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Tau numerical solution of the Volterra-Fredholm Hammerstein integro-differential equations with the Bernstein multi-scaling functions

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Abstract. This paper involves the development of the Tau method with Bernstein multi-scaling (BMS) functions basis for the numerical solution of the Volterra-Fredholm Hammerstein integro-differential equations (VFHIDEs). For this purpose at the beginning we define BMS functions and express briefly some properties of BMS functions and after function approximation by using BMS functions, will be presented. Then, the operator matrix representation for the differential and integral parts seeming in the equation using the operational Tau method base on BMS functions basis, will be displaced. The operational Tau method transforms the differential and integration parts of the desired VFHIDEs to some operational matrices. In fact, this method reduces VFHIDEs to a system of algebraic equations. Numerical examples demonstrate the validity and applicability of the proposed method with BMS functions basis.

Keywords: Bernstein multi-scaling functions , Operational Tau method , Hammerstein integro-differential equation, Algebraic equation, Fredholm, Volterra.

1. Introduction

Let us consider the general form of VFHIDE

$$Du(t)-\lambda_1\int_0^t k_1(t,s)G_1(s,u(s))ds-\lambda_2\int_0^1 k_2(t,s)G_2(s,u(s))ds = f(t) \quad 0\le t\le 1, \qquad (1)$$

with n_d independent boundary conditions

$$\sum_{s=1}^{n_d}[c_{js}^{(1)}u^{s-1}(t_1)+c_{js}^{(2)}u^{s-1}(t_2)] = d_j, \quad j=1,2,3,\dots,n_d \qquad (2)$$

where $f(t)$, $k_1(t,s)$ and $k_2(t,s)$ are given continuous functions. λ_1, λ_2 , $c_{js}^{(1)}$ and $c_{js}^{(2)}$, are given constants and $t_1,t_2\in[0,1]$. u(t) is the unknown function to be determined and $G_1(s,u(s)),G_2(s,u(s))$ are analytic functions of the unknown function u(s). n_d is order of the differential operator D with polynomial coefficients $p_i(t)$

$$D=\sum_{i=0}^{n_d}p_i(t)\frac{d^i}{dt^i}, \quad p_i(t)=\sum_{j=0}^{\alpha_i}p_{ij}t^j,$$

where α_i is the degree of $p_i(t)$.

In this section, some numerical methods that discuss about solutions of Volterra-Fredholm integro-differential equations will be presented. Ordokhani [1] has used walsh functions operational matrix with Newton-Cotes nodes for solving Fredholm-Hemmerstein integro-differential equations. Arikoglu et al. [2] by using differential transform method obtained numerical solution of integro-differential equations. Babolian in [3], obtained solutions of nonlinear Volterra-Fredholm integro-differential equations by using direct computational method and triangular functions. With in [4], hybrid Legendre polynomials and Block-

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Pulse functions are presented to approximate the solution of Volterra-Fredholm integro-differential equations. Saberi Nadjafi and Ghorbani in [5] have used his homotopy perturbation method for solving integral and integro-differential equations.

Also, In $[6-12]$ different numerical methods exist for resolving linear and nonlinear integro-differential equations.

Recently, the authors, have used the operational Tau method for the numerical solution of linear and nonlinear Fredholm and Volterra integral and integro-differential equations of second kind. Authors $[13-19]$, developed the Tau method to find numerical solutions of the Fredholm, Volterra and Fredholm-Volterra integral and integro-differential equations with arbitrary polynomial bases.

In this work, we are interested in solving VFHIDEs with an operational approach of the Tau method based on BMS functions. Because in the Tau method, we obtain a system of algebraic equations wherein its solution is easy. The paper is organized as follows: In Section 2, we define BMS functions and we give function approximation by using BMS functions. We drive matrix representation of differential, integral and supplementary conditions parts, in Section 3. Numerical examples are given in Section 4 to illustrate the accuracy of our method. Finally, concluding remarks are given in Section 5.

2. Basic definitions

2.1. Bernstein polynomials and their properties

For $m \geq 0$, the Bernstein polynomials (B-polynomials) defined on the interval $[0,1]$ as follows [20]

$$B_{i,m}(t)=\binom{m}{i}t^i(1-t)^{m-i}, \quad that \quad \binom{m}{i}=\frac{m!}{i!(m-i)!},$$

where

i) $B_{i,m}(t)=0$, if $i<0$ or $i>m$.

ii) $\{B_{i,m}(t), i=0,1,...,m\}$ in Hilbert space $L^2[0,1]$, is a complete non orthogonal set [21].

2.2. BMS functions and function approximation

For $m \geq 1$ and any positive integer $k>1$, the BMS functions $\psi_{i,n}, i=0,1,...,m$ and $n=0,1,...,k-1$ are defined on the interval $[0,1)$ as [22]

$$\psi_{i,n}(t)=\begin{cases} B_{i,m}(kt-n), & \frac{n}{k}\leq t<\frac{n+1}{k}, \\ 0 & otherwise. \end{cases} \tag{3}$$

In equation (3), m is the order of B-polynomials on the interval $[0,1]$, n is the translation argument and t is the normalized time.

