 , $a, b, c > 0$, Δ denotes the symmetric terms.

Proof: Let

$$\Phi = e^T P \frac{d^\alpha e}{dt^\alpha} = e_1 a \frac{d^{\alpha_1} e_1}{dt^{\alpha_1}} + e_2 b \frac{d^{\alpha_2} e_2}{dt^{\alpha_2}} + e_3 c \frac{d^{\alpha_3} e_3}{dt^{\alpha_3}}. \quad (11)$$

where $e = (e_1, e_2, e_3)^T$, $P = \text{diag}(a, b, c)$.

From Eqs. (9) and (11), we have

$$\begin{aligned} \Phi &= e_1 [36a(e_2 - e_1) - k_1 a e_1] + e_2 [20b e_2 - x_3 b e_1 - y_1 b e_3 - k_2 b e_2] + e_3 [y_1 c e_2 + x_2 c e_1 - 3c e_3 - k_3 c e_3] \\ &= 36a e_1 e_2 - 36a e_1^2 - k_1 a e_1^2 + 20b e_2^2 - y_1 b e_2 e_3 - x_3 b e_1 e_2 - k_2 b e_2^2 + y_1 c e_2 e_3 + x_2 c e_1 e_3 - (3+k_3) c e_3^2 \\ &\leq 36a e_1 e_2 - 36a e_1^2 - k_1 a e_1^2 + 20b e_2^2 + |y_1| b e_2 e_3 + |x_3| b e_1 e_2 - k_2 b e_2^2 + |y_1| c e_2 e_3 + |x_2| c e_1 e_3 - (3c + k_3 c) e_3^2 \\ &\leq 36a e_1 e_2 - 36a e_1^2 - k_1 a e_1^2 + 20b e_2^2 + |X_3| b e_1 e_2 - k_2 b e_2^2 + |Y_1| (c + b) e_2 e_3 + |X_2| c e_1 e_3 - (3c + k_3 c) e_3^2 \\ &= e^T \begin{pmatrix} -a(36+k_1) & (36a+b|X_3|)/2 & c|X_2|/2 \\ \Delta & -b(k_2-20) & (b+c)|Y_1|/2 \\ \Delta & \Delta & -c(3+k_3) \end{pmatrix} e < 0 \end{aligned}$$

On the other hand, from Eqs. (9) we also have

$$\frac{d^\alpha e}{dt^\alpha} = \begin{pmatrix} -(36+k_1) & 36 & 0 \\ -x_3 & 20-k_2 & -y_1 \\ x_2 & y_1 & -(3-k_3) \end{pmatrix} e \stackrel{\text{def}}{=} A e.$$

Supposing that λ is one of eigenvalues of matrix A , and there should be a nonzero vector $\xi = (\xi_1, \xi_2, \dots, \xi_n)^T$ being an eigenvector corresponding to the eigenvalue λ i.e. $A\xi = \lambda\xi$, $PA\xi = P\lambda\xi$. One can gain $\xi^T PA\xi = \lambda\xi^T P\xi$, $\xi^T A^T P\xi = \bar{\lambda}\xi^T P\xi$. So,

$$\xi^T (PA + A^T P) \xi = (\lambda + \bar{\lambda}) \xi^T P \xi, \text{ or } \lambda + \bar{\lambda} = \frac{\xi^T (PA + A^T P) \xi}{\xi^T P \xi}.$$

Since, $\Phi = e^T P \frac{d^\alpha e}{dt^\alpha} < 0$, so, $e^T P \frac{d^\alpha e}{dt^\alpha} + (\frac{d^\alpha e}{dt^\alpha})^T P^T e < 0$, i.e.

$$e^T PAe + (Ae)^T P^T e = e^T PAe + e^T A^T P e = e^T (PA + A^T P) e < 0,$$

that is, $PA + PA^T$ is a negative definite matrix, and then $\lambda + \bar{\lambda} = \frac{\xi^T (PA + A^T P) \xi}{\xi^T P \xi} < 0$.

