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A Second Kind Chebyshev Polynomial Approach for the Wave Equation Subject to an Integral Conservation Condition

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Abstract. The main purpose of this article is to present an approximate solution for the one dimensional wave equation subject to an integral conservation condition in terms of second kind Chebyshev polynomials. The operational matrices of integration and derivation are introduced and utilized to reduce the wave equation and the conditions into the matrix equations which correspond to a system of linear algebraic equations with unknown Chebyshev coefficients. Finally, some examples are presented to illustrate the applicability of the method.

Keywords: Wave equation, Non-local condition, Second kind Chebyshev polynomials, Operational matrix, Matrix form.

1. Introduction

Hyperbolic partial differential equations with an integral condition serve as models in many branches of physics and technology. There are many papers that deal with the numerical solution of the parabolic equation with integral conditions [1, 4, 5, 6, 7, 11, 13]. The present work focuses on the one-dimensional wave equation with the non-local boundary condition.

In this paper, we consider the following one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = F(x,t), \quad 0 \le x \le \ell,\ 0 < t \le T. \tag{1}$$

with initial conditions

$$u(x,0) = f_1(x),\ 0 \le x \le \ell, \tag{2}$$

and

$$u_t(x,0) = f_2(x),\ 0 \le x \le \ell, \tag{3}$$

and Dirichlet boundary condition

$$u(0,t) = g_1(t),\ 0 < t \le T, \tag{4}$$

and the non-local condition

$$\int_0^\ell u(x,t)dx = g_2(t),\ 0 < t \le T, \tag{5}$$

where F, f_1, f_2, g_1 and g_2 are known functions.

The existence and uniqueness of the solution of the problem (1)--(5) are discussed in [3]. Dehghan [8] presented finite difference schemes for the numerical solution of problem (1)--(5). In [15] the shifted Legendre Tau technique was developed for the solution of the studied model. Author of [2] developed a numerical technique based on an integro- differential equation and local interpolating functions for solving the one-dimensional wave equation subject to a non-local conservation condition and suitably prescribed initial-boundary conditions. Authors of [14] combined finite difference and spectral methods to solve the one-dimensional wave equation with an integral condition. In [9] variational iteration method was applied for

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solving the wave equation subject to an integral conservation condition. Authors of [10] presented a meshless method for numerical solution of problem (1)--(5). Also in [16] the method of lines was developed for the solution of the discussed problem.

Orthogonal functions have been used to solve various problems. The main characteristic of this technique is that it reduces problem to those of solving a system of algebraic equations thus greatly simplifying the problem. In the present paper, the numerical solution of the problem (1)--(5) is computed by using two variable shifted second kind Chebyshev orthogonal functions.

The paper is organized as follows: In Section 2, basic properties of the second kind Chebyshev polynomials are presented. In Section 3, we discuss how to approximate functions in terms of second kind Chebyshev polynomials and introduce operational matrices of integration and derivation. In section 4, we give an approximate solution for (1)--(5). Numerical examples are given in Section 5 to illustrate the accuracy of our method. Finally, concluding remarks are given in Section 6.

2. Properties of the Second Kind Chebyshev Polynomials

Second kind Chebyshev polynomials are total orthogonal basis for $L^2[-1,1]$ and can be determined with the aid of the following recursive formula [12].

$$U_0(t) = 1,$$
$$U_1(t) = 2t,$$
$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t), n \geq 2, -1 \leq t \leq 1.$$

For the case that the interval is not $[-1,1]$, say $[a,b]$, we can use the linear transformation

$$t' = \frac{2t-a-b}{b-a},$$

to transform the domain into $[-1,1]$.

The orthogonality property is as follows:

$$\int_{-1}^{1} \omega(t)U_i(t)U_j(t)dt = \begin{cases} \frac{\pi}{2}, & i = j, \\ 0, & i \neq j, \end{cases}$$

where $\omega(t) = \sqrt{1-t^2}$ is the weight function.