If $\phi(t)$ be a vector function of BMS functions on the interval $[0,1)$, as $\phi(t)=[\psi_{0,0}(t),\psi_{1,0}(t),...,\psi_{m-1,0}(t),\psi_{m,0}(t),...,\psi_{0,k-1}(t),\psi_{1,k-1}(t),...,\psi_{m-1,k-1}(t),\psi_{m,k-1}(t)]^T$, then by taking integration of the cross product of two of these vector functions, a matrix of $k(m+1)\times k(m+1)$ dimensional will be resulted which will be indicated as follow

$$D=<\phi,\phi>=\int_0^1\phi(t)\phi^T(t)dt. \tag{4}$$

This matrix is known by dual operational matrix of $\phi(t)$([22]).

A function f(t) defined over $[0,1]$ may be expanded in terms of BMS functions as

$$f(t)\cong\sum_{n=0}^{k-1}\sum_{i=0}^{m}f_{i,n}\psi_{i,n}(t)=F^T\phi(t),$$

where $\phi(t)$ is the vector function defined before and C is a $k(m+1)\times 1$ vector given by $F=[f_{0,0},f_{1,0},...,f_{m-1,0},f_{m,0},...,f_{0,k-1},f_{1,k-1},...,f_{m-1,k-1},f_{m,k-1}]^T$, and can be obtained by [22]

$$F^T = (\int_0^1 f(t)\phi^T(t)dt)D^{-1}. \quad (5)$$

We can write $f(t) = F^T \Phi X_t$, where Φ is a non-singular matrix given by $\phi(t) = \Phi X_t$ with a standard basic vector $X_t = [1, t, t^2, \cdots, t^{km+(k-1)}]^T$.

We can also approximate the function $k(t,s)$ as follows

$$k(t,s) \cong \phi^T(t) K \phi(s),$$

where K is a $k(m+1) \times k(m+1)$ matrix and can be calculated as

$$K = \begin{bmatrix} K_{0,0}^T \\ K_{1,0}^T \\ \vdots \\ K_{m,0}^T \\ \vdots \\ K_{0,k-1}^T \\ K_{1,k-1}^T \\ \vdots \\ K_{m,k-1}^T \end{bmatrix}, \quad (6)$$

and $\{K_{i,n}\}_{i=0,n=0}^{m,k-1}$ are $k(m+1)\times 1$ in order to calculate them firstly, $k(t,s)$ is approximated in terms of $\{\psi_{i,n}(s)\}_{i=0,n=0}^{m,k-1}$ as $k(t,s) \cong \xi^T(t)\phi(s)$, where $\xi(t) = [\xi_{0,0}(t), \xi_{1,0}(t), ..., \xi_{m,0}(t), ..., \xi_{0,k-1}(t), \xi_{1,k-1}(t), ...$ $, \xi_{m,k-1}(t)]^T$, and by using Eq. (5), the elements of vector $\xi(t)$ can be obtained for $i = 0,1,...,m$ and $n = 0,1,...,k-1$. Now, all functions are approximated $\{\xi_{i,n}(t)\}_{i=0,n=0}^{m,k-1}$ in terms of $\psi_{i,n}(t)$ for $i = 0,1,...,m$, $n = 0,1,...,k-1$ as

$$\xi_{i,n}(t) \cong \sum_{n=0}^{k-1}\sum_{i=0}^{m} k_{i,n}\psi_{i,n}(t) = K_{i,n}^T \phi(t), \quad (7)$$

where using Eq. (5), $\{K_{i,n}\}_{i=0,n=0}^{m,k-1}$ can be obtained from Eq. (7).

$k(t,s)$ can be expressed as:

$$k(t,s) \cong \phi^T(t) K \phi(s) = X_t^T \Phi^T K \Phi X_s,$$

where $\Phi = [\Phi_{i,j}]_{i,j=0}^{km+(k-1)}$ is a non-singular matrix given by $\phi(t) = \Phi X_t$ with a standard basic vector $X_t = [1, t, t^2, ..., t^{km+(k-1)}]^T$. If we take $\tilde{K} = \Phi^T K \Phi$, we can write

$$k(t,s) \cong X_t^T \tilde{K} X_s = \sum_{i=0}^{km+(k-1)} \sum_{j=0}^{km+(k-1)} \tilde{K}_{i,j} t^i s^j.$$

3. Matrix representation of 1 and 2

In this section we drive formulas for numerical solvability of integro-differential equation (1) with conditions (2) based on BMS functions of the operational Tau method. we can assume that in (1), the nonlinear analytic functions can be expanded as

$$G_1(s,u(s)) \cong \sum_{p=0}^{m} \gamma_p(s) u^p(s), \quad G_2(s,u(s)) \cong \sum_{p=0}^{m} \delta_p(s) u^p(s),$$

thus we can write (1) the following form

$$Du(t) - \lambda_1 \sum_{p=1}^{m} \int_0^t k_1(t,s)\gamma_p(s)u^p(s)d(s) - \lambda_2 \sum_{p=1}^{m} \int_0^1 k_2(t,s)\delta_p(s)u^p(s)d(s) = f(t) + \lambda_1 \int_0^t k_1(t,s)\gamma_0(s)d(s)$$
$$\lambda_2 \int_0^t k_2(t,s)\delta_0(s)d(s), t \in [0,1]. \quad (8)$$

