A collocation method for the solution of convection–diffusion parabolic problems

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Abstract. In this paper , we develop a collocation method based on cubic B-spline to the solution of singularly perturbed parabolic equation $\varepsilon^2 \frac{\partial^2 u}{\partial x^2} - c(x,t)u - p(x,t)\frac{\partial u}{\partial t} = f(x,t)$, subject to appropriate initial , and Dirichlet boundary conditions , where ε^2, is a small constant. We develop a new two-level three-point scheme of order $o\left(k^2 + h^2\right)$ .The convergence analysis of the method is proved. Numerical results are given to illustrate the efficiency of our method computationally.

Keywords: B-Spline, parabolic equation, convergence, singularly perturbed..

1. Introduction

In this paper , we consider singularly perturbed parabolic equation :

$$\varepsilon^2 \frac{\partial^2 u}{\partial x^2} - c(x,t)u - p(x,t)\frac{\partial u}{\partial t} = f(x,t) \qquad , \qquad (1)$$

on the domain D , where $D = (0,1) \times (0,T]$,
with initial condition

$$u(x,0) = u_0(x) \qquad , \qquad x \in (0,1) \qquad , \qquad (2)$$

and boundary donditions

$$u(0,t) = g_0(t) \qquad , \qquad u(1,t) = g_1(t) \qquad , \qquad t \in (0,T] \quad , \qquad (3)$$

We assume that
1. functions c(x,t) , p(x,t) and f(x,t) are sufficiently smooth on the D ,and
$$c(x,t) \geq 0 \quad , \quad p(x,t) \geq p_0 > 0 \quad ,(x,t) \in D$$
2. functions $g_0(t)$ and $g_1(t)$ are sufficiently smooth on the $[0,T]$ and $u_o(x)$ is smooth on $[0,1]$,
3. $\varepsilon \in (0,1]$,and
4. Compatibility conditions are satisfied at the corner points $(0,0)$ and $(1,0)$.

We suppose that $\varepsilon' \equiv \varepsilon^2$, $\kappa(x,t,u,u_x) = c(x,t)u + f(x,t) \Rightarrow \varepsilon' u_{xx} = p(x,t)u_t + \kappa(x,t,u,u_x)$

In this paper we have developed two-level implicit difference scheme by using cubic B-spline function for the solution of singularly perturbed parabolic problem (1). This paper is arranged as follows. In section2, we present a finite difference approximation to discretize the equation (1) and obtain the convergence of method in time variable. In section3 we applied cubic B-spline collocation method to solve the ordinary differential equations obtained in each time level.The uniform convergence of the method is proved in section4.In section5, numerical experiments are conducted to demonstrate the viability and the efficiency of the proposed method computationally.

2. Temporal discretization

Let us consider auniform mesh Δ with the grid points $\lambda_{i,j}$ to discretize the region $\Omega = (0,1) \times (0,T]$. Each $\lambda_{i,j}$ is the vertices of the grid point (x_i, t_j) where $x_i = ih, i = 0,1,2,...,N$ and $t_j = jk, j = 0,1,2,...$ and h and k are mesh sizes in the space and time directions respectively.

First we use the following finite difference approximation to discretize the time variable with uniform step size k,

$$u_t^n \cong \frac{\delta_t}{k(1-\gamma\delta_t)}u^n \quad , \quad n \geq 0 \qquad , \quad \gamma \neq 1 \quad , \tag{4}$$

where n is the step number in t direction and $\delta_t u^n = u^n - u^{n-1}, u^n = u(x,t_n)$ and $u^0 = u(x,0) = u_0(x)$, $(0 < x < 1)$.

