A Class of Quasi-Quartic Trigonometric BÉZier Curves and Surfaces

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Abstract. A new kind of quasi-quartic trigonometric polynomial base functions with a shape parameter λ over the space Ω=span {1, sin$t$, cos$t$, sin2t, cos2t} is presented, and the corresponding quasi-quartic trigonometric Bézier curves and surfaces are defined by the introduced base functions. The quasi-quartic trigonometric Bézier curves inherit most of properties similar to those of quartic Bézier curves, and can be adjusted easily by using the shape parameter λ. The jointing conditions of two pieces of curves with G^2 and C^4 continuity are discussed. With the shape parameter chosen properly, the defined curves can express exactly any plane curves or space curves defined by parametric equation based on{1, sin$t$, cos$t$, sin2t, cos2t} and circular helix with high degree of accuracy without using rational form. The corresponding tensor product surfaces can also represent precisely some quadratic surfaces, such as sphere, paraboloid, cylindrical surfaces, and some complex surfaces. The relationship between quasi-quartic trigonometric Bézier curves and quartic Bézier curves is also discussed, the larger is parameter λ, and the more approach is the quasi-quartic trigonometric Bézier curve to the control polygon. Examples are given to illustrate that the curves and surfaces can be used as an efficient new model for geometric design in the fields of CAGD.

Keywords: Bézier curves and surfaces, trigonometric polynomial, quasi-quartic, shape parameter, G^2 and C^4 continuity

1. Introduction

In Computer Aided Geometric Design (CAGD), lower order Bézier curves and B-spline curves have become the common tools for constructing free form curves and surfaces [1, 2]. But they cannot represent exactly some quadratic curves such as the circular arcs, parabolas, spheres, cylinders and the other conic curves and surfaces. Although rational Bézier curves and NURBS curves can construct some analytic curves and surfaces, such as conic curves and revolution surfaces, there are some defects because of theirs rational style, such as complexity of computing derivation and quadrature, the weights of selecting not easy to control [3, 4].

In recent years, people have gained interest in trigonometric polynomial curve spline and have started to search the represent method to construct curves and surfaces on the space of trigonometric functions, of which in [5] the famous C-curves is obtained based on {1, *t*, sin*t*, cos*t*}, quadratic and cubic trigonometric polynomial curves with two shape parameters are given respectively in [6] and [7], a group of T-Bézier curves with features of Bézier curves is proposed in [8], the adjustable quadratic trigonometric Bézier curves with a shape parameter are presented in [9,10]. These existing trigonometric curves have similar properties to polynomial curves.

In this paper, we present a class of new different trigonometric polynomial basis functions with a parameter based on the space Ω=span {1, sin*t*, cos*t*, sin2*t*, cos2*t*}, and the corresponding curves and tensor product surfaces named quasi-quartic trigonometric Bézier curves and surfaces are constructed based on the introduced basis functions. The quasi-quartic trigonometric Bézier curves not only inherit most of the similar

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properties to quartic Bézier curves, but also can express any plane curves or space curves defined by parametric equation based on {1, sin$t$, cos$t$, sint2t, cos2t} including some quadratic curves such as the circular arcs, parabolas, cardioid exactly and circular helix with high degree of accuracy under the appropriate conditions.

The rest of this paper is organized as follows. Section 2 defines the quasi-quartic trigonometric polynomial base functions and the corresponding curves and surfaces, theirs propositions are discussed. In section 3, we discussed the continuity conditions of quasi-quartic trigonometric Bézier curves. In section 4, we show the representations of some curves. Besides, some examples of shape modeling by using the quasi-quartic trigonometric Bézier surfaces are presented also. We devote Section 5 to giving the relationship between quasi-quartic trigonometric Bézier curves and quartic Bézier curves. The conclusions are given in section 6.

2. Construction and related properties of quasi-quartic trigonometric Bézier curves and surfaces

Definition 1 For $t \in [0, \frac{\pi}{2}]$, $b_{0,4}(t)$, $b_{1,4}(t)$, $b_{2,4}(t)$, $b_{3,4}(t)$ and $b_{4,4}(t)$ are called quasi-quartic trigonometric polynomial base functions with a shape parameter λ which can be defined to be

$$\begin{cases} b_{0,4} = (1+\frac{\lambda}{2}) - (1+\lambda)\sin t - \frac{\lambda}{2}\cos 2t \\ b_{1,4} = (1+\lambda)(-\frac{3}{2} + 2\sin t + \cos t - \frac{1}{2}\sin 2t + \frac{1}{2}\cos 2t) \\ b_{2,4} = 2(1+\lambda)(1 - \sin t - \cos t + \frac{1}{2}\sin 2t) \\ b_{3,4} = (1+\lambda)(-\frac{3}{2} + \sin t + 2\cos t - \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t) \\ b_{4,4} = (1+\frac{\lambda}{2}) - (1+\lambda)\cos t + \frac{\lambda}{2}\cos 2t \end{cases} \tag{1}$$

where $-1 \le \lambda \le 1.5$.