Let $F(t) = f(t) + \int_0^t k_{p1}(t,s)\gamma_0(s)d(s) + \int_0^1 k_{p2}(t,s)\delta_0(s)d(s)$ and consider $k_{p1}(t,s)$

$= k_1(t,s)\gamma_p(s)$ and $k_{p2}(t,s) = k_2(t,s)\delta_p(s)$ For $(p = 1,...,m)$,where $\gamma_p(s)$, $\delta_p(s)$, $(p = 1,...,m)$ are continuous functions. Therefore, equation (8) transform to following equation

$$Du(t) - \lambda_1 \sum_{p=1}^{m} \int_0^t k_{p1}(t,s)u^p(s)d(s) - \lambda_2 \sum_{p=1}^{m} \int_0^1 k_{p2}(t,s)u^p(s)d(s) = F(t), \quad t \in [0,1]. \qquad (9)$$

Now we convert equations (9) and (2) to the corresponding algebraic equations in the following three steps 3.1, 3.2 and 3.3.

3.1. Matrix representation of differential part

Let us $\phi(t) = [\psi_{0,0}(t), \psi_{1,0}(t),..., \psi_{m-1,0}(t), \psi_{m,0}(t),..., \psi_{0,k-1}(t), \psi_{1,k-1}(t),..., \psi_{m-1,k-1}(t), \psi_{m,k-1}(t)]^T$, be a polynomial basis vector given by $\phi(t) = \Phi X_t$, where Φ is a non-singular matrix. Also for any matrix P, $P_\Phi = \Phi P \Phi^{-1}$. Now we convert the operational approach to the Tau method proposed by Ortiz and samara $[23]$ is based on three simple matrices

$$\mu = \begin{bmatrix} 0 & 1 & 0 & 0 & \\ & 0 & 1 & 0 & \\ & & 0 & 1 & \vdots \\ & & & 0 & \\ & & \cdots & & \ddots \end{bmatrix}, \quad \eta = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ 0 & 2 & 0 & & \vdots \\ 0 & 0 & 3 & 0 & \\ & & \cdots & & \ddots \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 1 & 0 & 0 & \\ & 0 & \frac{1}{2} & 0 & \\ & & 0 & \frac{1}{3} & \vdots \\ & & & 0 & \\ & & \cdots & & \ddots \end{bmatrix}.$$

We recall the following properties from $[17]$.

Lemma 1. Let $u_m(t)$ be a polynomial as

$$u_m(t) = \sum_{i=0}^{m} a_i t^i = \vec{a}_m X_t,$$

where $\vec{a}_m = [a_0, a_1,..., a_m, 0,...]$, $X_t = [1, t, t^2,...]^T$, then we have

$$\frac{d^r}{dt^r} u_m(t) = \vec{a}_m \eta^r X_t, \qquad r = 1,2,...$$

$$t^r u_m(t) = \vec{a}_m \mu^r X_t, \qquad r = 1,2,...$$

$$\int_0^t u_m(t)dt = \vec{a}_m L X_t - \vec{a}_m L X_0, \quad X_0 = [1,0,0,...]^T.$$

Theorem 1. For any linear differential operator D defined by (1) and any series $u(t) \cong \mathbf{u}^T \phi(t)$, that $\mathbf{u} = [u_{0,0}, u_{1,0},..., u_{m,0},..., u_{0,k-1}, u_{1,k-1},..., u_{m,k-1}]^T$, we have $Du(t) \cong \mathbf{u}^T \Phi \Pi X_t = \mathbf{u}^T \Pi_\Phi \phi(t)$ where

$$\Pi = \sum_{i=0}^{nd} \eta^i p_i(\mu) = \sum_{i=0}^{n_d} \sum_{j=0}^{\alpha_i} p_{ij} \eta^i \mu^j,$$

and

$$\Pi_\Phi = \Phi \Pi \Phi^{-1}.$$

3.2. Matrix representation of integral part

Equation (9) shows that the using of the Tau method requires that $u^p(t)$ must be written as the product of a matrix and a vector. The following result is concerned with approximation of the nonlinear functions.

Lemma 2. Let $u(t) \cong \sum_{n=0}^{k-1} \sum_{i=0}^{m} u_{i,n} \psi_{i,n}(t) = \mathbf{u}^T \phi(t) = \mathbf{u}^T \Phi X_t$ be a polynomial with $\mathbf{u} = [u_{0,0}, u_{1,0},...,$ $u_{m,0},..., u_{0,k-1}, u_{1,k-1},..., u_{m,k-1}]^T$, $\Phi = [\Phi_{i,j}]_{i,j=0}^{km+(k-1)}$ and $X_t = [1, t, t^2,..., t^{km+(k-1)}]^T$, then for any natural number $p \in N$, we have

$$u^p(t) \cong \mathbf{u}^T \Phi B^{p-1} X_t,$$

where B is an upper triangular Toeplitz matrix having the following structure

$$B=\begin{bmatrix} \mathbf{u}^T\Phi_0 & \mathbf{u}^T\Phi_1 & \mathbf{u}^T\Phi_2 & \cdots & \mathbf{u}^T\Phi_{km+(k-1)} \\ 0 & \mathbf{u}^T\Phi_0 & \mathbf{u}^T\Phi_1 & \cdots & \mathbf{u}^T\Phi_{km+(k-2)} \\ 0 & 0 & \mathbf{u}^T\Phi_0 & \cdots & \mathbf{u}^T\Phi_{km+(k-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathbf{u}^T\Phi_0 \end{bmatrix},$$

with $\Phi_j=[\Phi_{0,j},\Phi_{1,j},\Phi_{2,j}...,\Phi_{km+(k-1),j}]^T$, $j=0,1,\cdots,km+(k-1)$.