Substituting the above approximation into equation (1) and discretizing in time variable we have:

$$\varepsilon' u_{xx}^n = p(x,t_n)\frac{\delta_t}{k(1-\gamma\delta_t)}u^n + \kappa(x,t_n,u^n,u_x^n) \quad , \tag{5}$$

thus we have,

$$\varepsilon' k(1-\gamma\delta_t)u_{xx}^n = p(x,t_n)\delta_t u^n + k(1-\gamma\delta_t)\kappa(x,t_n,u^n,u_x^n) \quad , \tag{6}$$

now by simplifying we can write equation (6) in the following form

$$-\varepsilon' u_{xx}^* + p^*(x)u^* + \kappa^*(x,u^*,u_x^*) = q^*(x) \quad , \tag{7}$$

where $u^* \equiv u^n$ and ,

$$\begin{cases} u_{xx}^* \equiv u_{xx}^n \quad , \\[2mm] p^*(x) = \dfrac{p(x,t_n)}{k(1-\gamma)} \quad , \\[4mm] q^*(x) = \dfrac{p(x,t_n)}{k(1-\gamma)}u^{n-1} + \dfrac{\varepsilon'\gamma}{1-\gamma}u_{xx}^{n-1} - \dfrac{\gamma}{1-\gamma}\kappa(x,t_n,u^{n-1},u_x^{n-1}) \quad , \\[4mm] \kappa^*(x,u^*,u_x^*) = \kappa(x,t_n,u^n,u_x^n) \quad . \end{cases}$$

with boundary conditions,

$$u^*(0) = g_0(t_n) \quad , u^*(1) = g_1(t_n) \quad . \tag{8}$$

Thus now in each time level we have a nonlinear ordinary differential equation in the form of (7) with the boundary conditions (8) which can be solved by using B-spline collocation method.

Theorem 2.1:
The above time discretization process that we use to discretize equation (1) in time variable is of the second order convergence.
Proof:
Let $u(t_i)$ be the exact solution and u^i the approximate solution of the problem (1) at the ith level time and also suppose that $e_i = u^i - u(t_i)$ be the local truncation error in (7). Then using equation (4) and replacing $\gamma = \frac{1}{2}$ it can be easily proved that,

$$|e_i| \leq c_i k^3 \quad , \tag{9}$$

where c_i is some finite constant.

Let E_{n+1} be the global error in time discretizing process then the global error in $(n+1)th$ level is

$$E_{n+1} = \sum_{i=1}^{n} e_i \qquad , \left(t \leq \frac{T}{n}\right)$$

thus with the help of (9) we have:

$$\left|E_{n+1}\right| = \left|\sum_{i=1}^{n} e_i\right| \le \sum_{i=1}^{n}\left|e_i\right| \le \sum_{i=1}^{n} c_i k^3 \le nC'k^3 \le C'n(\frac{T}{n})k^2 = Ck^2 \quad ,$$

where $C' = \max\,(c_i), 1 \le i \le n$, and $C = C'T$, which gives the second order convergent of the method in time direction easily.

3. B-spline collocation method

In this section we use B-spline collocation method to solve (7) with the boundary conditions (8) in each time level. Let $\Delta^* = \{a = x_0 < x_1 < ... < x_N = b\}$ be the partition in $[a,b]$. B-splines are the unique nonzero splines of smallest compact support with nodes at $x_0 < x_1 < ... < x_{N-1} < x_N$. We define the cubic B-splines for $i = -1,0,...,N+1$ by the following relation [2].

$$B_{3,i} = \frac{1}{h^3}\begin{cases}(x - x_{i-2})^3 & ,x \in [x_{i-2}, x_{i-1}) \quad, \\ h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 - 3(x - x_{i-1})^3 & ,x \in [x_{i-1}, x_i) \quad, \\ h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3 & ,x \in [x_i, x_{i+1}) \quad, \\ (x_{i+2} - x)^3 & ,x \in [x_{i+1}, x_{i+2}] \quad, \\ 0 & , otherwise \quad.\end{cases} \tag{10}$$

It can be easily seen that the set of functions $\Omega = \{B_{-1}(x), B_0(x),..., B_{N+1}(x)\}$ are linearly independent on $[a,b]$ thus $\Theta = Span(\Omega)$ is a subspace of $C^2[a,b]$, and Θ has $N+3$ dimension. Let us consider that $S^*(x) \in \Theta$ be the B-spline approximation to the exact solution of problem (7), thus we can write $S^*(x)$ in the following form:

$$S^*(x) = \sum_{i=-1}^{N+1} c_i^* B_i(x) \quad, \tag{11}$$

where c_i^* are unknown real coefficients and $B_i(x)$ are cubic B-spline functions.

