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A Spline based computational simulations for solving self-adjoint singularly perturbed two-point boundary value problems

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Abstract. In this paper, we proposed a spline based computational simulations for solving self-adjoint singularly perturbed two-point boundary value problems. The original problem is reduced to its normal form and the reduced boundary value problem is treated by using difference approximations via cubic splines in tension. The convergence of the method is analyzed. Some numerical examples are given to demonstrate the computational efficiency of the present method.

Keywords: Self-adjoint, Singularly Perturbed Problems, Difference Approximations, Cubic Spline in Tension.

1. Introduction

We consider the following self-adjoint singularly perturbed two-point boundary value problem:

$$Ly \equiv -\varepsilon(a(x)y')' + b(x)y = f(x) \text{ on } [0, 1] \text{ with } y(0) = \alpha,\ y(1) = \beta, \qquad (1.1)$$

where α, β are given constants and ε is a small positive parameter. We also assume that the coefficients *a(x), b(x)* are sufficiently smooth function satisfying

$$a(x) \geq \xi_0 > 1,\ a(x) \geq 0, b(x) \geq \xi_1 > 0 \qquad (1.2)$$

where ξ_0 and ξ_1 are some positive constants. Under these conditions operator L admits a maximum principle [1]. These type of problems arise frequently in fluid mechanics, aerodynamics, plasma dynamics, magneto hydrodynamics, oceanography, optimal control, chemical reactions, etc., In recent years, seeking numerical solutions of singularly perturbed boundary value problems has been the focus of a number of authors. Niijima [2, 3] produced a uniformly second order accurate difference schemes where as Miller [4] gave sufficient conditions for the uniform first order convergence of a general three-point difference schemes. Boglayev [5] discussed a variational difference scheme for solving boundary value problems with a small parameter in the highest derivative. Schatz and Wahlbin [6] used finite element techniques for solving singularly perturbed reaction diffusion problems in two and one dimension. In [7] a method based on spline collocation was presented for solving singularly perturbed boundary value problems. Cubic spline in compression for second order singularly perturbed boundary value problems was presented in [8]. Kadalbajoo and Devendra Kumar [9] presented a numerical method based on finite difference method with variable mesh for solving second order singular perturbed self-adjoint two-point boundary value problems. In [10] a fitted operator finite difference method via the standard Numerov's method was presented for solving self-adjoint singular perturbation problems. Riordan and Stynes [11] presented a uniformly accurate finite element method for solving singularly perturbed one dimensional reaction-diffusion problem. In [12], a spline approximation method was presented for solving self-adjoint singular perturbation problem on non-uniform grids. Lubuma and Patidar [13] presented uniformly convergent non-standard finite difference methods for solving self-adjoint singular perturbation problems. Mishra et.al [14] extended the initial value

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technique to self-adjoint singularly perturbed boundary value problems. Recently, variable mesh finite difference method [15] was extended for solving self-adjoint singularly perturbed two-point boundary value problems.

In order to solve the self-adjoint singularly perturbed problem, first we reduce equation (1.1)-(1.2) to its normal form and then the reduced problem is treated by spline method. In general finding numerical solution of a second order boundary value problem with y' term is more difficult as compared to a second order boundary value problem with absence of y' term. Therefore, it is better to convert the second order boundary value problem with y' term to the second order boundary value problem without y' term i.e to its normal form. In this paper, we have presented computational simulations of self-adjoint singular perturbed two-point boundary value problems via spline method. Convergence of the method is analyzed and some numerical evidences are included to show the applicability and efficiency of the method..

2. Description of the Method

We consider the following self-adjoint singularly perturbed two-point boundary value problem:

$$-\varepsilon(a(x)y')' + b(x)y = f(x) \text{ on } [0,1] \text{ with } y(0)=\alpha,\ y(1)=\beta, \tag{2.1}$$

where α, β are given constants and ε is a small positive parameter. We also assume that the coefficients *a(x), b(x)* are sufficiently smooth function satisfying

$$a(x) \geq \xi_0 > 1, a(x) \geq 0, b(x) \geq \xi_1 > 0 \tag{2.2}$$

where ξ_0 and ξ_1 are some positive constants. Equation (2.1) can be written as

$$-\varepsilon a(x)y''(x) - \varepsilon a'(x)y'(x) + b(x)y(x) = f(x) \qquad \text{or}$$

$$y''(x) + \frac{a'(x)}{a(x)}y'(x) - \frac{b(x)}{\varepsilon a(x)}y(x) = -\frac{f(x)}{\varepsilon a(x)} \qquad \text{or}$$

$$y''(x) + p(x)y'(x) + q(x)y(x) = r(x) \tag{2.3}$$

where $p(x) = \dfrac{a'(x)}{a(x)},\ q(x) = -\dfrac{b(x)}{\varepsilon a(x)}$ *and* $r(x) = -\dfrac{f(x)}{\varepsilon a(x)}$