Proof. The validity of the lemma for p=1 is obvious. Let $u^2(t)\cong(\mathbf{u}^T\Phi X_t)\times(\mathbf{u}^T\Phi X_t)=\mathbf{u}^T\Phi(X_t\times(\mathbf{u}^T\Phi X_t))$. Now, it is shown that $X_t\times(\mathbf{u}^T\Phi X_t)=BX_t$.

If $\mathbf{u}=[u_{0,0},u_{1,0},...,u_{m,0},...,u_{0,k-1},u_{1,k-1},...,u_{m,k-1}]^T=[u_0,u_1,u_2,...,u_{km+(k-1)}]^T$, we can set

$$X_t\times(\mathbf{u}^T\Phi X_t)=X_t\times\left(\sum_{s=0}^{km+(k-1)}\sum_{r=0}^{km+(k-1)}u_r\Phi_{r,s}t^s\right)$$

$$=\left[\sum_{s=0}^{km+(k-1)}\sum_{r=0}^{km+(k-1)}u_r\Phi_{r,s}t^{s+i}\right]_{i=0}^{km+(k-1)^T},$$

and

$$BX_t=\left[\sum_{j=0}^{km+(k-1)}B_{ij}t^j\right]_{i=0}^{km+(k-1)^T}=\left[\sum_{j=0}^{km+(k-1)}\sum_{r=0}^{km+(k-1)}u_r\Phi_{r,j-i}t^j\right]_{i=0}^{km+(k-1)^T},$$

concerning $B_{ij}=0,$ for $i>j$, it follows that

$$BX_t=\left[\sum_{j=0}^{km+(k-1)}\sum_{r=0}^{km+(k-1)}u_r\Phi_{r,j}t^{j+i}\right]_{i=0}^{km+(k-1)^T},$$

which states the lemma hold for p=2. So we assume the validity of the proposition for k and transit to $k+1$ are as follows:

$$u^{k+1}(t)=u^k(t)u(t)\cong(\mathbf{u}^T\Phi B^{k-1}X_t)\times(\mathbf{u}^T\Phi X_t)=\mathbf{u}^T\Phi B^{k-1}(X_t)\times(\mathbf{u}^T\Phi X_t))$$

$$=\mathbf{u}^T\Phi B^{k-1}(BX_t)=\mathbf{u}^T\Phi B^kX_t.$$

Following the structure of matrix *B* in Lemma 2, we can write

$$B=\begin{bmatrix} \mathbf{u}^T\Phi\mathbf{e}_1 & \mathbf{u}^T\Phi\mathbf{e}_2 & \mathbf{u}^T\Phi\mathbf{e}_3 & \cdots & \mathbf{u}^T\Phi\mathbf{e}_{k(m+1)} \\ 0 & \mathbf{u}^T\Phi\mathbf{e}_1 & \mathbf{u}^T\Phi\mathbf{e}_2 & \cdots & \mathbf{u}^T\Phi\mathbf{e}_{k(m)+(k-1)} \\ 0 & 0 & \mathbf{u}^T\Phi\mathbf{e}_1 & \cdots & \mathbf{u}^T\Phi\mathbf{e}_{k(m)+(k-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathbf{u}^T\Phi\mathbf{e}_1 \end{bmatrix},$$

where $\Phi_{i-1}=\Phi\mathbf{e}_i$, $\mathbf{e}_i$ and Φ are unit and non-singular matrices respectively and $i=1,2,...,k(m+1)$.If we take $\tilde{\mathbf{u}}=\Phi^T\mathbf{u}$ that $\tilde{\mathbf{u}}=[\tilde{u}_{0,0},\tilde{u}_{1,0},...,\tilde{u}_{m,0},...,\tilde{u}_{0,k-1},\tilde{u}_{1,k-1},...,\tilde{u}_{m,k-1}]^T$, the matrix *B* can be represented as an upper triangular Toeplitz form

$$B=\begin{bmatrix}\tilde{u}_{0,0} & \tilde{u}_{0,1} & \tilde{u}_{0,2} & \cdots & \tilde{u}_{m,k-1}\\ 0 & \tilde{u}_{0,0} & \tilde{u}_{0,1} & \cdots & \tilde{u}_{m,k-2}\\ 0 & 0 & \tilde{u}_{0,0} & \cdots & \tilde{u}_{m,k-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \tilde{u}_{0,0}\end{bmatrix}.$$

Now, we present the operational Tau representation of the integration terms of (9) in Theorems 2 and 3 . Using Theorems 2 and $3,$ we obtain operational Tau matrix for Volterra and Fredholm integration terms of (9) . We give the following theorems whose proof is based mainly on Lemma 2.