By forcing $S^*(x)$ to satisfy the collocation equations plus the boundary conditions, we have:

$$LS^*(x_i) = q^*(x_i) \quad, \qquad 0 \le i \le N \quad, \tag{12}$$
$$S^*(x_0) = g_0(t_n) \quad, \qquad S^*(x_N) = g_1(t_n)$$

where $Lu^* \equiv -\varepsilon'u_{xx}^* + p^*(x)u^* + \kappa^*(x,u^*,u_x^*)$.

Substituting (11) into (12) and using the properties of B-spline functions we have:

$$-\frac{6\varepsilon'}{h^2}(c_{i-1}^* - 2c_i^* + c_{i+1}^*) + p^*(x_i)(c_{i-1}^* + 4c_i^* + c_{i+1}^*) + \kappa^*(x_i,(c_{i-1}^* + 4c_i^* + c_{i+1}^*),\frac{3}{h}(c_{i+1}^* - c_{i-1}^*))$$
$$= q^*(x_i) \quad, \qquad\qquad\qquad\qquad 0 \le i \le N \quad,$$

Simplifying the above relation leads to the following system of $(N-1)$ non-linear equations in $(N+3)$ unknowns:

$$(-6\varepsilon' + h^2 p_i^*)c_{i-1}^* + (12\varepsilon' + 4h^2 p_i^*)c_i^* + (-6\varepsilon' + h^2 p_i^*)c_{i+1}^* + h^2\kappa_i^*$$
$$= h^2 q_i^* \quad, \qquad\qquad\qquad\qquad 1 \le i \le N-1 \quad, \tag{13}$$

where $p_i^* = p^*(x_i)$, $q_i^* = q^*(x_i)$ and $\kappa_i^* = \kappa^*(x_i,(c_{i-1}^* + 4c_i^* + c_{i+1}^*),\frac{3}{h}(c_{i+1}^* - c_{i-1}^*))$.

To be able to solving the system, and obtaining a unique solution to $C^* = \left(c_{-1}^*, c_0^*, c_1^*,..., c_N^*, c_{N+1}^*\right)$ we need to obtain four extra equations to be associated with (13). This can be done by using the boundary conditions and then eliminating c_{-1}^* and c_{N+1}^*.

Using the left boundary condition we have,

$$u(0,t_n) = S^*(0) = g_0(t_n) = c_{-1}^* + 4c_0^* + c_1^* \ ,$$

Now eliminating c_{-1}^* from the above equation and substituting into equation (13) for $i = 0$ we have,

$$(-6\varepsilon' + h^2 p_0^*)(g_0(t_n) - 4c_0^* - c_1^*) + (12\varepsilon' + 4h^2 p_0^*)c_0^* + (-6\varepsilon' + h^2 p_0^*)c_1^* + h^2\kappa_0^* = h^2 q_0^* \ .$$

Simplifying the above equation we obtain the following relation.

$$36\varepsilon' c_0^* + h^2\kappa_0^* = h^2 q_0^* + (6\varepsilon' - h^2 p_o^*)g_0(t_n) \quad , \quad i = 0 \tag{14}$$

where

$$\kappa_0^* = \kappa^*(x_0, g_o(t_n), \frac{3}{h}(c_1^* - c_{-1}^*)) = \kappa^*(x_0, g_0(t_n), \frac{3}{h}(4c_0^* + 2c_1^* - g_0(t_n))).$$