From Eq. (1), it is easy to check that

1) Weight property: $b_{0,4}(t) + b_{1,4}(t) + b_{2,4}(t) + b_{3,4}(t) + b_{4,4}(t) \equiv 1$;

2) Symmetry: $b_{i,4}(\frac{\pi}{2} - t) = b_{4-i,4}(t)$, $i = 0,1,2,3,4$

3) Nonnegative property: when $-1 \le \lambda \le 1$, $b_{i,4}(t) \ge 0$ $(i = 0,1,2,3,4)$.

The above results show that the quasi-quartic trigonometric polynomial base functions have the most of properties similar to quartic Bernstein basis functions.

Definition 2 Let P_0, P_1, P_2, P_3 and P_4 be given control points, the following curves are called quasi-quartic trigonometric Bézier curve,

$$B(t) = \sum_{j=0}^{4} b_{j,4}(t)P_j \tag{2}$$

where $b_{j,4}(t)$ $(j = 0,1,2,3,4)$ are quasi-quartic trigonometric polynomial base functions.

Let $[t]$ = [1 sin$t$ cos$t$ sint2t cos2t], $[P]$ = [P_0 P_1 P_2 P_3 P_4] and

$$[b] = \begin{bmatrix} 1+\frac{\lambda}{2} & -\frac{3(1+\lambda)}{2} & 2(1+\lambda) & -\frac{3(1+\lambda)}{2} & 1+\frac{\lambda}{2} \\ -(1+\lambda) & 2(1+\lambda) & -2(1+\lambda) & 1+\lambda & 0 \\ 0 & 1+\lambda & -2(1+\lambda) & 2(1+\lambda) & -(1+\lambda) \\ 0 & -\frac{1+\lambda}{2} & 1+\lambda & -\frac{1+\lambda}{2} & 0 \\ -\frac{\lambda}{2} & \frac{1+\lambda}{2} & 0 & -\frac{1+\lambda}{2} & \frac{\lambda}{2} \end{bmatrix}$$

then Eq. (2) can be transformed into matrix form as follows:

$$B(t) = [\boldsymbol{t}][\boldsymbol{b}][\boldsymbol{P}]^{\mathrm{T}}$$

It's not difficult to prove that the rank of matrix $[\boldsymbol{b}]$ equal 5 for an arbitrary parameter λ, thus the matrix $[\boldsymbol{b}]$ is linear independent.

From the properties of quasi-quartic trigonometric polynomial base functions, the properties of quasi-quartic trigonometric Bézier curves can be obtained as follows.

1) **Geometric properties at the endpoints:**

$$\boldsymbol{B}(0) = \boldsymbol{P}_0 , \quad \boldsymbol{B}(\frac{\pi}{2}) = \boldsymbol{P}_4 , \quad \boldsymbol{B}'(0) = (1+\lambda)(\boldsymbol{P}_1 - \boldsymbol{P}_0) , \quad \boldsymbol{B}'(\frac{\pi}{2}) = (1+\lambda)(\boldsymbol{P}_4 - \boldsymbol{P}_3) \tag{3}$$

2) **Symmetry:** Assume we keep the location of control points $\boldsymbol{P}_i(i=0,1,2,3,4)$ fixed, invert their orders, and then the obtained curve coincides with the former one with opposite directions. In fact, from the symmetry of quasi-quartic trigonometric polynomial base functions, we have

$$\boldsymbol{B}(\frac{\pi}{2}-t) = \overset{4}{\underset{i=0}{\Sigma}} \boldsymbol{P}_i b_{i,4}(\frac{\pi}{2}-t) = \overset{4}{\underset{i=0}{\Sigma}} \boldsymbol{P}_{4-i} b_{i,4}(t) = \boldsymbol{B}(t)$$

3) **Affine invariance:** The shapes of quasi-quartic trigonometric Bézier curves are independent of the choice of coordinates. This property can be easily verified by considering an affine map $\Phi(x) = Ax + v$ where A is 4×4 matrix and v is in R^3. Now

$$\overset{4}{\underset{i=0}{\Sigma}} \Phi(\boldsymbol{P}_i) b_{4,i}(t) = \overset{4}{\underset{i=0}{\Sigma}} (A\boldsymbol{P}_i + v) b_{4,i}(t) = A \overset{4}{\underset{i=0}{\Sigma}} \boldsymbol{P}_i b_{4,i}(t) + v = \Phi(\overset{4}{\underset{i=0}{\Sigma}} \boldsymbol{P}_i b_{4,i}(t))$$

4) **V.D. property:** No plane more intersections with the curve than with the control polygon in a quasi-quartic trigonometric Bézier curve segment.