Theorem 2. Let the analytic functions $u(s)$ and $k_{p1}(t,s),\ p=1,2,...,m$ be expressed as :

$$u(s)\cong\sum_{n=0}^{k-1}\sum_{i=0}^{m}u_{i,n}\psi_{i,n}(s)=\mathbf{u}^T\phi(s)=\mathbf{u}^T\Phi X_s,$$

$$k_{p1}(t,s)\cong\phi^T(t)K_{p1}\phi(s)=X_t^T\Phi^T K_{p1}\Phi X_s=\sum_{i=0}^{km+(k-1)}\sum_{j=0}^{km+(k-1)}\tilde{K}_{p1_{(i,j)}}s^i t^j,$$

where $\mathbf{u}=[u_{0,0},u_{1,0},...,u_{m,0},...,u_{0,k-1},u_{1,k-1},...,u_{m,k-1}]^T,\Phi=[\Phi_{i,j}]_{i,j=0}^{km+(k-1)}$ is a non-singular matrix and $X_s=[1,s,s^2,...,s^{km+(k-1)}]^T$, then we have

$$\int_0^t k_{p1}(t,s)u^p(s)ds\cong\mathbf{u}^T\Phi B^{p-1}M_{p1}X_t,$$

where M_{p1} for $p=1,2,...,m$ is in the following form

$$M_{p1}=\begin{bmatrix}0 & \tilde{K}_{p1_{(0,0)}} & \tilde{K}_{p1_{(0,1)}}+\frac{1}{2}\tilde{K}_{p1_{(1,0)}} & \tilde{K}_{p1_{(0,2)}}+\frac{1}{2}\tilde{K}_{p1_{(1,1)}}+\frac{1}{3}\tilde{K}_{p1_{(2,0)}} & \cdots\\ 0 & 0 & \frac{1}{2}\tilde{K}_{p1_{(0,0)}} & \frac{1}{2}\tilde{K}_{p1_{(0,1)}}+\frac{1}{3}\tilde{K}_{p1_{(1,0)}} & \cdots\\ 0 & 0 & 0 & \frac{1}{3}\tilde{K}_{p1_{(1,0)}} & \cdots\\ 0 & 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \frac{1}{m(k)+(k-1)}\tilde{K}_{p1_{(0,0)}}\\ 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

and B has been given in Lemma 2.

Proof. using Lemma 2 , we have:

$$u^p(s)\cong\mathbf{u}^T\Phi B^{p-1}X_s,$$

also $k_{p1}(t,s)u^p(s)\cong\mathbf{u}^T\Phi B^{p-1}[k_{p1}(t,s),sk_{p1}(t,s),...,s^{km+(k-1)}k_{p1}(t,s)]^T$. We can write:

$$k_{p1}(t,s)s^n=\sum_{i=0}^{km+(k-1)}\sum_{j=0}^{km+(k-1)}\tilde{K}_{p1_{(i,j)}}t^j s^{n+i},$$

the integration term can be written as:

$$\int_0^t k_{p1}(t,s)u^p(s)d(s)\cong\mathbf{u}^T\Phi B^{p-1}\left[\int_0^t k_{p1}(t,s)s^n d(s)\right]_{n=0}^{km+(k-1)^T}$$

$$=\mathbf{u}^T\Phi B^{p-1}\left[\sum_{i=0}^{km+(k-1)}\sum_{j=0}^{km+(k-1)}\tilde{K}_{p11_{(i,j)}}t^j\,\frac{t^{n+i+1}}{n+i+1}\right]_{n=0}^{km+(k-1)^T}.$$

On the other hand, we have

$$\sum_{i=0}^{km+(k-1)}\sum_{j=0}^{km+(k-1)}\widetilde{K}_{p1_{(i,j)}}\frac{t^{n+j+i+1}}{n+i+1}=\left[\frac{1}{n+i+1}\right]_{i=0}^{km+(k-1)}\widetilde{\mathbf{k}}_{p1}(n)X_t,$$

such that $\widetilde{\mathbf{k}}_1(n)$ is a matrix having the following entries

$$\widetilde{\mathbf{k}}_{p1_{(i,j)}}(n)=\begin{cases}\widetilde{K}_{p1_{(i,j-i-1-n)}}, & j>n+i,\\ 0, & j\le n+i.\end{cases}$$

Therefore, we can write

$$\begin{aligned}\int_0^t k_{p1}(t,s)u^p(s)d(s) &\cong \mathbf{u}^T\Phi B^{p-1}\left[\left[\frac{1}{n+i+1}\right]_{i=0}^{km+(k-1)}\widetilde{\mathbf{k}}_{p1}(n)X_t\right]_{n=0}^{km+(k-1)^T}\\ &= \mathbf{u}^T\Phi B^{p-1}\left[\left[\frac{1}{n+i+1}\right]_{i=0}^{km+(k-1)}\widetilde{\mathbf{k}}_{p1}(n)\right]_{n=0}^{km+(k-1)}X_t\\ &= \mathbf{u}^T\Phi B^{p-1}M_{p1}X_t.\end{aligned}$$