Consider the transformation

$$y(x) = U(x)V(x) \tag{2.4}$$

Then equation (2.3) can be written as its normal form as

$$V''(x) + A(x)V(x) = G(x) \tag{2.5}$$

$$\text{with } V(0) = \frac{y(0)}{U(0)} = \gamma_0,\ \ V(1) = \frac{y(1)}{U(1)} = \gamma_1, \qquad \gamma_0, \gamma_1 \in R \tag{2.6}$$

where $A(x) = q(x) - \dfrac{1}{2}p'(x) - \dfrac{1}{4}(p(x))^2$

$$G(x) = r(x)exp\left(\frac{1}{2}\int_0^x p(\psi)d\psi\right) \tag{2.7}$$

$$U(x) = exp\left(-\frac{1}{2}\int_0^x p(\psi)d\psi\right). \tag{2.8}$$

Multiplying equation (2.5) throughout by $-\varepsilon$ we get,

$$-\varepsilon V''(x) + W(x)V(x) = Z(x), \tag{2.9}$$

with boundary conditions

$$V(0) = \gamma_0,\ V(1) = \gamma_1 \tag{2.10}$$

where $W(x) = -\varepsilon A(x)$, $Z(x) = -\varepsilon G(x)$ and $w(x) \geq w^* > 0$.

Now, since

$$Z(x) = -\varepsilon G(x) = -\varepsilon r(x)exp\left(\frac{1}{2}\int_0^x p(\psi)d\psi\right) = \frac{f(x)}{a(x)}exp\left(\frac{1}{2}\int_0^x p(\psi)d\psi\right) \tag{2.11}$$

Equation (2.11) shows that *Z(x)* is independent on ε .However *W(x)* may or may not be depend on ε .

3. Derivation of the Scheme

We develop a smooth approximate solution of (2.9) using spline in tension. For this purpose we discretize the interval [*0*, *1*] divided into a set of grid points $x_i = ih$, *i* = 0,1,…, *N* with $h = \frac{1}{N}$. A function $S(x,\tau)$ of class $C^2[a,b]$ which interpolates *y(x)* at the mesh point x_i depends on a parameter τ, reduces to cubic spline in [*a*, *b*] as $\tau \to 0$ is termed as parametric cubic-spline function. The spline function $S(x,\tau) = S(x)$ satisfying in $[x_i, x_{i+1}]$, the differential equation,

$$S''(x) - \tau S(x) = \left[S''(x_i) - \tau S(x_i)\right]\frac{(x_{i+1} - x)}{h} + \left[S''(x_{i+1}) - \tau S(x_{i+1})\right]\frac{(x - x_i)}{h} \tag{3.1}$$

where $S(x_i) = V_i$ and $\tau > 0$ is termed as cubic spline in tension. Solving the equation (3.1) and determining the arbitrary constants from the interpolatory conditions $S(x_i) = V_i$ and $S(x_{i+1}) = V_{i+1}$. After writing $\lambda = h\sqrt{\tau}$, we get

$$S(x) = \frac{h^2}{\lambda^2 sinh\lambda}\left[M_{i+1} sinh\lambda\frac{(x - x_i)}{h} + M_i sinh\lambda\frac{(x_{i+1} - x)}{h}\right]$$
$$-\frac{h^2}{\lambda^2}\left[\frac{(x - x_i)}{h}\left(M_{i+1} - \frac{\lambda^2}{h^2}V_{i+1}\right) + \frac{(x_{i+1} - x)}{h}\left(M_i - \frac{\lambda^2}{h^2}V_i\right)\right] \tag{3.2}$$

Differentiating equation (3.2) and using continuity conditions which lead to the tridiagonal system.

$$h^2\left(\lambda_1 M_{i-1} + 2\lambda_2 M_i + \lambda_1 M_{i+1}\right) = V_{i+1} - 2V_i + V_{i-1} \tag{3.3}$$

where $\lambda_1 = \frac{1}{\lambda^2}\left(1 - \frac{\lambda}{\sinh\lambda}\right)$, $\lambda_2 = \frac{1}{\lambda^2}\left(\lambda\coth\lambda - 1\right)$, $M_i = S''(x)$, *i=1(1)N-1*

The condition (3.3) ensures the continuity of the first order derivatives of the spline $S(x,\tau)$ at interior nodes. We rewrite (2.9) in the form $\varepsilon M_i = W(x_i)V(x_i) - Z(x_i)$ and substituting into equation (3.3), we get the following three term recurrence relation, which gives the approximation $V_1, V_2, \ldots, V_{N-1}$ of the solution *V(x)* at the points $x_1, x_2, \ldots x_{N-1}$.

$$\begin{aligned}&(-\varepsilon + \lambda_1 h^2 W_{i-1})V_{i-1} + (2\varepsilon + 2\lambda_2 h^2 W_i)V_i + (-\varepsilon + \lambda_1 h^2 W_{i+1} V_{i+1}),\\ &= -h^2(\lambda_1 Z_{i-1} + 2\lambda_2 Z_i + \lambda_1 Z_{i+1}) \qquad i = 1, 2, \ldots N-1\end{aligned} \tag{3.4}$$

Using (3.4) with (2.10) we get the approximate solution of *V(x)* at the grid points x_i, since *U(x)* is known, therefore the solution of the original problem (2.1) at these grid points will be obtained by using (2.4) .