Similarly by using the right boundary condition we have,

$$u(1,t_n) = S^*(1) = g_1(t_n) = c_{N-1}^* + 4c_N^* + c_{N+1}^* \quad ,$$

and eliminating c_{N+1}^* from the last equation (13) for $i = N$,we find

$$36\varepsilon' c_N^* + h^2\kappa_N^* = h^2 q_N^* + (6\varepsilon' - h^2 p_N^*)g_1(t_n) \qquad , i = N \ , \tag{15}$$

where

$$\kappa_N^* = \kappa^*(x_N, u(x_N), u_x(x_N)) = \kappa^*(x_N, g_1(t_n), \frac{3}{h}(c_{N+1}^* - c_{N-1}^*)) =$$

$$\kappa^*(x_N, g_1(t_n), -\frac{3}{h}(4c_N^* + 2c_{N-1}^* - g_1(t_n))).$$

Equations (14) , (15) can be associated with (13) which gives the following $(N+1)$ in $(N+1)$ non- linear system ,

$$AC^* + h^2 B^* = h^2 Q^* \tag{16}$$

where ,

$$A = \begin{pmatrix} 36\varepsilon' & 0 & 0 \\ -6\varepsilon' + h^2 p_1^* & 12\varepsilon' + 4h^2 p_1^* & -6\varepsilon' + h^2 p_1^* \\ & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot \\ & -6\varepsilon' + h^2 p_1^* & 12\varepsilon' + 4h^2 p_1^* & -6\varepsilon' + h^2 p_1^* \\ & 0 & 0 & 36\varepsilon' \end{pmatrix}, \tag{17}$$

and ,

$$B^* = \begin{pmatrix} \kappa_0^* \\ \kappa_1^* \\ \cdot \\ \cdot \\ \cdot \\ \kappa_N^* \end{pmatrix} \qquad , Q^* = \begin{pmatrix} q_0^* \\ q_1^* \\ \cdot \\ \cdot \\ \cdot \\ q_N^* \end{pmatrix} \qquad , C^* = \begin{pmatrix} c_0^* \\ c_1^* \\ \cdot \\ \cdot \\ \cdot \\ c_N^* \end{pmatrix} \qquad , \tag{18}$$

4. Convergence analysis

Let $u^*(x)$ be the exact solution of the equation (7) with the boundary conditions (8) and also $S^*(x) = \sum_{i=-1}^{N+1} c_i^* B_i(x)$ be the B-spline collocation approximation to $u^*(x)$. Due to round off errors in computations we assume that $\hat{S}(x)$ be the computed spline for $S^*(x)$ so that $\hat{S}(x) = \sum_{i=-1}^{N+1} \hat{c}_i B_i(x)$ where

$$\hat{C} = (\hat{c}_0, \hat{c}_1, ..., \hat{c}_{N-1}, \hat{c}_N) \quad .$$

To estimate the error $\left\| u^*(x) - \hat{S}(x) \right\|_\infty$ we must estimate the errors $\left\| u^*(x) - S^*(x) \right\|_\infty$ and $\left\| S^*(x) - \hat{S}(x) \right\|_\infty$ separately.

Following (16) for $\hat{S}(x)$ we have,

$$A\hat{C} + h^2 \hat{B} = h^2 \hat{Q} \qquad , \qquad (19)$$

where $\hat{B} = [\hat{\kappa}_0, \hat{\kappa}_1, ..., \hat{\kappa}_N]^T$, $\hat{Q} = (\hat{q}_0, \hat{q}_1, ..., \hat{q}_N)^T$, $\hat{\kappa}_i = \kappa^*(x_i, (\hat{c}_{i-1} + 4\hat{c}_i + \hat{c}_{i+1}), \frac{3}{h}(\hat{c}_{i+1} - \hat{c}_{i-1}))$.

Substracting (19) and (16) we have,

$$A(C^* - \hat{C}) + h^2(B^* - \hat{B}) = h^2(Q^* - \hat{Q}) \ . \qquad (20)$$

First we need to recall some theorems.