Theorem 3. Let the analytic functions u(s) and $k_{p2}(t,s),\ p=1,2,...,m$ be expressed as:

$$u(s)\ \cong\ \sum_{n=0}^{k-1}\sum_{i=0}^{m}u_{i,n}\psi_{i,n}(s)=\mathbf{u}^T\phi(s)=\mathbf{u}^T\Phi X_s,$$

$$k_{p2}(t,s)\cong\phi^T(s)K_{p2}\phi(t)=X_s^T\Phi^TK_{p2}\Phi X_t=\sum_{i=0}^{km+(k-1)}\sum_{j=0}^{km+(k-1)}\widetilde{K}_{p2_{(i,j)}}t^is^j,$$

where $\mathbf{u}=[u_{0,0},u_{1,0},...,u_{m,0},...,u_{0,k-1},u_{1,k-1},...,u_{m,k-1}]^T$, $\Phi=[\Phi_{i,j}]_{i,j=0}^{km+(k-1)}$ is a non-singular matrix and $X_s=[1,s,s^2,...,s^{km+(k-1)}]^T$, then we have

$$\int_0^1 k_{p2}(t,s)u^p(s)ds\cong\mathbf{u}^T\Phi B^{p-1}M_{p2}X_t,$$

such that M_{p2} for $p=1,2,...,m$ is a matrix having the following form

$$M_{p2}=\begin{bmatrix}\sum_{j=0}^{k(m)+(k-1)}\frac{\widetilde{K}_{p2_{(0,j)}}}{j+1} & \cdots & \sum_{j=0}^{k(m)+(k-1)}\frac{\widetilde{K}_{p2_{(k(m)+(k-1),j)}}}{j+1}\\ \vdots & \vdots & \vdots\\ \sum_{j=0}^{k(m)+(k-1)}\frac{\widetilde{K}_{p2_{(0,j)}}}{j+k(m+1)} & \cdots & \sum_{j=0}^{k(m)+(k-1)}\frac{\widetilde{K}_{p2_{(k(m)+(k-1),j)}}}{j+k(m+1)}\end{bmatrix},$$

and B has been given in Lemma 2.

Proof. According to Lemma 2 :

$$u^p(s)\cong\mathbf{u}^T\Phi B^{p-1}X_s,$$

also

$$k_{p2}(t,s)u^p(s)\cong\mathbf{u}^T\Phi B^{p-1}[k_{p2}(t,s),sk_{p2}(t,s),...,s^{km+(k-1)}k_{p2}(t,s)]^T.$$

We can write:

$$k_{p2}(t,s)s^n=\sum_{i=0}^{km+(k-1)}\sum_{j=0}^{km+(k-1)}\widetilde{K}_{p2_{(i,j)}}t^is^{n+j},$$

the integration term can be written as:

$$\int_0^1 k_{p2}(t,s)u^p(s)d(s) = \mathbf{u}^T\Phi B^{p-1}\left[\int_0^1 k_{p2}(t,s)s^n d(s)\right]_{n=0}^{km+(k-1)^T}$$

$$= \mathbf{u}^T\Phi B^{p-1}\left[\sum_{i=0}^{km+(k-1)}\sum_{j=0}^{km+(k-1)}\tilde{K}_{p2_{(i,j)}} t^i \frac{1}{n+j+1}\right]_{n=0}^{km+(k-1)^T},$$

on the other hand, we will have

$$\left[\sum_{i=0}^{km+(k-1)}\sum_{j=0}^{km+(k-1)}\tilde{K}_{p2_{(i,j)}} t^i \frac{1}{n+j+1}\right]_{n=0}^{km+(k-1)^T} = M_{p2}X_t,$$

so that M_{p2} For $p = 1,2,...,m$ having has the following form:

$$M_{p2} = \begin{bmatrix} \sum_{j=0}^{k(m)+(k-1)} \frac{\tilde{K}_{p2_{(0,j)}}}{j+1} & \cdots & \sum_{j=0}^{k(m)+(k-1)} \frac{\tilde{K}_{p2_{(k(m)+(k-1),j)}}}{j+1} \\ \vdots & \vdots & \vdots \\ \sum_{j=0}^{k(m)+(k-1)} \frac{\tilde{K}_{p2_{(0,j)}}}{j+k(m+1)} & \cdots & \sum_{j=0}^{k(m)+(k-1)} \frac{\tilde{K}_{p2_{(k(m)+(k-1),j)}}}{j+k(m+1)} \end{bmatrix}.$$

3.3. Matrix representation for the supplementary conditions

Replacing $u(t) \cong \sum_{i=0}^{m}\sum_{n=0}^{k-1} u_{i,n}\psi_{i,n} = \mathbf{u}^T\Phi X_t$ on the left hand side of (2), it can be written as

$$\sum_{s=1}^{n_d}[c_{js}^{(1)}u^{s-1}(t_1) + c_{js}^{(2)}u^{s-1}(t_2)] \cong \mathbf{u}^T\Phi\sum_{s=1}^{n_d}[c_{js}^{(1)}\eta^{s-1}X_{t_1} + c_{js}^{(2)}\eta^{s-1}X_{t_2}].$$

Let $A_j = \sum_{s=1}^{n_d}[c_{js}^{(1)}\eta^{s-1}X_{t_1} + c_{js}^{(2)}\eta^{s-1}X_{t_2}]$ where $X_{t_1} = [1, t_1, t_1^{\,2}, ..., t_1^{\,km+(k-1)}]^T$ and $X_{t_2} = [1, t_2, t_2^{\,2}, ..$ $, t_2^{\,km+(k-1)}]^T$. Thus if we take $\tilde{\mathbf{u}} = \Phi^T\mathbf{u}$, the (jth) condition of (2) is converted to $\tilde{\mathbf{u}}^T A_j = d_j$, $j = 1,2,$ $..., n_d$. Now by setting A as the matrix with columns A_j, $j = 1,2,..., n_d$ and by setting $d = [d_1, d_2, ..., d_{n_d}]^T$. as the vector that contains right-hand side of supplementary conditions, they take the form $\tilde{\mathbf{u}}^T A = d$.