Some properties of the second kind Chebyshev polynomials are as follows:

$$\int_{-1}^{1} U_n(t)dt = \begin{cases} \frac{2}{n+1}, & n = 2k, \\ 0, & n = 2k+1, \end{cases} \tag{6}$$

$$\int_{-1}^{t} U_n(t')dt' = \frac{1}{n+1}[(-1)^n U_0(t) - \frac{1}{2}U_{n-1}(t) + \frac{1}{2}U_{n+1}(t)],\ U_{-1}(t) = 0, \tag{7}$$

$$U'_n(t) = 2\sum_{i=0}^{[n/2]}(n-2i)U_{n-2i-1}(t), \tag{8}$$

$$U_n(-1) = (-1)^n(n+1). \tag{9}$$

3. Function Approximation

3.1. Approximation of one-variable and two-variable functions

A function $y(t)$ defined over $[0,T]$ may be expanded by the shifted second kind Chebyshev functions as

$$y(t) = \sum_{j=0}^{\infty} a_j U_j(\frac{2}{T}t - 1) = \sum_{j=0}^{\infty} a_j \varphi_j(t), \tag{10}$$

where

$$\varphi_j(t)=U_j(\frac{2}{T}t-1),$$

$$a_j=\frac{4}{\pi T}\int_0^T \omega(t)y(t)\varphi_j(t)dt, \tag{11}$$

and

$$\omega(t)=\sqrt{1-(\frac{2}{T}t-1)^2}.$$

If the infinite series in (10) is truncated, then (10) can be written as

$$y(t)\cong y_n(t)=\sum_{j=0}^{n}a_j\varphi_j(t)=A^T\varphi(t),$$

where

$$A=[a_0,a_1,\dots,a_n]^T,$$

$$\varphi(t)=[\varphi_0(t),\varphi_1(t),\dots,\varphi_n(t)]^T. \tag{12}$$

Similarly, a function $h(x,t)$ defined over $[0,\ell]\times[0,T]$ may be expanded by the shifted second kind Chebyshev functions as

$$h(x,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}c_{ij}U_i(\frac{2}{\ell}x-1)U_j(\frac{2}{T}t-1)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}c_{ij}\psi_{ij}(x,t), \tag{13}$$

so that

$$\psi_{ij}(x,t)=U_i(\frac{2}{\ell}x-1)U_j(\frac{2}{T}t-1),$$

$$c_{ij}=\frac{16}{\pi^2T\ell}\int_0^T\int_0^\ell \omega(x,t)h(x,t)\psi_{ij}(x,t)dxdt, \tag{14}$$

and

$$\omega(x,t)=\sqrt{1-(\frac{2}{T}t-1)^2}\sqrt{1-(\frac{2}{\ell}x-1)^2}.$$

If the infinite series in (13) is truncated, then it can be written as:

$$h(x,t)\cong h_{m,n}(x,t)=\sum_{i=0}^{m}\sum_{j=0}^{n}c_{ij}\psi_{ij}(x,t)=C^T\Psi(x,t),$$

such that

$$C=[c_{00},c_{01},\dots,c_{0n},c_{10},c_{11},\dots,c_{1n},\dots,c_{m0},c_{m1},\dots,c_{mn}]^T,$$

$$\Psi(x,t)=[\psi_{00}(x,t),\psi_{01}(x,t),\dots,\psi_{0n}(x,t),\dots,\psi_{m0}(x,t),\psi_{m1}(x,t),\dots,\psi_{mn}(x,t)]^T. \tag{15}$$

In order to calculate the integral part of (11) and (14) we transform the intervals $[0,\ell]$ and $[0,T]$ into the interval $[-1,1]$ by means of the transformations

$$x'=\frac{2}{\ell}x-1,\quad t'=\frac{2}{T}t-1,$$

and then use the second kind Gauss-Chebyshev quadrature formula [12].