Theorem 4.1

Suppose that $f(x) \in C^4[0,1]$ and $\left| f^{(4)}(x) \right| \le L \qquad , \forall x \in [0,1]$ and $\Delta = \{0 = x_0 < x_1 < ... < x_N = 1\}$ be the equally spaced partition of $[0,1]$ with step size h. If $S_\Delta(x)$ be the unique spline function interpolate $f(x)$ at nodes $x_0, x_1, ..., x_N \in \Delta$, then there exist a constant $\lambda_j \le 2$ such that $\forall x \in [0,1]$,

$$\left\| f^{(j)}(x) - S_\Delta^{(j)}(x) \right\| \le \lambda_j L h^{4-j} \quad , \ j = 0,1,2,3 \ . \qquad (21)$$

where $\|.\|$ represents the ∞-norm.

Proof: For the proof see Stoer and Bulirsch [5](p.105).

Now we want to find a bound on $\left\| Q^* - \hat{Q} \right\|_\infty$ first. Applying (12) we have,

$$\left| q^*(x_i) - \hat{q}(x_i) \right| = \left| LS^*(x_i) - L\hat{S}(x_i) \right| = \left| \begin{array}{l} -\varepsilon' S^{*''}(x_i) + \varepsilon' \hat{S}''(x_i) + p^*(x_i) S^*(x_i) - p^*(x_i) \hat{S}(x_i) \\ + \kappa^*(x_i, S^*(x_i), S^{*'}(x_i)) - \kappa^*(x_i, \hat{S}(x_i), \hat{S}'(x_i)) \end{array} \right|$$

thus we have,

$$\left| q^*(x_i) - \hat{q}(x_i) \right| \le \left| -\varepsilon' S^{*''}(x_i) + \varepsilon' \hat{S}''(x_i) \right| + \left| p^*(x_i) \right| - \left| S^*(x_i) - \hat{S}(x_i) \right| + \\ \left| \kappa^*(x_i, S^*(x_i), S^{*'}(x_i)) - \kappa^*(x_i, \hat{S}(x_i), \hat{S}'(x_i)) \right| \ . \qquad (22)$$

From (22) and by following Theorem 4.1 and [4](page218) we obtain,

$$\left\| Q^* - \hat{Q} \right\|_\infty \le \varepsilon' \lambda_2 L h^2 + \left\| p^*(x) \right\|_\infty \lambda_0 L h^4 + M \left(\left| S^*(x) - \hat{S}(x) \right| + \right. \\ \left. \left| S^{*'}(x) - \hat{S}'(x) \right| \right) \le \varepsilon' \lambda_2 L h^2 + \left\| p^*(x) \right\|_\infty \lambda_0 L h^4 + M \lambda_0 L h^4 + M \lambda_1 L h^3 , \qquad (23)$$

where $\left\| \kappa'(z) \right\|_\infty \le M$. Thus we can rewrite (23) as follows,

$$\left\| Q^* - \hat{Q} \right\|_{\infty} \leq M_1 h^2 \quad , \tag{24}$$

where $M_1 = \varepsilon' \lambda_2 L + \left\| p^*(x) \right\|_{\infty} \lambda_0 L h^2 + M L \lambda_0 h^2 + M \lambda_1 L h$.

Also we consider the nonlinear term in left hand side of (20) as follows

$$h^2(B^* - \hat{B}) =$$

$$h^2 \begin{bmatrix} \kappa^*(x_0, g_0(t), \dfrac{3}{h}(4c_0^* + 2c_1^* - g_0(t))) \\ \cdot \\ \cdot \\ \cdot \\ \kappa^*(x_i, (c_{i-1}^* + 4c_i^* + c_{i+1}^*), \dfrac{3}{h}(c_{i+1}^* - c_{i-1}^*)) \\ \cdot \\ \cdot \\ \cdot \\ \kappa^*(x_N, g_1(t), -\dfrac{3}{h}(4c_N^* + 2c_{N-1}^* - g_1(t))) \end{bmatrix} - \begin{bmatrix} \kappa^*(x_0, g_0(t), \dfrac{3}{h}(4\hat{c}_0 + 2\hat{c}_1 - g_0(t))) \\ \cdot \\ \cdot \\ \cdot \\ \kappa^*(x_i, (\hat{c}_{i-1} + 4\hat{c}_i + \hat{c}_{i+1}), \dfrac{3}{h}(\hat{c}_{i+1} - \hat{c}_{i-1})) \\ \cdot \\ \cdot \\ \cdot \\ \kappa^*(x_N, g_1(t), -\dfrac{3}{h}(4\hat{c}_N + 2\hat{c}_{N-1} - g_1(t))) \end{bmatrix} .$$