4. Convergence Analysis

The above system (3.4) can be written in the matrix form as follows

$$\text{BV} - \text{C} = 0, \tag{4.1}$$

Where

$$B = \begin{bmatrix} 2(\varepsilon + h^2\lambda_1 W_1) & -\varepsilon + \lambda_1 h^2 W_2 & 0 & \cdots & 0 \\ -\varepsilon + \lambda_1 h^2 W_1 & 2(\varepsilon + h^2\lambda_1 W_2) & -\varepsilon + \lambda_1 h^2 W_3 & \cdots & 0 \\ \ddots & \ddots & \ddots & & 0 \\ 0 & \cdots & -\varepsilon + \lambda_1 h^2 W_{N-3} & 2(\varepsilon + h^2\lambda_1 W_{N-2}) & -\varepsilon + \lambda_1 h^2 W_{N-1} \\ 0 & \cdots & 0 & -\varepsilon + \lambda_1 h^2 W_{N-2} & 2(\varepsilon + h^2\lambda_1 W_{N-1}) \end{bmatrix}$$

and $C = \left[-F_1 - (-\varepsilon + \lambda_1 h^2 W_0)V_0, F_2, F_3, \ldots, F_{N-1}, -F_N - (-\varepsilon + \lambda_1 h^2 W_{N+1})V_N\right]^T$,

with $F_k = -h^2(\lambda_1 Z_{i-1} + 2\lambda_2 Z_i + \lambda_1 Z_{i+1})$, *i=1(1)N-1.*

Now, consider the above system with the exact solution $\overline{V} = [\overline{V}_1, \ldots, \overline{V}_{N-1}]^T$, we get

$$B\overline{V} - C + T_{ih} = 0 \tag{4.2}$$

Where $T_{ih} = [T_{1h}, \ldots T_{(N-1)h}]^T$

with truncating error

$$T_{ih} = \left[-1 + 2\varepsilon(\lambda_1 + \lambda_2)\right]\varepsilon h^2 V''(x_i) + \left(\lambda_1 - \frac{1}{12}\right)\varepsilon h^4 V^{iv}(x_i) + \left(\frac{\lambda_1}{12} - \frac{1}{360}\right)\varepsilon h^6 V^{vi}(x_i) + O(h^8)$$

Let $e_i = V_i - \overline{V}_i$, $i = 1(1)N-1$, be the discretization error, subtracting (4.1) and (4.2), we have

$$BE = T_{ih} \tag{4.3}$$

where $E = [e_1,...,e_{N-1}]^T$. For $W(x) > 0$, we can choose h sufficiently small so that the matrix B is irreducible and Monotone [16]. It follows that B^{-1} exists and its elements are non-negative. From (4.3) we have $E = B^{-1}T_{ih}$. Following [17],

$$\|E\| = \|B^{-1}T_{ih}\| \Rightarrow \|E\| \le \|B\|\|T\|.$$

Therefore, $\|E\| = O(h^2)$ for any choice of λ_1 and λ_2 whose sum is $\frac{1}{2}$ and $\|E\| = O(h^4)$ for $\lambda_1 = \frac{1}{12}, \lambda_2 = \frac{5}{12}$. Thus, we summarize the following.

Theorem: The method given by (3.4) for solving the boundary-value problem (2.9)-(2.10) for $W(x) \ge 0$ and sufficiently small *h*, gives a second-order convergent solution for arbitrary λ_1 and λ_2 with $\lambda_1 + \lambda_2 = \frac{1}{2}$ and a fourth-order convergent solution for $\lambda_1 = \frac{1}{12}, \lambda_2 = \frac{5}{12}$. After knowing the value of $V(x)$ in the given domain, we can calculate the value of $y(x)$ using (2.4).

5. Computational Results

To show the computational competence of proposed numerical method, we implemented the present method on three self-adjoint singularly perturbed problems. The rate of convergence is determined as given in [17].

Rate of convergence, $r_{k,\varepsilon} = \log_2\left(\frac{I_{k,\varepsilon}}{I_{k+1,\varepsilon}}\right),\ k = 0,1,2,...$

where $I_{k,\varepsilon} = \max_j \left| y_j^{\frac{h}{2^k}} - y_{2j}^{\frac{h}{2^{k+1}}} \right|,\ k = 0,1,2,...$

Example 1: First, we consider the problem

$$-\varepsilon y''(x) + \frac{4}{(x+1)^4}\left(1+\sqrt{\varepsilon}(x+1)\right)y = f(x)$$

subject to the boundary conditions

$$y(0) = 2,\ y(1) = -1$$

where *f(x)* is chosen such that the exact solution is given by

$$y(x) = -cos\left(\frac{4\pi x}{x+1}\right) + \frac{3\left(e^{-\frac{2x}{\sqrt{\varepsilon}(x+1)}} - e^{-\frac{1}{\sqrt{\varepsilon}}}\right)}{1-e^{\frac{-1}{\sqrt{\varepsilon}}}}.$$

In Table 1(a)-1(d) we have compared the maximum absolute errors for the present method with the methods in [11, 12, 13, 15] for different values of ε and N. Maximum absolute errors and order of convergence for the present method are given in Table 1(e) and Table 1(f) respectively.