Proof. We use the method presented by [11]. First, we need to prove that the base functions (1) fulfill the Descartes' rule of signs on $[0,\frac{\pi}{2}]$, that is, for an arbitrary set of const sequence $\{c_0, c_1, c_2, c_3, c_4\}$,

$$Zeros(0,\frac{\pi}{2})\{\overset{4}{\underset{k=0}{\Sigma}} c_k B_k(t)\} \le SA(c_0, c_1, c_2, c_3, c_4) \tag{4}$$

where $Zeros(0,\frac{\pi}{2})\{f(t)\}$ means the number of zeros of function $f(t)$ on $(0,\frac{\pi}{2})$, $f(t) = \overset{4}{\underset{k=0}{\Sigma}} c_k B_k(t)$. SA $(c_0, c_1, c_2, c_3, c_4)$ denotes the number of sign changed. Let's suppose $c_0 > 0$, SA $(c_0, c_1, c_2, c_3, c_4)$ would be 4, 3,2,1,0.

1) When SA $(c_0, c_1, c_2, c_3, c_4)=4$, $c_4 > 0$. On the other hand, $f(t)$ is continuous on $[0,\frac{\pi}{2}]$, thus $f(0) = c_0$, $f(\frac{\pi}{2}) = c_4$. Suppose $f(t)$ has five zeros over $(0,\frac{\pi}{2})$, then $f(\frac{\pi}{2}) = c_4 < 0$, it is contradictious with pervious result, hence Eq. (4) is true.

2) When SA $(c_0, c_1, c_2, c_3, c_4)=3,2,1$, we can prove that Eq. (4) is true using the same method.

3) When SA $(c_0, c_1, c_2, c_3, c_4)=0$, it is obvious that Eq. (4) comes true. So we can come to a conclusion ultimately that Eq. (4) is true.

Next we will prove the V.D. property. Let $\boldsymbol{L}$ is a line (or plane) that through point Q and whose normal vector is $\boldsymbol{v}$, N_p and N_c are numbers of intersection points between control polygon$<P_0 P_1 P_2 P_3 P_4>$ and curve segment $\boldsymbol{B}_i(t)$ with $\boldsymbol{L}$ respectively.

If $\boldsymbol{L}$ intersects the control polygon $<P_0 P_1 P_2 P_3 P_4>$ at the side $P_k P_{k+1}$, then P_k, P_{k+1} must be flanked by $\boldsymbol{L}$, hence $\boldsymbol{v} \cdot (P_k - Q)$ and $\boldsymbol{v} \cdot (P_{k+1} - Q)$ have contrary sign. So,

$$SA\{\boldsymbol{v} \cdot (P_0 - Q), \boldsymbol{v} \cdot (P_1 - Q), \boldsymbol{v} \cdot (P_2 - Q), \boldsymbol{v} \cdot (P_3 - Q), \boldsymbol{v} \cdot (P_4 - Q)\} \le N_p \tag{5}$$

on the other hand, according the Descartes' rule of signs,

$$N_c = Zeros(0,\frac{\pi}{2})\{\overset{4}{\underset{k=0}{\Sigma}} b_{k,4}(t)(P_k - Q) \cdot \boldsymbol{v}\} \le SA\{\boldsymbol{v} \cdot (P_0 - Q), \boldsymbol{v} \cdot (P_1 - Q), \boldsymbol{v} \cdot (P_2 - Q), \boldsymbol{v} \cdot (P_3 - Q), \boldsymbol{v} \cdot (P_4 - Q)\} \tag{6}$$

from Eq.(5) and (6), we can get easily that $N_c \le N_p$. This completed the proof.

5) **Convexity preserving**: From V.D. property, when the control polygon is convex, because the maximum of intersection points between a line with control polygon not exceeds 2, the maximum of

intersection points between an arbitrary line in a plane with curve not exceeds 2, hence the corresponding quasi-quartic trigonometric Bézier curve is convex.

6) **Approximation**: for $t \in [0, \frac{\pi}{2}]$, when parameter λ increases gradually, then $b_{0,4}(t)$ and $b_{4,4}(t)$ decrease gradually, while $b_{1,4}(t)$, $b_{2,4}(t)$ and $b_{3,4}(t)$ increases gradually; thus when parameter λ increases gradually, the curve approach to P_1P_2 and P_2P_3. Especially, the curve goes back to the straight line when λ=-1. (See Fig. 1)
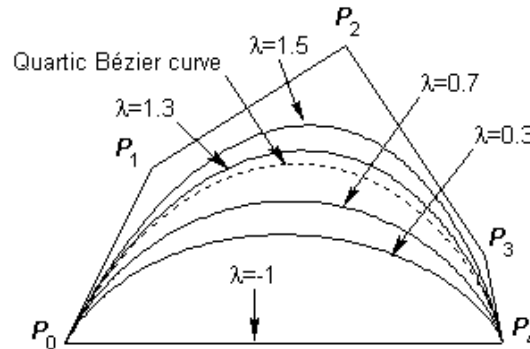