Formulation $|\arg(\lambda)| > \pi/2 > \alpha\pi/2$ is obvious. According to the stability theory of fractional-order system [37], the system (9) is stable, therefore, the fractional order systems (5) and (6) can realize synchronization.

3.1. Simulation and results

In this section, computer simulations are used to verify and demonstrate the effectiveness of the above method. In all simulation, $\alpha_1 = 0.97, \alpha_2 = 0.98, \alpha_3 = 0.99$. The initial conditions of the master and slave systems are $(-1 \ 2 \ 15)$ and $(2 \ 6 \ 5)$, respectively. By estimating simulations, we let $|X_2| = 30, |X_3| = 50, |Y_1| = 28$.

The conditions $a, b, c > 0$ can be expressed as

$$\begin{pmatrix} -a & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & -c \end{pmatrix} < 0. \quad (12)$$

So, inequality (10) and (12) are linear matrix inequality in $a, b, c, ak_1, bk_2, ck_3$. By solving the LMI (10) and (12), the following solutions are obtained:

$$a = 2.7649, b = 1.3880, c = 2.0564, ak_1 = 48.7577, bk_2 = 195.4376, ck_3 = 111.1031.$$

This yields $k_1 = 17.6345, k_2 = 140.8052, k_3 = 54.0280$. Therefore, the controller (7) will drive the slave system (6) to synchronize the master system (5) as desired, the synchronous errors are shown in Figs. 3.

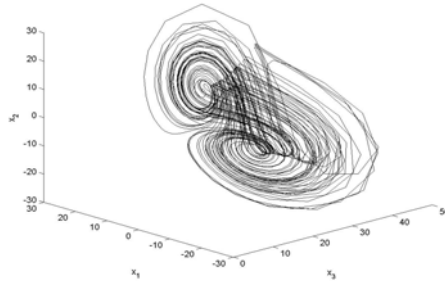


Fig.1. Chaotic attractor of the fractional-order Lü system (3)

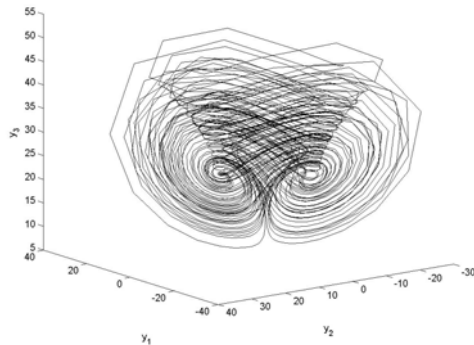


Fig.2. Chaotic attractor of the fractional-order Chen system (4)

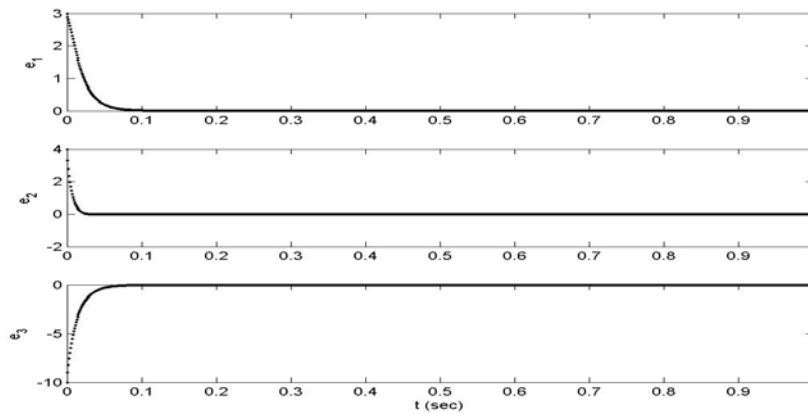


Fig.3. Synchronization errors between two identical fractional order Lü systems

4. Chaos synchronization between Lü and Chen systems of fractional order

In this section we study the synchronization between Lü and Chen systems of fractional order. Assuming that the Lü system drives the Chen system, we define the drive (master) and response (slave) systems as follows:

$$\begin{cases} \frac{d^{\alpha_1} x_1}{dt^{\alpha_1}} = 36(x_2 - x_1), \\ \frac{d^{\alpha_2} x_2}{dt^{\alpha_2}} = -x_1 x_3 + 20x_2, \\ \frac{d^{\alpha_3} x_3}{dt^{\alpha_3}} = x_1 x_2 - 3x_3, \end{cases} \quad (13)$$

and

$$\begin{cases} \frac{d^{\alpha_1} y_1}{dt^{\alpha_1}} = 35(y_2 - y_1) + u_1, \\ \frac{d^{\alpha_2} y_2}{dt^{\alpha_2}} = (28 - 35)y_1 + 28y_2 - y_1 y_3 + u_2, \\ \frac{d^{\alpha_3} y_3}{dt^{\alpha_3}} = y_1 y_2 - 3y_3. \end{cases} \quad (14)$$

Eq. (8) together with (13) and (14) yields the error system

$$\begin{cases} \frac{d^{\alpha_1} e_1}{dt^{\alpha_1}} = 35(e_2 - e_1) - x_2 + x_1 + u_1, \\ \frac{d^{\alpha_2} e_2}{dt^{\alpha_2}} = -7y_1 + 20e_2 + 8y_2 - x_3 e_1 - y_1 e_3 + u_2, \\ \frac{d^{\alpha_3} e_3}{dt^{\alpha_3}} = y_1 e_2 - 3e_3 + x_2 e_1. \end{cases} \quad (15)$$

We define control functions u_1, u_2 as

$$u_1 = x_2 - x_1, \quad u_2 = 7y_1 - 8y_2 - k_1 e_1 - k_2 e_2 - k_3 e_3, \quad (16)$$

where $k_1 > 0, k_2 > 0, k_3 > 0$.

Theorem 2. The fractional order systems (13) and (14) can realize synchronization using the following linear matrix inequality:

$$\begin{pmatrix} -35a & (35a+b|X_3|-k_1b)/2 & c|X_2|/2 \\ \Delta & -b(k_2-20) & [(b+c)|Y_1|-bk_3]/2 \\ \Delta & \Delta & -3c \end{pmatrix} < 0, \quad (17)$$

where $|X_2|$, $|X_3|$ and $|Y_1|$ are the upper bounds of the absolute values of the states x_2 , x_3 and y_1 , $a, b, c > 0$, Δ denotes the symmetric terms.

Proof: Similar to proof of Theorem 1, let

$$\Phi = e^T P \frac{d^\alpha e}{dt^\alpha} = e_1 a \frac{d^{\alpha_1} e_1}{dt^{\alpha_1}} + e_2 b \frac{d^{\alpha_2} e_2}{dt^{\alpha_2}} + e_3 c \frac{d^{\alpha_3} e_3}{dt^{\alpha_3}}. \quad (18)$$