Applying the mean value theorem we have

$$h^2(B^* - \hat{B}) = [h^2 (\frac{\partial \kappa^*}{\partial u})(\xi_1) J + h^2 (\frac{\partial \kappa^*}{\partial u_x})(\xi_2) D](C^* - \hat{C}) \quad , \tag{25}$$

where ξ_1 and ξ_2 are in $(0,1)$ and J and D are given by the following $(N+1) \times (N+1)$ matrixes.

$$J = \begin{pmatrix} 0 & 0 & & & & \\ 1 & 4 & 1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & 1 & 4 & 1 \\ & & & & 0 & 0 \end{pmatrix} \quad , \quad D = \begin{pmatrix} \dfrac{12}{h} & \dfrac{6}{h} & & & & \\ -\dfrac{3}{h} & 0 & \dfrac{3}{h} & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & -\dfrac{3}{h} & 0 & \dfrac{3}{h} \\ & & & -\dfrac{6}{h} & -\dfrac{12}{h} \end{pmatrix} .$$

Substituting (25) into (20) and simplifying we have,

$$R(C^* - \hat{C}) = h^2(Q^* - \hat{Q}) \quad , \tag{26}$$

where $R = A + h^2 (\frac{\partial \kappa^*}{\partial u})(\xi_1) J + h^2 (\frac{\partial \kappa^*}{\partial u_x})(\xi_2) D$.

It is obvious that the matrix R is a strictly diagonally dominant matrix and hence it is non singular, thus from (26) we have,

$$(C^* - \hat{C}) = h^2 R^{-1}(Q^* - \hat{Q}) \tag{27}$$

Taking the infinity norm and then by using (24) we find,

$$\left\|\left(C^* - \hat{C}\right)\right\|_\infty \leq h^2 \left\|R^{-1}\right\|_\infty \left\|\left(Q^* - \hat{Q}\right)\right\|_\infty \leq h^2 \left\|R^{-1}\right\|_\infty (h^2 M_1) = \left\|R^{-1}\right\|_\infty M_1 h^4. \tag{28}$$

Now suppose that χ_i is the summation of the ith row of the matrix $R = [r_{ij}]_{(N+1)\times(N+1)}$, then we have:

$$\chi_0 = \sum_{j=0}^{N} r_{0,j} = 36\varepsilon' + 18h \frac{\partial \kappa^*}{\partial u_x} \quad , \qquad i = 0 \quad , \tag{29}$$

$$\chi_i = \sum_{j=0}^{N} r_{i,j} = 6h^2 p_i^* + 6h^2 \frac{\partial \kappa^*}{\partial u} \quad , \qquad i = 1,...,N-1 \quad , \tag{30}$$

$$\chi_N = \sum_{j=0}^{N} r_{N,j} = 36\varepsilon' - 18h \frac{\partial \kappa^*}{\partial u_x} \quad , \qquad i = N \quad , \tag{31}$$

From the theory of matrices we have:

$$\sum_{i=0}^{N} r_{k,i}^{-1} \chi_i = 1, \qquad\qquad k = 0,...,N, \tag{32}$$

where $r_{k,i}^{-1}$ are the elements of R^{-1}. There for ,

$$\left\|R^{-1}\right\|_\infty = \sum_{i=0}^{N} \left|r_{k,i}^{-1}\right| \leq \frac{1}{\min_{0 \leq i \leq N} \chi_i} = \frac{1}{h^2 \mu_l} \leq \frac{1}{h^2 |\mu_l|}, \tag{33}$$

where l is some index between 0 and N. Now substituting (33) into (28) and simplifying we have,

$$\left\|C^* - \hat{C}\right\|_\infty \leq \frac{M_1 h^4}{h^2 |\mu_l|} \leq M_2 h^2, \tag{34}$$

where M_2 is some finite constant. Now we will be able to prove the convergence of our presented method. We recall a following lemma first,

Lemma 4.2

The B-splines $\{B_{-1}, B_0,..., B_{N+1}\}$ are satisfies the following inequality:

$$\left|\sum_{i=-1}^{N+1} B_i(x)\right| \leq 10, \quad (0 \leq x \leq 1). \tag{35}$$

Proof: For proof see [1].