Table 1(a): Maximum absolute errors for the Example 1 with $\varepsilon = \left(\frac{1}{N}\right)^{0.25}$.

N	Stynes[11]	Patidar[12]	Lubuma [13]	Method in [15]	Present method	
					$\lambda_1 = \frac{1}{6}, \lambda_2 = \frac{1}{3}$	$\lambda_1 = \frac{1}{12}, \lambda_2 = \frac{5}{12}$
16	9.5E-02	5.8E-02	3.8E-02	2.0E-02	5.1E-02	1.3E-03
32	2.3E-02	1.3E-02	9.6E-03	4.7E-03	1.3E-02	7.9E-05
64	5.6E-03	3.2E-03	2.4E-04	1.1E-03	3.2E-03	4.8E-06
128	1.3E-03	7.8E-04	6.0E-04	2.6E-04	7.8E-04	2.9E-07
256	3.1E-04	1.9E-04	1.5E-04	6.1E-05	1.9E-04	1.7E-08

Table 1(b): Maximum absolute errors for the Example 1 with $\varepsilon = \left(\frac{1}{N}\right)^{0.5}$.

N	Stynes[11]	Patidar[12]	Lubuma [13]	Method in [15]	Present method	
					$\lambda_1 = \frac{1}{6}, \lambda_2 = \frac{1}{3}$	$\lambda_1 = \frac{1}{12}, \lambda_2 = \frac{5}{12}$
16	7.8E-02	4.8E-02	2.5E-02	1.7E-02	4.8E-02	1.1E-03
32	1.8E-02	1.2E-02	6.3E-03	4.0E-03	1.2E-02	5.9E-05
64	4.2E-03	3.1E-03	1.6E-03	9.1E-04	2.9E-03	3.0E-06
128	1.0E-03	1.0E-03	3.9E-04	2.0E-04	7.1E-04	1.2E-07
256	2.5E-04	1.9E-04	9.8E-05	5.0E-05	1.8E-04	4.5E-09

Table 1(c): Maximum absolute errors for the Example 1 with $\varepsilon = \left(\frac{1}{N}\right)^{0.75}$.

N	Stynes[11]	Patidar[12]	Lubuma [13]	Method in [15]	Present method	
					$\lambda_1 = \frac{1}{6}, \lambda_2 = \frac{1}{3}$	$\lambda_1 = \frac{1}{12}, \lambda_2 = \frac{5}{12}$
16	6.6E-02	4.8E-02	1.6E-02	1.5E-02	4.6E-02	7.5E-04
32	1.6E-02	1.2E-02	4.3E-03	3.4E-02	1.1E-02	1.9E-05
64	4.0E-03	3.0E-03	1.1E-03	9.3E-04	2.9E-03	2.2E-06
128	1.0E-03	7.6E-04	2.7E-04	2.4E-04	8.3E-04	5.5E-07
256	2.6E-04	2.1E-04	6.9E-05	6.4E-05	2.8E-04	9.1E-08

Table 1(d): Maximum absolute errors for the Example 1 with $\varepsilon = \left(\frac{1}{N}\right)^{1.0}$.

N	Stynes[11]	Patidar[12]	Lubuma [12]	Method in [15]	Present method	
					$\lambda_1 = \frac{1}{6}, \lambda_2 = \frac{1}{3}$	$\lambda_1 = \frac{1}{12}, \lambda_2 = \frac{5}{12}$
16	6.4E-02	4.8E-02	1.4E-02	1.4E-02	4.6E-02	2.9E-04
32	1.7E-02	1.2E-02	7.9E-03	4.1E-03	1.3E-02	9.5E-05
64	4.2E-03	3.4E-03	2.4E-03	1.1E-03	4.4E-03	2.3E-05
128	1.3E-03	1.0E-03	6.2E-04	3.2E-04	1.9E-03	4.8E-06
256	3.7E-04	3.1E-04	1.6E-04	9.6E-05	8.8E-04	9.9E-07

Table 1(e): Maximum absolute errors for the present method of Example 1

N	128		256		512	
ε	$\lambda_1=\frac{1}{6},\lambda_2=\frac{1}{3}$	$\lambda_1=\frac{1}{12},\lambda_2=\frac{5}{12}$	$\lambda_1=\frac{1}{6},\lambda_2=\frac{1}{3}$	$\lambda_1=\frac{1}{12},\lambda_2=\frac{5}{12}$	$\lambda_1=\frac{1}{6},\lambda_2=\frac{1}{3}$	$\lambda_1=\frac{1}{12},\lambda_2=\frac{5}{12}$
2^{-4}	7.1E-04	7.2E-08	1.8E-04	4.5E-09	4.4E-05	2.8E-10
2^{-5}	7.8E-04	3.8E-07	1.9E-04	2.4E-08	4.9E-05	1.5E-09
2^{-6}	1.1E-03	1.5E-06	2.8E-04	9.1E-08	6.9E-05	5.7E-09
2^{-7}	1.9E-03	4.8E-06	4.8E-04	2.9E-07	1.2E-04	1.9E-08
2^{-8}	3.5E-03	1.6E-05	8.8E-04	9.9E-07	2.2E-04	6.2E-08