Fig.1 A family of quasi-quartic trigonometric Bézier curves

Definition 3 Given the control mesh $[\boldsymbol{P}_{rs}]$ $(r=i,\ldots,i+3; \; s=j,\ldots,j+3)$, $(i=0,1,\ldots,n-1; j=0,1,\ldots,m-1)$, tensor product quasi-biquartic trigonometric Bézier surfaces can be defined as

$$\boldsymbol{B}_{i,j}(u,v) = \sum_{r=i}^{4} \sum_{s=j}^{4} b_{r,4}(\lambda_1,u) b_{s,4}(\lambda_2,v) \boldsymbol{P}_{rs} \quad (u,v) \in [0,\frac{\pi}{2}] \times [0,\frac{\pi}{2}]$$

where $b_{r,4}(\lambda_1,u)$ $b_{s,4}(\lambda_2,v)$ are quasi-quartic trigonometric polynomial base functions. Many properties of quasi-quartic trigonometric Bézier curves can be extended to the quasi-quartic trigonometric surfaces. For example, the symmetry, affine invariance and the convexity preserving property also hold for the surface scheme.

3. Jointing of quasi-quartic trigonometric Bézier curves

Suppose there are two segment of quasi-quartic trigonometric Bézier curves $\boldsymbol{P}(t) = \sum_{i=0}^{4} \boldsymbol{P}_i b_{i,4}(t)$, $\boldsymbol{Q}(t) = \sum_{i=0}^{4} \boldsymbol{Q}_i b_{i,4}(t)$; where $\boldsymbol{P}_4 = \boldsymbol{Q}_0$, parameters of $\boldsymbol{P}_i(t)$ and $\boldsymbol{Q}_i(t)$ are λ_1 and λ_2 respectively.

To achieve G^1 continuity of the two curve segments, it is required that not only the last control point of $\boldsymbol{P}_i(t)$ and the first control point of $\boldsymbol{Q}_i(t)$ must be the same, but also the direction of the first order derivative at jointing point should be the same, namely

$$\boldsymbol{P}'(\frac{\pi}{2}) = k\boldsymbol{Q}'(0) \quad (k>0)$$

Substituting Eq. (3) into the above equitation, one can get

$$(1+\lambda_1)(\boldsymbol{P}_4 - \boldsymbol{P}_3) = k(1+\lambda_2)(\boldsymbol{Q}_1 - \boldsymbol{Q}_0)$$

Let $\delta = \dfrac{k(1+\lambda_2)}{1+\lambda_1}$, substituting it into the above equitation, then

$$(\boldsymbol{P}_4 - \boldsymbol{P}_3) = \delta(\boldsymbol{Q}_1 - \boldsymbol{Q}_0) \quad (\delta > 0)$$

Especially, for k=1, namely, $\delta = \dfrac{1+\lambda_2}{1+\lambda_1}$, the first order derivative of two segment of curves is equal .Thus,

G^1 continuity has transformed into C^1 continuity. Then we can get following theorem1.

Theorem1 If $\boldsymbol{P}_3\boldsymbol{P}_4$ and $\boldsymbol{Q}_0\boldsymbol{Q}_1$ is collinear and have the same directions, i.e.

$$(\boldsymbol{P}_4 - \boldsymbol{P}_3) = \delta(\boldsymbol{Q}_1 - \boldsymbol{Q}_0) \quad (\delta > 0) \tag{7}$$

Then curves of $\boldsymbol{P}(t)$ and $\boldsymbol{Q}(t)$ will reach G^1 continuity at a jointing point and when $\delta = \dfrac{1+\lambda_2}{1+\lambda_1}$, they will

get C^1 continuity.

Then we will discuss continuity conditions of G^2 and C^4 when $\lambda_1 = \lambda_2 = 1$.

First, we'll discuss conditions of G^2 continuity which is required to have common curvature, namely

$$\frac{\left| \boldsymbol{P}'(\pi/2) \times \boldsymbol{P}''(\pi/2) \right|}{\left| \boldsymbol{P}'(\pi/2) \right|^3} = \frac{\left| \boldsymbol{Q}'(0) \times \boldsymbol{Q}''(0) \right|}{\left| \boldsymbol{Q}'(0) \right|^3} \tag{8}$$

Let $\lambda_1 = \lambda_2 = 1$, second derivatives of two segments of curves can be get

$$\boldsymbol{P}''(\pi/2) = 4\boldsymbol{P}_2 - 6\boldsymbol{P}_3 + 2\boldsymbol{P}_4 , \ \boldsymbol{Q}''(0) = 2\boldsymbol{Q}_0 - 6\boldsymbol{Q}_1 + 4\boldsymbol{Q}_2 \tag{9}$$