Also in the righthand side of (9), we assume that

$$F(t) \cong \sum_{n=0}^{k-1}\sum_{i=0}^{m} F_{i,n}\psi_{i,n} = F^T\Phi X_t,$$

that $F = [F_{0,0}, F_{1,0}, ..., F_{m,0}, ..., F_{0,k-1}, F_{1,k-1}, ..., F_{m,k-1}]^T$. We take $\tilde{F} = \Phi^T F$, thus $F(t) \cong \tilde{F}^T X_t$.

Consequently, using Theorem (1) and the results of 3.1, 3.2 and 3.3 parts, we obtain equations (9) and (2) as following :

$$\begin{cases} \tilde{\mathbf{u}}^T[\Pi - \lambda_1\sum_{p=1}^{m}B^{p-1}M_{p1} - \lambda_2\sum_{p=1}^{m}B^{p-1}M_{p2}] = \tilde{F}^T, \\ \tilde{\mathbf{u}}^T A = d^T. \end{cases} \tag{10}$$

Now setting

$$\overline{\Pi} = \Pi - \lambda_1\sum_{p=1}^{m}B^{p-1}M_{p1} - \lambda_2\sum_{p=1}^{m}B^{p-1}M_{p2},$$

$$G = [A_1, A_2, ..., A_{n_d}, \overline{\Pi}_1, \overline{\Pi}_2, ..., \overline{\Pi}_{k(m+1)-n_d}],$$

and

$$g = [d_1, d_2, ..., d_{n_d}, \tilde{F}_0, \tilde{F}_1, ..., \tilde{F}_{km+(k-1)-n_d}],$$

where $\overline{\Pi}_i$ denotes the (ith) column of $\overline{\Pi}$, system of (10) can be written as $\tilde{\mathbf{u}}^T G = g$ which must be solved for the unknown coefficients, $\tilde{u}_{0,0}, \tilde{u}_{1,0}, ..., \tilde{u}_{m,0}, ..., \tilde{u}_{0,k-1}, \tilde{u}_{1,k-1}, ..., \tilde{u}_{m,k-1}$.

4. Illustrative Examples

We apply the present method in this section and solve some examples given in different papers. The computations associated with the examples were performed using Mathematica.

Example 4.1. Consider the following VFHIDE [24]

$$tu''(t) - tu'(t) + 2u(t) - \int_0^t (t-s)u(s)ds - \int_0^1 (t+s)u(s)ds = \frac{t^4}{12} - \frac{t^3}{6} - \frac{t^2}{2} - \frac{13t}{6} + \frac{17}{12}, \quad 0 \le t \le 1,$$

with the initial condition $u(0) = 1, u'(0) - 2u(1) + 2u(0) = 1$.

In this example we have $n_d = 2, p_0(t) = 2, p_1(t) = -t, p_2(t) = t, \lambda_1 = \lambda_2 = 1, k_1(t,s) = (t-s)$ $, k_2(t,s) = t - s, G_1(s,u(s)) = G_2(s,u(s)) = u(s)$, and the exact solution is $u(t) = 1 + t - t^2$. For computational details and numerical implementation of the proposed Tau method, we take $m = 2, k = 1$, so the following simple matrices in the case of BMS functions will be obtained

$$\Phi = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_{11} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M_{12} = \begin{bmatrix} \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{3} & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & -2 \end{bmatrix}, \quad \Pi = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix},$$

$$\tilde{F}^T = \begin{bmatrix} \tilde{F}_0 & \tilde{F}_1 & \tilde{F}_2 \end{bmatrix} = \begin{bmatrix} \frac{17}{12} & \frac{-13}{6} & \frac{-1}{2} \end{bmatrix}, \quad d^T = \begin{bmatrix} d_1 & d_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

and using the given relations, we obtain the system of equations follows

$$\begin{cases} \tilde{u}_{0,0} = 1, \\ -\tilde{u}_{0,1} - 2\tilde{u}_{0,2} = 1, \\ \frac{3}{2}\tilde{u}_{0,0} - \frac{1}{3}\tilde{u}_{0,1} - \frac{1}{4}\tilde{u}_{0,2} = \frac{17}{12}, \end{cases}$$

with the exact solution, $\tilde{u}_{0,0} = 1, \tilde{u}_{0,1} = 1, \tilde{u}_{0,2} = -1$.

Thus we have $\tilde{\mathbf{u}} = [1,1,-1]^T$. Using the computation $\mathbf{u} = (\Phi^T)^{-1}\tilde{\mathbf{u}}$, we can be given the approximate solution as $\mathbf{u}^T \phi(t)$ that it is $1 + t - t^2$, which is the exact solution of this example.