Table 1(f): Order of convergence for the present method of Example 1

ε	N=128,256		N=256,512	
	$\lambda_1=\frac{1}{6},\lambda_2=\frac{1}{3}$	$\lambda_1=\frac{1}{12},\lambda_2=\frac{5}{12}$	$\lambda_1=\frac{1}{6},\lambda_2=\frac{1}{3}$	$\lambda_1=\frac{1}{12},\lambda_2=\frac{5}{12}$
2^{-4}	2.00	3.99	1.99	4.00
2^{-5}	2.00	3.99	2.00	3.99
2^{-6}	2.00	3.99	2.00	3.99
2^{-7}	2.00	3.99	2.00	3.99
2^{-8}	2.01	3.99	2.00	3.99

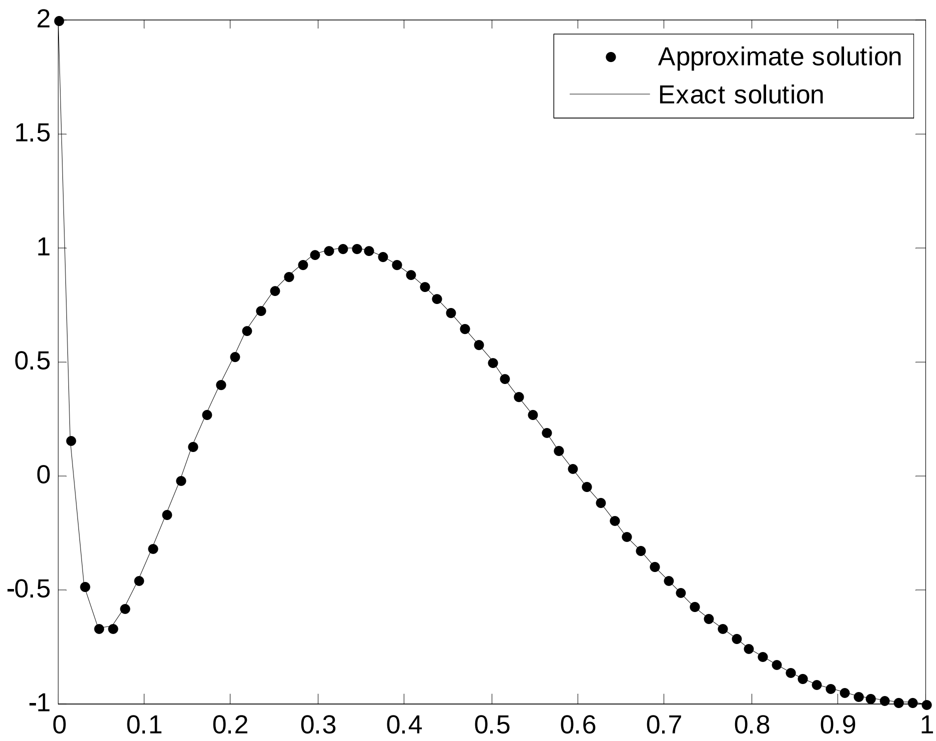

Fig1 : Numerical solution for Example 1 with $\varepsilon=10^{-3}$ and N=*64*.

Example 2: Next, we consider the problem

$$-\varepsilon y''+y=f(x),$$

subject to the boundary conditions

$$y(0)=0,\ y(1)=0$$

where *f(x)* is chosen such that the exact solution of the problem is given by

$$y(x)=e^{x}+e^{\frac{-x}{\sqrt{\varepsilon}}}-x(e+e^{\frac{-1}{\sqrt{\varepsilon}}})-2(1-x).$$

We have compared maximum absolute errors for the present method with the method [6] in Table 2(a). Maximum absolute errors for different values of *N*, ε, λ_1 and λ_2 are given in Table 2(b)-2(c).

Table 2(a): Comparison of Maximum absolute errors for Example 2 with $\varepsilon=5^{-6}$

N	Method in [6]	Method in [15]	Present method			
			$\lambda_1=\frac{1}{6},\lambda_2=\frac{1}{3}$	order	$\lambda_1=\frac{1}{12},\lambda_2=\frac{5}{12}$	order
80	1.6E-03	1.0E-03	4.1E-02	2.10	3.8E-03	3.81
160	1.9E-04	9.1E-05	9.7E-03	2.04	2.7E-04	3.96

Table 2(b) Maximum absolute errors and order of convergence for Example 2 with $\lambda_1=\frac{1}{12},\lambda_2=\frac{5}{12}$ using present method

ε	N= 100	N= 200	N= 400
1/100	7.661148e-008	4.789632e-009	2.994225e-010
Order	3.999	3.999	4.065
1/200	3.063084e-007	1.915564e-008	1.197452e-009
Order	3.999	3.999	3.998
1/400	1.224333e-006	7.661119e-008	4.7896136e-009
Order	3.998	3.999	3.999
1/800	4.851047e-006	3.063083e-007	1.915563e-008
Order	3.98	3.999	3.999
1/1600	1.915765e-005	1.224333e-006	7.661118e-008
Order	3.97	3.99	3.99