3.2. Operational matrix of integration

Using equation (7) the integration of the vector $\Psi(x,t)$ defined in equation (15) in direction t can be approximated by

$$\int_0^t\Psi(x,t')dt'\cong P\Psi(x,t), \tag{16}$$

such that P is an $(m+1)(n+1)\times(m+1)(n+1)$ matrix as follows:

$$P=\frac{T}{2}\begin{bmatrix} P_0 & O & O & \dots & O \\ O & P_0 & O & \dots & O \\ O & O & P_0 & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \dots & P_0 \end{bmatrix},$$

where O is zero matrix with dimension $n+1$ and

$$P_0=\begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{-3}{4} & 0 & \frac{1}{4} & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{3} & \frac{-1}{6} & 0 & \frac{1}{6} & 0 & \dots & 0 & 0 \\ \frac{-1}{4} & 0 & \frac{-1}{8} & 0 & \frac{1}{8} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{(-1)^n}{n+1} & 0 & 0 & 0 & 0 & \dots & \frac{-1}{2(n+1)} & 0 \end{bmatrix}.$$

3.3. Operational matrix of derivation

By the definition of $\psi_{ij}(x,t)$ we have

$$\frac{\partial\psi_{ij}(x,t)}{\partial x}=(\frac{d}{dx}U_i(\frac{2}{l}x-1))U_j(\frac{2}{T}t-1)=\frac{2}{l}U_i'(\frac{2}{l}x-1)U_j(\frac{2}{T}t-1),$$

and using equation (8) we get

$$\frac{\partial\psi_{ij}(x,t)}{\partial x}=\frac{4}{l}\sum_{k=0}^{[i/2]}(i-2k)\psi_{(i-2k-1)j}(x,t),$$

therefore we obtain

$$\frac{\partial\Psi(x,t)}{\partial x}=M\Psi(x,t),$$

where M is an $(m+1)(n+1)\times(m+1)(n+1)$ matrix as

$$M=\frac{4}{\ell}\begin{bmatrix} O & O & O & O & \dots & O & O \\ I & O & O & O & \dots & O & O \\ O & 2I & O & O & \dots & O & O \\ I & O & 3I & O & \dots & O & O \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ M_1 & M_2 & M_3 & M_4 & \dots & mI & O \end{bmatrix},$$

So that, M_1, M_2, M_3 and M_4 are respectively I, O, $3I$ and O, for odd m and O, $2I$, O and $4I$, for even m. I and O are identity and zero matrix with dimension $n+1$ respectively.

Similarly we have

$$\frac{\partial^2\Psi(x,t)}{\partial x^2}(x,t)=M^2\Psi(x,t). \tag{17}$$

4. Numerical Solution of the Wave Equation

4.1. Matrix form of the wave equation

We approximate $u(x,t)$ and $F(x,t)$ in equation (1) respectively as

$$u(x,t) \cong A^T\Psi(x,t), \tag{18}$$

$$F(x,t) \cong F^T\Psi(x,t), \tag{19}$$

such that

$$A = [\,a_{00}, a_{01}, \ldots, a_{0n}, a_{10}, a_{11}, \ldots, a_{1n}, \ldots, a_{m0}, a_{m1}, \ldots, a_{mn}\,]^T,$$

where a_{ij}, $i = 0,1,\cdots,m$, $j = 0,1,\cdots,n$ are unknown second kind Chebyshev coefficients and $\Psi(x,t)$ is as (15) and m and n are chosen positive integers.

Twice integrating equation (1) from 0 to t and using equations (2), (3), (16)--(19) we have

$$A^T\Psi(x,t) - f_1(x) - tf_2(x) - A^T M^2 P^2 \Psi(x,t) = F^T P^2 \Psi(x,t). \tag{20}$$

Suppose that

$$F_1(x,t) = f_1(x),$$
$$F_2(x,t) = tf_2(x),$$

then F_1 and F_2 can be approximated as:

$$F_1(x,t) \cong F_1^T\Psi(x,t), \tag{21}$$

$$F_2(x,t) \cong F_2^T\Psi(x,t). \tag{22}$$

Substituting equations (21) and (22) into equation (20) we get

$$WA = V, \tag{23}$$

so that W is an $(m+1)(n+1)\times(m+1)(n+1)$ matrix and V is an $(m+1)(n+1)\times 1$ vector as follows:

$$W = I - (M^2P^2)^T,$$

$$V = P^{2^T}F + F_1 + F_2,$$

and I is identity matrix of dimension $(m+1)(n+1)$.