Now observe that we have:

$$S^*(x) - \hat{S}(x) = \sum_{i=-1}^{N+1} (c_i^* - \hat{c}_i) B_i(x) \quad , \tag{36}$$

thus taking the infinity norm and using (34) and (35) we get,

$$\left\|S^*(x) - \hat{S}(x)\right\|_\infty = \left\|\sum_{i=-1}^{N+1} (c_i^* - \hat{c}_i) B_i(x)\right\|_\infty \leq \left\|(c_i^* - \hat{c}_i)\right\|_\infty \left|\sum_{i=-1}^{N+1} B_i(x)\right| \leq 10 M_2 h^2, \tag{37}$$

Theorem 4.3

Let $u^*(x)$ be the exact solution of (7) with boundary conditions (8) and let $\hat{S}(x)$ be the B-spline collocation approximation to $u^*(x)$ then the method has second order convergence

$$\left\|u^*(x) - \hat{S}(x)\right\|_\infty \leq \omega h^2, \tag{38}$$

where $\varpi = \lambda_0 L h^2 + 10 M_2$ is some finite constant.

Proof:

From Theorem 4.1 we have:

$$\left\| u^*(x) - S^*(x) \right\| \le \lambda_0 L h^4 , \tag{39}$$

thus substituting from (37) and (39) we have ,

$$\left\| u^*(x) - \hat{S}(x) \right\| \le \left\| u^*(x) - S^*(x) \right\| + \left\| S^*(x) - \hat{S}(x) \right\| \le \lambda_0 L h^4 + 10 M_2 h^2 = \omega h^2 ,$$

where $\varpi = \lambda_0 L h^2 + 10 M_2$.

We suppose that $u(x,t)$ be the solution of equation (1) and $U(x,t)$ be the approximate solution by our presented method then we have,

$$\left\| u(x,t_n) - U(x,t_n) \right\| \le \rho(k^2 + h^2) , \quad (\rho \text{ is some finite constant}), \tag{40}$$

thus the order of convergence of our process is $O(k^2 + h^2)$.

5. Numerical illustrations

In order to test the viability of our method based on B-spline collocation, and to demonstrate it's convergence computationally we consider some test problems of partial differential equations.
We solve these problems for various values of h and ε . The computed results are compared with the theoretical to demonstrate the efficiency and performance of our method computationally.
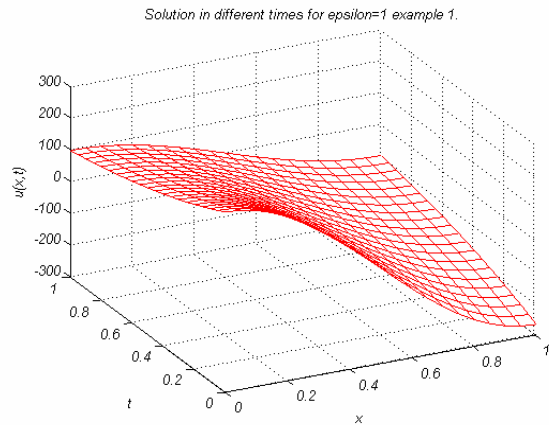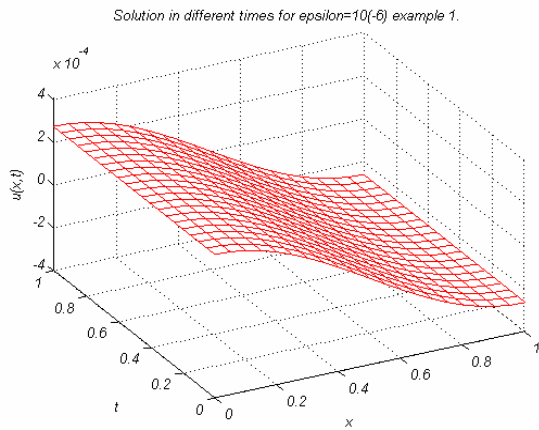**Example 1.**