Table 2(c) Maximum absolute errors and order of convergence for Example 2 with $\lambda_1=\frac{1}{6},\lambda_2=\frac{1}{3}$ using present method

ε	N= 100	N= 200	N= 400
1/100	1.534858e-004	3.834506e-005	9.584619e-006
Order	2.00	2.00	2.00
1/200	3.071443e-004	7.668021e-005	1.916345e-005
Order	2.00	2.00	2.00
1/400	6.154031e-004	1.534273e-004	3.833045e-005
Order	2.00	2.00	2.00
1/800	1.225290e-003	3.071167e-004	7.667331e-005
Order	1.99	2.00	2.00
1/1600	2.443767e-003	6.153897e-004	1.534239e-004
Order	1.98	2.00	2.00

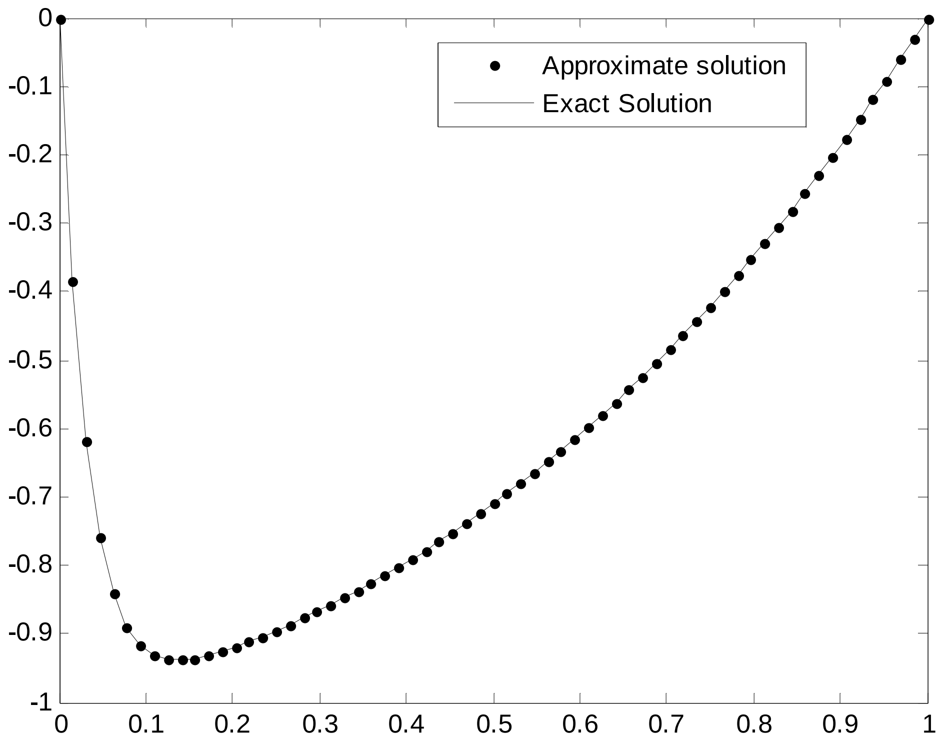

Fig: 2 Numerical Solution for Example 2 with $\varepsilon = 10^{-3}$ and N=64.

Example 3: Finally, we consider the problem

$$-\varepsilon y'' + (1 + x(1-x))y = f(x)$$

subject to the boundary conditions

$$y(0)=0, y(1)=0$$

where *f(x)* is chosen in such a way that the exact solution is given by

$$y(x) = 1 + (x-1)e^{\frac{-x}{\sqrt{\varepsilon}}} - xe^{-\frac{(-x)}{\sqrt{\varepsilon}}}.$$

We have compared the maximum absolute errors for the present method with the method [15] in Table 3(a). Maximum absolute errors presented in Table 3(b)-3(c). In Table 3(d), we have presented maximum absolute errors and order of convergence for different choices of $\lambda_1 + \lambda_2 = \frac{1}{2}$.

Table 3(a): Maximum absolute errors for the Example 3 with *N=128*.

ε	Results in [15]		Results in [15]		Present method	
	Uniform mesh	Variable mesh	Uniform mesh	Variable mesh	$\lambda_1 = \frac{1}{6}, \lambda_2 = \frac{1}{3}$	$\lambda_1 = \frac{1}{12}, \lambda_2 = \frac{5}{12}$
2^{-3}	2.30E-05	2.60E-05	2.27E-05	4.59E-05	2.27E-05	7.47E-10
2^{-4}	3.10E-05	3.90E-05	3.06E-05	2.44E-05	3.06E-05	1.94E-09
2^{-5}	4.50E-05	5.30E-05	4.54E-05	2.92E-05	4.54E-05	5.47E-09
2^{-6}	7.80E-05	7.00E-05	7.84E-05	3.74E-05	7.84E-05	1.80E-08
2^{-7}	1.50E-04	9.00E-05	1.45E-04	4.58E-05	1.45E-04	6.45E-08
2^{-8}	2.80E-04	1.10E-04	2.74E-04	5.29E-05	2.75E-04	2.37E-07
2^{-9}	5.30E-04	1.40E-04	5.26E-04	5.47E-05	5.29E-04	8.87E-07
2^{-10}	1.00E-03	2.00E-04	1.02E-03	5.82E-05	1.03E-03	3.39E-06
2^{-11}	2.00E-03	3.00E-04	1.99E-03	1.28E-04	2.03E-03	1.30E-05

Table 3(b): Maximum absolute errors for the Example 3 with $\lambda_1 = \frac{1}{12}, \lambda_2 = \frac{5}{12}$ using present method.