4.2. Matrix form of the Direchlet boundary condition

The Dirichlet boundary condition (4) can be written as

$$A^T\Psi(0,t) = g_1(t). \tag{24}$$

Using equation (9) we obtain

$$\psi_{ij}(0,t) = U_i(-1)U_j(\frac{2}{T}t - 1) = (-1)^i(i+1)\varphi_j(t),$$

therefore

$$A^T\Psi(0,t) = A^T W_1^T \varphi(t), \tag{25}$$

where $\varphi(t)$ is as (12) and W_1 is an $(n+1)\times(m+1)(n+1)$ matrix as

$$W_1 = \begin{bmatrix} I & -2I & 3I & \ldots & (-1)^m(m+1)I \end{bmatrix}.$$

and I is identity matrix with dimension $n+1$.

We approximate $g_1(t)$ as

$$g_1(t) = G_1^T\varphi(t). \tag{26}$$

Now, using equations (24)--(26), the matrix representation of the boundary condition can be written as

$$W_1 A = G_1. \tag{27}$$

4.3. Matrix form of the non-local condition

By using equation (18) the non-local condition (5) can be written as

$$A^T \int_0^{\ell} \Psi(x,t)dx = g_2(t). \quad (28)$$

Using equation (6) we get

$$\begin{aligned}\int_0^{\ell} U_i(\frac{2}{\ell}x-1)U_j(\frac{2}{T}t-1)dx &= (\int_0^{\ell} U_i(\frac{2}{\ell}x-1)dx)\varphi_j(t)\\ &= \frac{\ell}{2}(\int_{-1}^{1} U_i(x')dx')\varphi_j(t)\\ &= \begin{cases}\frac{\ell}{i+1}\varphi_j(t), & i=2k,\\ 0, & i=2k+1.\end{cases}\end{aligned}$$

Therefore

$$A^T \int_0^{\ell} \Psi(x,t)dx = A^T W_2^{\,T} \varphi(t), \quad (29)$$

such that W_2 is an $(n+1)\times(m+1)(n+1)$ matrix as

$$W_2 = \begin{bmatrix} B_0 & B_1 & B_2 & \dots & B_m \end{bmatrix},$$

and B_i, $(i=0,1,\dots,m)$ are $(n+1)\times(n+1)$ matrices as

$$B_i = \begin{cases}\frac{\ell}{i+1}I, & i=2k,\\ O, & i=2k+1.\end{cases}$$

We approximate $g_2(t)$ as

$$g_2(t) = G_2^{\,T}\varphi(t). \quad (30)$$

Using equations (28)--(30), we obtain the matrix representation of the non-local condition as

$$W_2 A = G_2. \quad (31)$$

4.4. Method of solution

To obtain the solution of equation (1) under the conditions (2)--(5), we replace $2(n+1)$ rows of the augmented matrix $[W;V]$ with the rows of the augmented matrices $[W_1;G_1]$ and $[W_2;G_2]$. In this way, the second kind shifted Chebyshev polynomials coefficients are determined by solving the new linear algebraic system.

5. Illustrative Examples

In this section, two examples are given to demonstrate the applicability and accuracy of our method. In order to demonstrate the error of the method we introduce the notation:

$$e_{m,n}(x,t) = |u(x,t) - u_{m,n}(x,t)|,$$

where $u(x,t)$ is the exact solution and $u_{m,n}(x,t)$ is the computed result with m and n.

Example 5.1. Consider (1)--(5) with $l=1$, $T=1$ and [10]

$$F(x,t) = (\frac{1}{4}+\pi^2)\exp(\frac{-t}{2}\sin(\pi x), 0<x<1, 0<t<1,$$

$$f_1(x) = \sin(\pi x), f_2(x) = \frac{-1}{2}\sin(\pi x), 0<x<1,$$

$$g_1(t) = 0, g_2(t) = (\frac{2}{\pi})\exp(\frac{-t}{2}), 0<t<1,$$

for which the exact solution of this problem is $u(x,t) = \exp(\frac{-t}{2})\sin(\pi x)$.