We consider the following problem,

$$\varepsilon' \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + \left(\frac{\partial u}{\partial x} \right)^2 - u^2 + f(x,t)$$

$$u(x,0) = \varepsilon' \sin(\pi x)\cos(\pi x)$$

$$u(0,t) = 0 , \; u(1,t) = 0$$

We get $f(x,t)$ by the exact solution of this problem is $u(x,t) = \varepsilon' e^{-\varepsilon'^2 t} \sin(\Pi x)\cos(\Pi x)$.

We solve this problem using B-spline collocation method (13) with $k = \dfrac{1}{100}$ and various values of h and ε' .The RMS errors in solutions at $t = 1.0$ are shown in table 1.

Table 1:   RMS errors in solutions of problem by our method

ε N+1	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
08	3.308(-5)	3.620(-7)	3.6544(-9)	3.658(-11)	3.662(-13)
16	8.167(-6)	8.924(-8)	9.005(-10)	9.013(-12)	9.017(-13)
32	2.034(-6)	2.222(-8)	2.242(-10)	2.245(-12)	2.245(-14)
64	5.080(-7)	5.551(-9)	5.601(-10)	5.606(-12)	5.607(-14)

Solution in different times for epsilon=10(-6) example 1.

Solution in different times for epsilon=1 example 1.
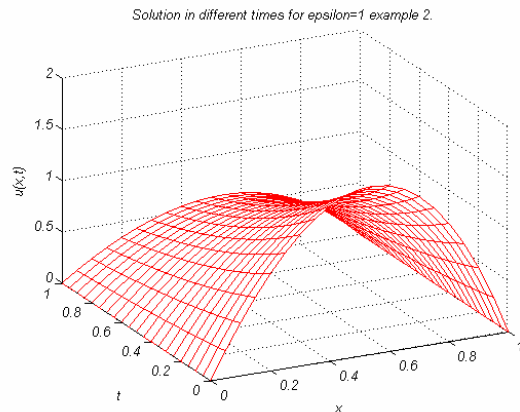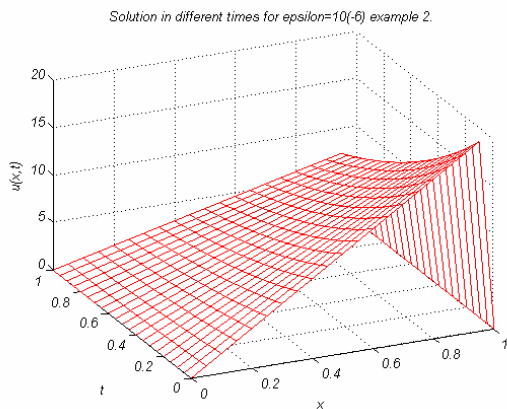
Example 2.

We consider the following problem,

$$\frac{\partial u}{\partial t} - \varepsilon' \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} = f(x,t)$$

$$u(x,0) = u_0(x),$$

$$u(0,t) = 0 \ , \ u(1,t) = 0,$$

The exact solution of this problem is $u(x,t) = e^{-t}(e^{\frac{-1}{\varepsilon}} + (1 - e^{\frac{-1}{\varepsilon}})x - e^{\frac{x-1}{\varepsilon}})$.

Solution in different times for epsilon=10(-6) example 2.

Solution in different times for epsilon=1 example 2.

5. Conclusion

A numerical method is developed to solve a singularly perturbed parabolic initial-boundary value problem. This process is based on the B-spline collocation method. It has been found that the proposed algorithm gives highly accurate numerical results. Numerical experiments demonstrated that the method converges faster for the small values of ε' .

6. References

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