Substituting Eq. (3) and (9) into Eq. (8), simplifying it, then

$$\frac{\left| (\boldsymbol{P}_4 - \boldsymbol{P}_3) \times (\boldsymbol{P}_3 - \boldsymbol{P}_2) \right|}{\left| \boldsymbol{P}_4 - \boldsymbol{P}_3 \right|^3} = \frac{\left| (\boldsymbol{Q}_1 - \boldsymbol{Q}_0) \times (\boldsymbol{Q}_2 - \boldsymbol{Q}_1) \right|}{\left| \boldsymbol{Q}_1 - \boldsymbol{Q}_0 \right|^3} \tag{10}$$

Substituting Eq. (7) into the above equitation, one can get

$$h_1 = \delta^2 h_2$$

where h_1 is the distance from $\boldsymbol{P}_2$ to $\boldsymbol{P}_3\boldsymbol{P}_4$ and h_2 is the distance from $\boldsymbol{Q}_2$ to $\boldsymbol{Q}_0\boldsymbol{Q}_1$. Hence we can get theorem 2.

Theorem 2 Let parameters λ_1, λ_2 are all equal one, if they satisfy Eq. (7) and (10), five points $\boldsymbol{P}_2, \boldsymbol{P}_3, \boldsymbol{P}_4, \boldsymbol{Q}_1, \boldsymbol{Q}_2$ are coplanar and $\boldsymbol{P}_2, \boldsymbol{Q}_2$ are in the same side of the common tangent, then jointing of curves $\boldsymbol{P}(t)$ and $\boldsymbol{Q}(t)$ reach G^2 continuity. (See Fig. 2)
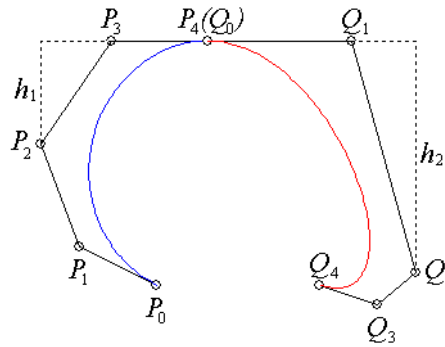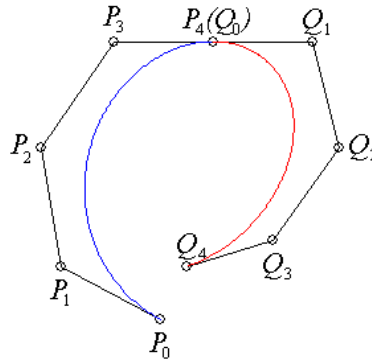

Fig.2 G^2 continuity of two pieces of curves (δ=1.5, h_2=2.25h_1)

Next, we will discuss the conditions of C^4 continuity. When curves $\boldsymbol{P}(t)$ and $\boldsymbol{Q}(t)$ reach C^1 continuity at the linked point, i.e. $\boldsymbol{Q}_0 = \boldsymbol{P}_4$ and $\boldsymbol{Q}_1 - \boldsymbol{Q}_0 = \boldsymbol{P}_4 - \boldsymbol{P}_3$. Under such circumstances, if $\boldsymbol{P}''(\pi/2) = \boldsymbol{Q}''(0)$, $\boldsymbol{P}'''(\pi/2) = \boldsymbol{Q}'''(0)$, $\boldsymbol{P}^{(4)}(\pi/2) = \boldsymbol{Q}^{(4)}(0)$, then two curves will become C^4 continuity. Combine all the above conditions, we get

$$\begin{aligned} \boldsymbol{Q}_2 &= \boldsymbol{P}_2 - 3\boldsymbol{P}_3 + 3\boldsymbol{P}_4 \\ \boldsymbol{Q}_3 &= -\boldsymbol{P}_1 + 4\boldsymbol{P}_2 - 6\boldsymbol{P}_3 + 4\boldsymbol{P}_4 \\ \boldsymbol{Q}_4 &= \boldsymbol{P}_0 - 4\boldsymbol{P}_1 + 8\boldsymbol{P}_2 - 8\boldsymbol{P}_3 + 4\boldsymbol{P}_4 \end{aligned} \tag{11}$$

Theorem 3 Let parameters λ_1, λ_2 are all equal one and satisfy (11) in theorem 1, curves $\boldsymbol{P}(t)$ and $\boldsymbol{Q}(t)$ will reach C^4 continuity at the linked point. (See Fig. 3)

Fig.3 C^4 continuity of two pieces of curves

4. Applications of quasi-quartic trigonometric Bézier curves and surfaces

4.1. Exact Expressions of some curves

Theorem 4 When $\lambda \neq -1$, quasi-quartic trigonometric Bézier curves can express any plane curves or space curves defined by parametric equation based on {1, sin$t$, cos$t$, sin2t, cos2t}.