We applied the presented method in this paper for this problem with different values of m and n. In Table 1 we give the error $\|u-u_{m,n}\|_2$, for $n=5$ and $m=2,4,6,8,10$. From Table 1 we see that for fixed

n as m is increased the error decreases. Also the error function $e_{m,n}(x,1)$ for different values of m and n is shown in Fig. 1.

Table 1. Values of $\|u-u_{m,n}\|_2$ for $n=5$ and some values of m for Example 5.1.

m	2	4	6	8
$\|u-u_{m,n}\|_2$	4.463×10^{-2}	7.39×10^{-4}	6.649×10^{-6}	5.338×10^{-8}

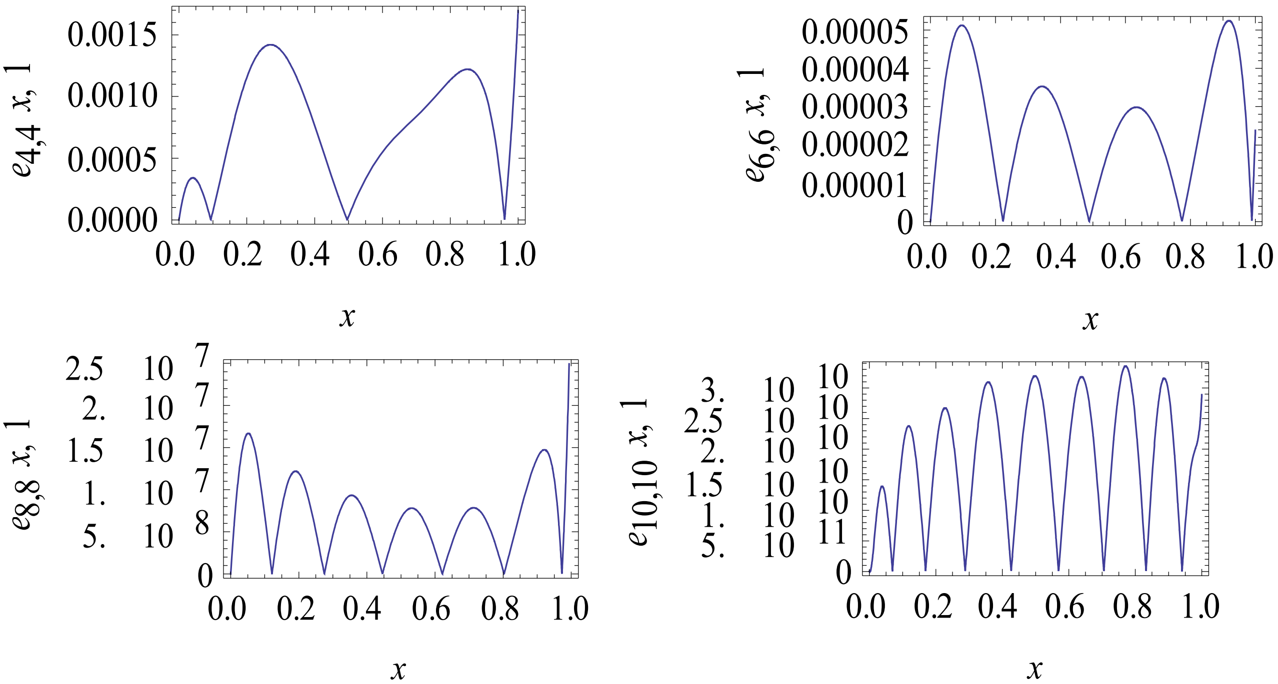

Fig. 1: Graph of the $e_{m,n}(x,1)$ with $m=n=4,6,8,10$ for Example 5.1.

Example 5.2. In this example, consider (1)–(5) with $l=1$, $T=0.25$ and [9]

$$F(x,t)=0, 0<x<1, 0<t<0.25,$$

$$f_1(x)=\cos(\pi x), f_2(x)=0, 0<x<1,$$

$$g_1(t)=\cos(\pi t), g_2(t)=0, 0<t<0.25.$$

The exact solution of this problem is $u(x,t)=\cos(\pi x)\cos(\pi t)$. We applied the presented method in this paper with $m=n$ and show some numerical results in Table 2 and Fig. 2. From Table 2 we see that the approximate solution computed by different values of m and n converges to the exact solution.