ε	N = 32	N = 64	N = 128	N = 256	N = 512	N = 1024
2^{-4}	1.91161e-07	1.19511e-08	7.47050e-10	4.68597e-11	3.74512e-12	1.82321e-12
2^{-8}	6.00665e-05	3.78340e-06	2.36939e-07	1.48356e-08	9.27317e-10	5.78503e-11
2^{-12}	7.90488e-04	7.90488e-04	5.06336e-05	3.18766e-06	1.99605e-07	1.24813e-08
2^{-16}	7.76309e-02	4.00486e-02	8.41165e-03	7.53698e-04	4.82468e-05	3.03692e-06
2^{-20}	9.68476e-02	9.40491e-02	7.87372e-02	4.00593e-02	8.34189e-03	7.44467e-04

Table 3(c): Maximum absolute errors for the Example 3 with $\lambda_1 = \frac{1}{6}, \lambda_2 = \frac{1}{3}$ using present method.

ε	N = 32	N = 64	N = 128	N = 256	N = 512	N = 1024
2^{-4}	4.90837e-04	1.22567e-04	3.06328e-05	7.65783e-06	1.91442e-06	4.78606e-07
2^{-8}	4.50429e-03	1.10520e-03	2.75023e-04	6.87774e-05	1.71893e-05	4.29716e-06
2^{-12}	6.50850e-02	1.75260e-02	4.06705e-03	9.98832e-04	2.48611e-04	6.20846e-05
2^{-16}	2.23619e-01	1.51883e-01	6.39205e-02	1.70407e-02	3.95820e-03	9.72355e-04
2^{-20}	2.57993e-01	2.53971e-01	2.26820e-01	1.52156e-01	6.36337e-02	1.69194e-02

Table 3(d): Maximum absolute errors and order of convergence for Example 3 with $\varepsilon = 2^{-10}$ using present method.

λ_1 , λ_2	N=256	N=512	N=1024
$\frac{1}{18}$, $\frac{4}{9}$	8.5439e-05	2.1403e-05	5.3559e-06
Order	1.99	1.99	1.99
$\frac{1}{14}$, $\frac{3}{7}$	3.6500e-05	9.1655e-06	2.2949e-06
Order	1.99	1.99	1.99
$\frac{1}{24}$, $\frac{11}{24}$	1.2825e-04	3.2110e-05	8.0343e-06
Order	1.99	1.99	1.99
$\frac{1}{30}$, $\frac{14}{30}$	1.5393e-04	3.8534e-05	9.6413e-06
Order	1.99	1.99	1.99
$\frac{1}{6}$, $\frac{1}{3}$	2.5739e-04	6.4277e-05	1.6072e-05
Order	2.00	1.99	2.00
$\frac{1}{12}$, $\frac{5}{12}$	2.1212e-07	1.3264e-08	8.2944e-10
Order	3.99	3.99	4.00