From Eqs. (15) and (18), we have

$$\begin{aligned} \Phi &= 35ae_1(e_2 - e_1) + e_2b[20e_2 - x_3e_1 - y_1e_3 - k_1e_1 - k_2e_2 - k_3e_3] + e_3c(y_1e_2 + x_2e_1 - 3e_3) \\ &= 35ae_1e_2 - 35ae_1^2 + (20 - k_2)be_2^2 - x_3be_1e_2 - y_1be_2e_3 - k_1be_1e_2 - k_3be_2e_3 + y_1ce_2e_3 + x_2ce_1e_3 - 3ce_3^2 \\ &\leq 35ae_1e_2 - 35ae_1^2 + (20 - k_2)be_2^2 + |x_3|be_1e_2 + |y_1|be_2e_3 - k_1be_1e_2 - k_3be_2e_3 + |y_1|ce_2e_3 + |x_2|ce_1e_3 - 3ce_3^2 \\ &\leq 35ae_1e_2 - 35ae_1^2 + (20 - k_2)be_2^2 + |X_3|be_1e_2 + |Y_1|be_2e_3 - k_1be_1e_2 - k_3be_2e_3 + |Y_1|ce_2e_3 + |X_2|ce_1e_3 - 3ce_3^2 \\ &= e^T \begin{pmatrix} -35a & (35a+b|X_3|-k_1b)/2 & c|X_2|/2 \\ \Delta & -b(k_2-20) & [(b+c)|Y_1|-bk_3]/2 \\ \Delta & \Delta & -3c \end{pmatrix} e < 0 \end{aligned}$$

On the other hand, from Eqs. (15) we also have

$$\frac{d^\alpha e}{dt^\alpha} = \begin{pmatrix} -35 & 35 & 0 \\ -x_3 - k_1 & 20 - k_2 & -y_1 - k_3 \\ x_2 & y_1 & -3 \end{pmatrix} e \stackrel{\text{def}}{=} Be.$$

The undone reasoning process is similar to proof of Theorem 1, thus we leave out the following proofs here.

4.1. Simulation and results

In this section, computer simulations are used to verify and demonstrate the effectiveness of the above method. The initial conditions of the master and slave systems are $(-1 \ 2 \ 15)$ and $(2 \ 6 \ 5)$, respectively. By estimating simulations, we also let $|X_2| = 30$, $|X_3| = 50$, $|Y_1| = 28$. So, inequality (12) and (17) are linear matrix inequality in $a, b, c, ak_1, bk_2, ck_3$. By solving the LMI (12) and (17), the following solutions are obtained:

$$a = 0.0295, b = 1.0035, c = 0.0087, ak_1 = 7.7786, bk_2 = 23.7544, ck_3 = 15.0975.$$

This yields $k_1 = 26.7858, k_2 = 143.3579, k_3 = 55.3225$. Therefore, the controller (16) will drive the slave system (14) to synchronize the master system (13) as desired, the synchronous errors are shown in Figs. 4.

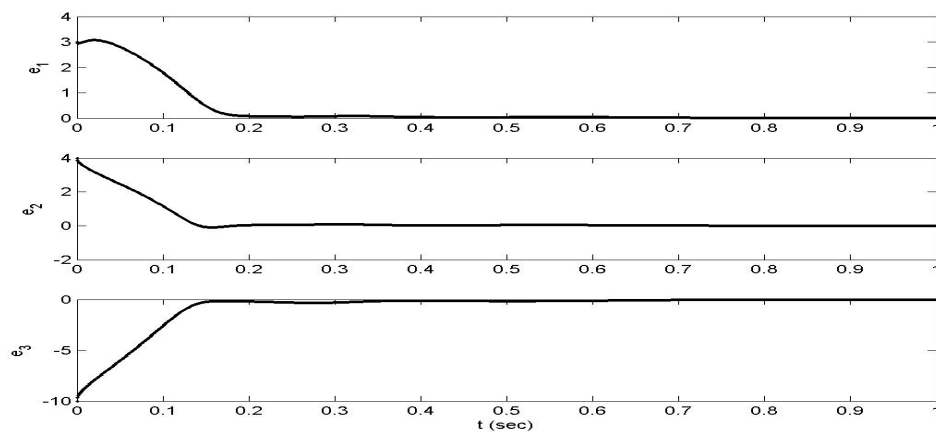


Fig.4. Synchronization errors between fractional order Lü and Chen systems

5. Conclusion

This paper discusses chaos synchronization between two identical or different fractional order chaotic systems. The simple state feedback controllers for fractional-order chaos synchronisation are obtained based on linear matrix inequality, simulation results are presented to demonstrate the application of theoretical results.

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