Table 2. Values of $\|u-u_{m,n}\|_2$ with $m=n$ for Example 5.2.

$m=n$	2	4	6	8
$\|u-u_{m,n}\|_2$	5.479×10^{-2}	2.062×10^{-3}	4.377×10^{-5}	3.302×10^{-7}

6. Conclusion

In this article we presented a numerical scheme for solving the second-order wave equation subject to an integral condition. Properties of the second kind Chebyshev polynomials were employed. The matrices and have many zeroes, hence making second kind Chebyshev functions computationally very attractive. Chebyshev coefficients of the solution are found very easily by using the computer programs without any computational effort and this process is very fast. The new described method doesn't need any collocation point and produces very accuracy results.

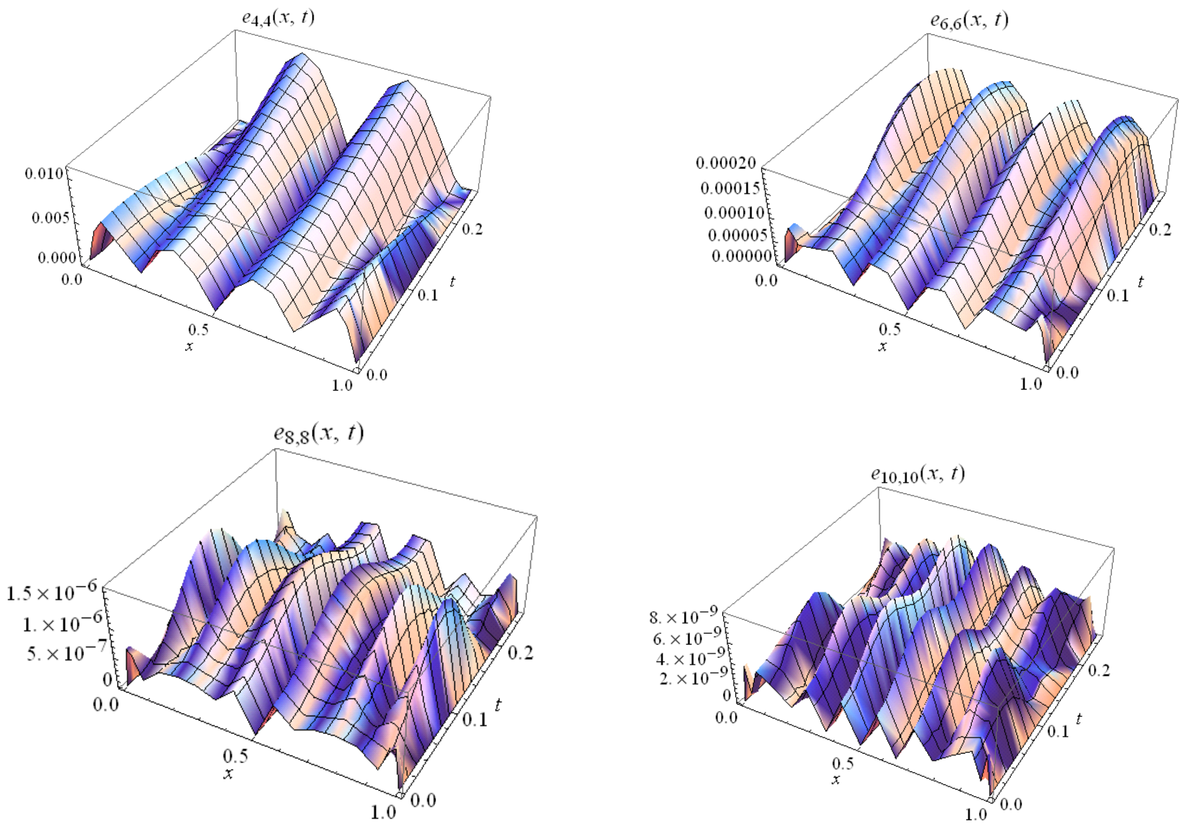

Fig. 2: Graph of the $e_{m,n}(x,t)$ with $m = n = 4,6,8,10$ for Example 5.2.

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