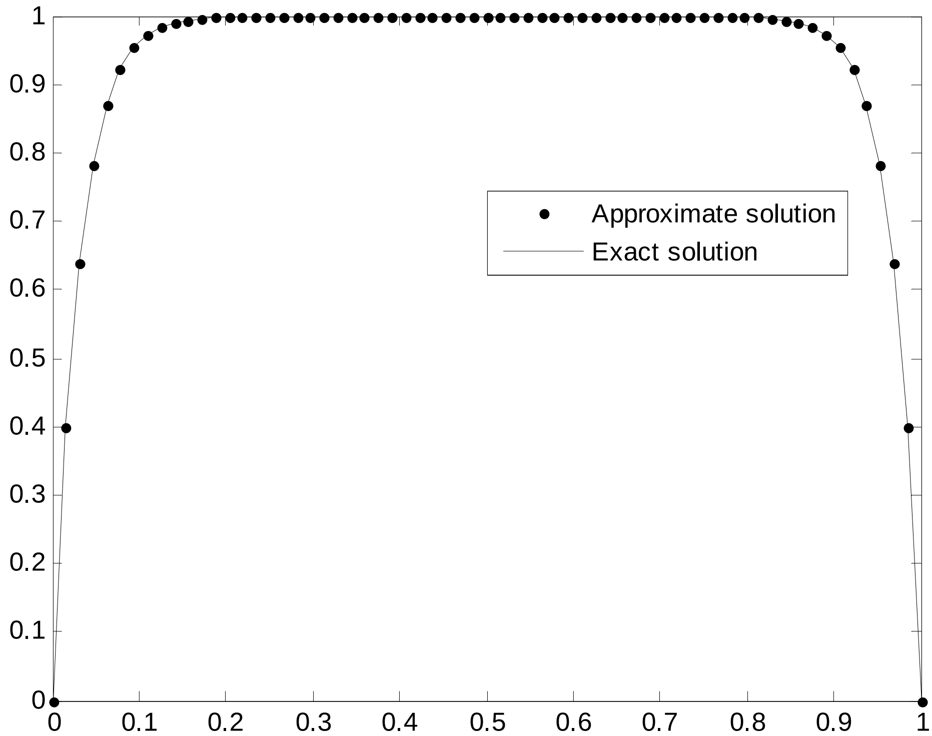

Fig: 3 Numerical solution for Example 3 with $\varepsilon = 10^{-3}$ and *N=64*.

6. Conclusions

We have presented spline based computational simulations for solving self-adjoint singularly perturbed boundary value problems. We have analyzed the convergence of the method and it is found to be fourth order for specific choice of parameters. Three examples are given to demonstrate the efficiency of the proposed method. It is observed from the tables that the present method is more efficient than the methods in [6, 11, 12, 13, 15]. The computational result shows that the present method is fourth order convergence only for $\lambda_1 = \frac{1}{12}$ and $\lambda_2 = \frac{5}{12}$. Also it is shown that for any other choice of $\lambda_1 + \lambda_2 = \frac{1}{2}$, the order of convergence is two. To further authenticate the applicability of the present method, the graphs have been plotted between exact solution and approximate solution of all the three examples for a fixed $\varepsilon = 10^{-3}$ and *N=64*. It can be seen from Fig. 1-3 the computed solutions are in very good agreement with the exact solution.

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7. References

[1] M. H. Protter, H. F. Weinberger, "Maximum Principles in Differential Equations", Prentice-Hall, New Jersey, 1967.

[2] K. Niijima, "On a three-point difference scheme for a singular perturbation problem without a first derivative term. II", Mem. Numer. Math., 7 (1980), pp. 11-27.

[3] K. Niijima, On a three-point difference scheme for a singular perturbation problem without a first derivative term. I, Mem. Numer. Math., 7 (1980)1-10.

[4] J. J. H. Miller, "On the convergence, uniformly in ε of difference schemes for a two-point boundary singular perturbation problem, Numerical Analysis of singular perturbation problems", Academic Press, New York, 1979, pp. 467-474.

[5] I. P. Boglayev, " A variational difference scheme for boundary value problems with a small parameter in the highest derivative", USSR, Comput. Math. Phys., 21(4) (1981), pp. 71-81.

[6] A. H. Schatz, L. B . Wahlbin, "On the finite element method for singularly perturbed reaction diffusion problems in two and one dimension", Math. Comput., 40 (1983), pp. 47-89.

[7] K. Surla, V. Jerkovic, "Some possibilities of applying spline collocations to singular perturbation problems", Numer. Methods Approxim. Theory, Vol.II, Novisad, (1985), pp. 19-25.

[8] Tariq Aziz, Arshad Khan, "A Spline method for second-order singularly perturbed boundary-value problems", J. Comp. Appl. Math., 147 (2002), pp. 445-452.

[9] Mohan K. Kadalbajoo, Devendra Kumar, "Geometric mesh FDM for self-adjoint singular perturbation boundary value problems", Appl. Math. Comput., 190(2)(2007), pp. 1646-1656.

[10] Kailash C. Patidar, "High order fitted operator numerical method for self-adjoint singular perturbation problems", Appl. Math. Comput., 171(2005), pp. 547-566.

[11] E.O' Riordan and M. Stynes, "A uniformly accurate finite element method for a singularly perturbed one dimensional reaction-diffusion problem", Math. Comput., 47(1986), pp. 555-570.

[12] M. K. Kadalbajoo and K. C. Patidar , "Spline approximation method for solving self-adjoint singular perturbation problems on non-uniform grids", J. Comput. Anal. Appl., 5 (2003), pp. 425-451.

[13] J. M. S. Lubuma and K. C. Patidar, "Uniformly convergent non-standard finite difference methods for self-adjoint singular perturbation problems", J. Comput. Anal. Appl., 191(2006), pp.228-238.

[14] H. K. Mishra, M. Kumar, P. Singh, "Initial-value technique for self-adjoint singular perturbation boundary value problems", Comput. Math. Modelling., 20(2) (2009), pp. 207-217.

[15] M. K. Kadalbajoo and Devendra Kumar, "Variable mesh finite difference method for self-adjoint singularly perturbed two-point boundary value problems", J. Comput. Math., 28(5)(2010), pp.711-724.

[16] P. Henrici, " Discrete Variable Methods in ordinary differential Equations", John Wiley, New York, 1962.

[17] E. P. Doolan, J. J. H. Miller, W. H. A. Schilders, "Uniform Numerical Methods for Problems with Initial and Boundary Layer", Boole Press, Dublin, Ireland, 1980.