Proof. Let any plane curves defined by parametric equation based on {1, sin$t$, cos$t$, sin2t, cos2t} defined as

$$\boldsymbol{p}(t) = (\boldsymbol{p}_x(t), \boldsymbol{p}_y(t))$$

and let $\boldsymbol{p}_x(t) = [\boldsymbol{t}][x_a, x_b, x_c, x_d, x_e]^T$, $\boldsymbol{p}_y(t) = [\boldsymbol{t}][y_a, y_b, y_c, y_d, y_e]^T$, where $[\boldsymbol{t}]$=[1 sin$t$ cos$t$ sin2t cos2t], $x_a, ..., x_e \in R$, $y_a, ..., y_e \in R$, and both vector are non-zero vector.

If quasi-quartic trigonometric Bézier curves $\boldsymbol{B}(t)$ can express $\boldsymbol{p}(t)$, i.e. $\boldsymbol{B}(t) = \boldsymbol{p}(t)$.

So, for x, y coordinate, we have $\boldsymbol{B}_x(t) = \boldsymbol{p}_x(t)$ and $\boldsymbol{B}_y(t) = \boldsymbol{p}_y(t)$, namely,

$$[\boldsymbol{t}][\boldsymbol{b}][\boldsymbol{P}_x] = [\boldsymbol{t}][x_a, ..., x_e]^T, \quad [\boldsymbol{t}][\boldsymbol{b}][\boldsymbol{P}_y] = [\boldsymbol{t}][y_a, ..., y_e]^T$$

Simplifying them, one can get

$$[\boldsymbol{b}][\boldsymbol{P}_x] = [x_a, ..., x_e]^T, \quad [\boldsymbol{b}][\boldsymbol{P}_y] = [y_a, ..., y_e]^T \qquad (12)$$

Because $|b| \neq 0$ when $\lambda \neq -1$, and the rank of matrix $[\boldsymbol{b}]$ equal 5, the above two systems of linear inhomogeneous equations must have solutions, and the solutions are unique when the expression of solutions without λ. Therefore, quasi-quartic trigonometric Bézier curves $\boldsymbol{B}(t)$ can express $\boldsymbol{p}(t)$.

For curve in space, the proof is the same as the plane curve, and this completed the proof.

Then, only one example which exact expression of circular or ellipse is to be given as follow.

Let $\boldsymbol{p}(t) = (u\cos t, v\sin t)$ or $\boldsymbol{p}(t) = (u\cos 2t, v\sin 2t)$, where $0 \leq t \leq \pi/2$. Obviously, $\boldsymbol{p}(t)$ is circular parametric or ellipse equation. We discuss the second form of $\boldsymbol{p}(t)$ here.

It is easy to get

$$\boldsymbol{p}_x(t) = u\cos 2t = [\boldsymbol{t}][0\ 0\ 0\ 0\ u]^T, \quad \boldsymbol{p}_y(t) = v\sin 2t = [\boldsymbol{t}][0\ 0\ 0\ v\ 0]^T$$

then

$$[x_a, ..., x_e] = [0\ 0\ 0\ 0\ u], \quad [y_a, ..., y_e] = [0\ 0\ 0\ v\ 0]$$

Substituting the above equations into Eqs. (12), we can solve the system of linear equations and get

$$[\boldsymbol{P}_x] = [u\ \ u\ \ 0\ \ -u\ \ -u]^T, \quad [\boldsymbol{P}_y] = [0\ \ \frac{2v}{1+\lambda}\ \ \frac{3v}{1+\lambda}\ \ \frac{2v}{1+\lambda}\ \ 0]^T$$

Clearly, when $\lambda \neq -1$, given the five control points determined by above $[\boldsymbol{P}_x]$ and $[\boldsymbol{P}_y]$ the curve express exactly the up-semicircular or up-semi ellipse. In the same way, one can get the five control points expressing the down-semicircular or down-semi ellipse

$$[\boldsymbol{P}_x] = [-u\ \ -u\ \ 0\ \ u\ \ u]^T, \quad [\boldsymbol{P}_y] = [0\ \ -\frac{2v}{1+\lambda}\ \ -\frac{3v}{1+\lambda}\ \ -\frac{2v}{1+\lambda}\ \ 0]^T$$

Note：It's important that the joining of two arcs of circular or ellipse is C^4 continuous because of satisfying

the theorem 3, and it's not difficulty to verify it.

4.2. Approximation of Helix

Theorem 5 Let $\lambda = 1$, $P_0 = (a,0,0)$, $P_1 = (a,a,b)$, $P_2 = (0,\frac{3}{2}a,\frac{3}{2}b)$, $P_3 = (-a,a,\frac{7}{3}b)$, $P_4 = (-a,0,\frac{8}{3}b)$ are given, then the curve defined by (2) is the approximate expression of circular helix.

Proof. Substituting P_0, P_1, P_2, P_3, P_4 into Eq. (2), then

$$B(t) = (a\cos 2t, a\sin 2t, \frac{8}{3}b\sin t - \frac{1}{3}b\sin 2t)$$

Clearly, the curve $B(t)$ must be on the cylinder $x^2 + y^2 = a^2$.

And we have

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots (-1)^{n+1}\frac{t^{2n-1}}{(2n-1)!} + \cdots, \quad \sin 2t = 2t - \frac{8t^3}{3!} + \frac{32t^5}{5!} - \cdots (-1)^{n+1}\frac{(2t)^{2n-1}}{(2n-1)!} + \cdots,$$

then we can get

$$\frac{8}{3}\sin t - \frac{1}{3}\sin 2t = 2t + O(t^5)$$

Control the error in the given $\varepsilon(t)$ by controlling the scope of t, then $B(t)$ can be transformed into:

$$\begin{cases} x = a\cos 2t \\ y = a\sin 2t \\ z \approx 2bt \end{cases}$$

This is an approximate equation of helix, and this was to be proved.

When $t \in [0, \frac{\pi}{4}]$, $a = b = 1$, a segment of curves which approximate expression of helix is presented in Fig. 4 (the red lines).
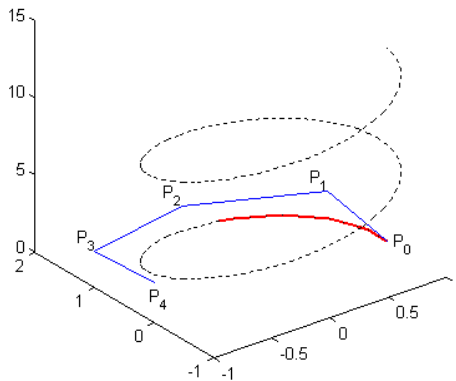

Fig.4 Approximation of Helix

4.3. Some Modeling examples of surfaces

Because the quasi-quartic trigonometric Bézier curves can represent precisely straight line segment, circular arc, elliptic arc, parabola, cardioid; therefore the corresponding tensor product quasi-biquartic trigonometric Bézier surfaces can represent some quadratic surfaces such as cylindrical surfaces, sphere, ellipsoid, parabolic surfaces, torus. Furthermore, some complex surfaces can be constructed by these basic surfaces exactly. While the method of traditional quartic Bézier curves needs joining with many patches of surface in order to satisfy the precision of users for designing. Therefore the method presented by this paper can raise the efficient of constituting surfaces and precision of representation in a large extent.
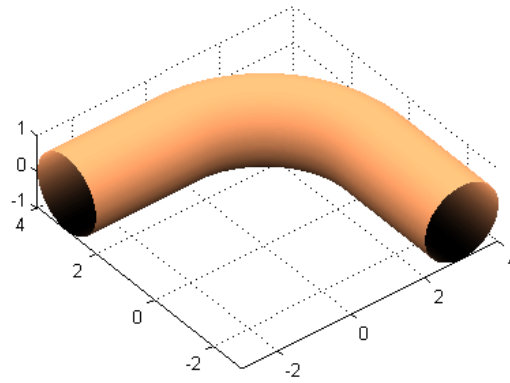
Fig.5 A right angle pipe modeled by twelve pieces of C^1 continuous quasi-biquartic trigonometric Bézier surfaces
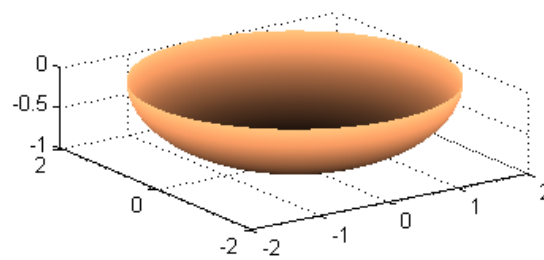

Fig.6 A bowl-shaped surfaces modeled by four pieces of C^4 continuous quasi-biquartic trigonometric Bézier surfaces
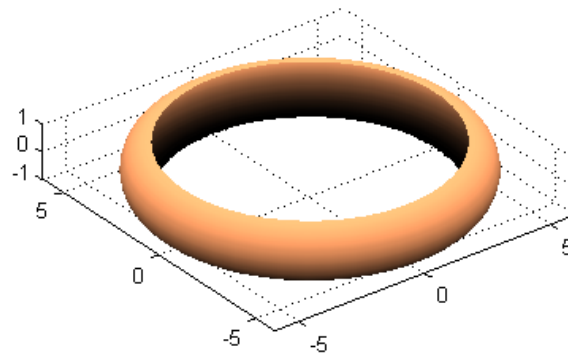

Fig.7 A tread surface modeled by four pieces of C^4 continuous quasi-biquartic trigonometric Bézier surfaces

Some modeling examples of by tensor product quasi-biquartic trigonometric Bézier surfaces are given as follows: Fig.5 shows the precise reconstruction of right angle pipe with twelve pieces of C^1 continuous quasi-biquartic trigonometric Bézier surfaces. Fig.6 denotes a bowl-shaped surface by four pieces of C^4 continuous quasi-biquartic trigonometric Bézier surfaces. Fig.7 depicts a tread surface by four pieces of C^4 continuous quasi-biquartic trigonometric Bézier surfaces.

5. Relationship between quasi-quartic trigonometric Bézier curves and quartic Bézier curves

Suppose $\boldsymbol{Q}_i$, $\boldsymbol{Q}_{i+1}$, $\boldsymbol{Q}_{i+2}$ and $\boldsymbol{Q}_{i+3}$ are not collinear, considering the symmetry of basis functions of quasi-quartic trigonometric Bézier curves, to obtain the relationships between quasi-quartic trigonometric Bézier curves segment $\boldsymbol{P}(t)$ and quartic Bézier curves segment $\boldsymbol{B}(t)$, we can set

$$\begin{cases} \boldsymbol{P}(\frac{\pi}{4}) - a\boldsymbol{Q}_i - b\boldsymbol{Q}_{i+1} - c\boldsymbol{Q}_{i+2} - b\boldsymbol{Q}_{i+3} - a\boldsymbol{Q}_{i+4} = (x + y\lambda)(d\boldsymbol{Q}_i + e\boldsymbol{Q}_{i+1} + f\boldsymbol{Q}_{i+2} + e\boldsymbol{Q}_{i+3} + d\boldsymbol{Q}_{i+4}) \\ \boldsymbol{B}(\frac{1}{2}) - a\boldsymbol{Q}_i - b\boldsymbol{Q}_{i+1} - c\boldsymbol{Q}_{i+2} - b\boldsymbol{Q}_{i+3} - a\boldsymbol{Q}_{i+4} = z(d\boldsymbol{Q}_i + e\boldsymbol{Q}_{i+1} + f\boldsymbol{Q}_{i+2} + e\boldsymbol{Q}_{i+3} + d\boldsymbol{Q}_{i+4}) \end{cases} \tag{13}$$

For $\boldsymbol{Q}_i$, from Eq. (2) and (13), we can get

$$(1+\frac{\lambda}{2})-\frac{\sqrt{2}}{2}(1+\lambda)-a=(x+y\lambda)d \quad , \quad \frac{1}{16}-a=zd$$

For $\boldsymbol{Q}_{i+1}$, from Eq. (2) and (13), we can get

$$(1+\lambda)(\frac{3\sqrt{2}}{2}-2)-b=(x+y\lambda)e \quad , \quad \frac{1}{4}-b=ze$$

For $\boldsymbol{Q}_{i+2}$, from Eq. (2) and (13), we can get

$$2(1+\lambda)(\frac{3}{2}-\sqrt{2})-c=(x+y\lambda)f \quad , \quad \frac{3}{8}-c=zf$$

We can get a group of approximate solution of the above system of equations by solving non-linear system using MATLAB2010b is

$$a = 0.7071 , \quad b = \text{-}0.1213 , \quad c = \text{-}0.1716 ,$$
$$d = \text{-}0.1953 , \quad e = \text{-}0.1144 , \quad f = 0.1618$$
$$x = 2.1212 , \quad y = 1.0606$$

Let $\boldsymbol{P}(\frac{\pi}{4}) = \boldsymbol{B}(\frac{1}{2})$, Substituting it into Eq. (13), one can get

$$\lambda = 1.1124$$

So, when $\lambda = 1.1124$, the quasi-quartic trigonometric Bézier curves segment $\boldsymbol{P}(t)$ is most close to the quartic Bézier curves segment $\boldsymbol{B}(t)$, when $1.1124 < \lambda \leq 1.5$, the larger is λ, and the more approach is the quasi-quartic trigonometric Bézier curves segment $\boldsymbol{P}(t)$ to the control polygon.

6. Conclusion

In this paper, we proposed a class of quasi-quartic trigonometric Bézier curves and surfaces which inherit most of properties similar to those of quartic Bézier curves and surfaces. With the shape parameter chosen properly, the curves and surfaces can express exactly some quadratic curves and surfaces without using rational form. The jointing of two pieces of curves can reach G^2 and C^4 continuity under the appropriate conditions. Finally, the relationship between quasi-quartic trigonometric Bézier curves and quartic Bézier curves is discussed.

7. Acknowledgement

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