Example 4.2. Consider the first-order nonlinear VFHIDE [3]

$$u'(t) - u(t) + 2\int_0^t tse^{-u^2(s)} ds = 1 - te^{-t^2}, \quad 0 \le t \le 1$$

with the initial condition $u(0) = 0$.

In this example we have $n_d = 1, p_0(t) = -1, p_1(t) = 1, \lambda_1 = 2, \lambda_2 = 0, k_1(t,s) = ts, G_1(s,u(s)) =$ $e^{-u^2(s)}$ and the exact solution is $u(t) = t$. By applying the Tau method for $m = 1, k = 1$, will be obtained

$$\Phi=\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}, M_{11}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, B=\begin{bmatrix}1\\ 0\end{bmatrix}, \Pi=\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}, \tilde{F}^T=\begin{bmatrix}\tilde{F}_0 & \tilde{F}_1\end{bmatrix}=\begin{bmatrix}1 & -1\end{bmatrix}, d^T=[d_0]=[0],$$

and

$$\begin{cases}\tilde{u}_{0,0}=0,\\ -\tilde{u}_{0,0}+\tilde{u}_{0,1}=1.\end{cases}$$

Thus, we can obtain the approximate solution as $\mathbf{u}^T\phi(t)=t$, which is the exact solution of this example.

Example 4.3. Consider the second-order nonlinear VFHIDE [13]

$$u''(t)+tu'(t)-tu(t)-\int_0^1 \sin(t)e^{-2s}u^2(s)ds=e^t-\sin(t), \quad 0\le t\le 1,$$

with the initial conditions $u(0)=u'(0)=1$. The exact solution of this example is $u(t)=e^t$.

We solve this example by using the Tau method. The comparison between the present method and method of [13] is shown in Table 2. As we see in this table, it is clear that the result obtained by the present method is very superior to that by the method of [13]. Also, the result for $m=7, k=2$, in this table will be presented. As we observed in this table with increasing the value of m, k, particular m, the resultant accuracy increased as well. Figure 3.

Example 4.4. Consider the first-order nonlinear problem [25]

$$u'(t)-\int_0^1 \frac{1}{5}e^{t+s}u(s)\cos(u(s))ds=1-\frac{e^{t+1}\cos(1)}{10}, \quad 0\le t\le 1,$$

with the initial condition $u(0)=0$. The exact solution of this example is $u(t)=t$.

The absolute difference errors for $m=2,3,4, k=2$ in Table 1 are being observed. In addition the last columns of this table indicates the existed result in [25]. As you can observe in the presented method the less basic function the more accuracy with respective method [25], can be seen. Figure 1 shows a plot of the exact and approximate solutions of this example for $m=2, k=2$ in (c) and display a plot of the absolute difference errors of this example for the variant value of m, k, in (d).

Table 1. Absolute errors of Example 4.4

t	Present method			Method of [25]	
	m=2	m=3	m=4	j=9	j=17
	k=2				
0.0	0	0	0	0	0
0.1	6.51×10^{-7}	1.74×10^{-8}	9.47×10^{-11}	1.38×10^{-5}	2.99×10^{-6}
0.2	2.75×10^{-6}	7.32×10^{-8}	3.99×10^{-10}	2.52×10^{-5}	5.59×10^{-6}
0.3	6.52×10^{-6}	1.73×10^{-7}	9.46×10^{-10}	3.59×10^{-5}	9.77×10^{-6}
0.4	1.21×10^{-5}	3.25×10^{-7}	1.77×10^{-9}	4.86×10^{-5}	1.29×10^{-5}
0.5	2.03×10^{-5}	5.36×10^{-7}	2.92×10^{-9}	6.82×10^{-5}	1.73×10^{-5}
0.6	3.06×10^{-5}	8.16×10^{-7}	4.44×10^{-9}	9.06×10^{-5}	2.23×10^{-5}
0.7	4.39×10^{-5}	1.17×10^{-6}	6.39×10^{-9}	1.09×10^{-4}	2.66×10^{-5}
0.8	6.08×10^{-5}	1.62×10^{-6}	8.83×10^{-9}	1.26×10^{-4}	3.34×10^{-5}
0.9	8.15×10^{-5}	2.17×10^{-6}	1.18×10^{-8}	1.47×10^{-4}	3.87×10^{-5}

Table 2. Absolute errors of Example 4.3

t	Present method			Method of [13]	Present method
	m=4		m=5	m=5	m=7
	k=2	k=3	k=3	k=5	k=2
0.0	0	0	0	0	0
0.2	7.03×10^{-8}	4.80×10^{-10}	1.41×10^{-10}	4.00×10^{-7}	6.20×10^{-13}
0.4	3.05×10^{-7}	6.99×10^{-8}	2.12×10^{-10}	8.10×10^{-6}	7.41×10^{-12}
0.6	3.89×10^{-6}	2.46×10^{-7}	8.67×10^{-9}	7.73×10^{-5}	2.54×10^{-11}
0.8	8.98×10^{-6}	2.14×10^{-8}	8.24×10^{-10}	4.24×10^{-4}	7.34×10^{-11}

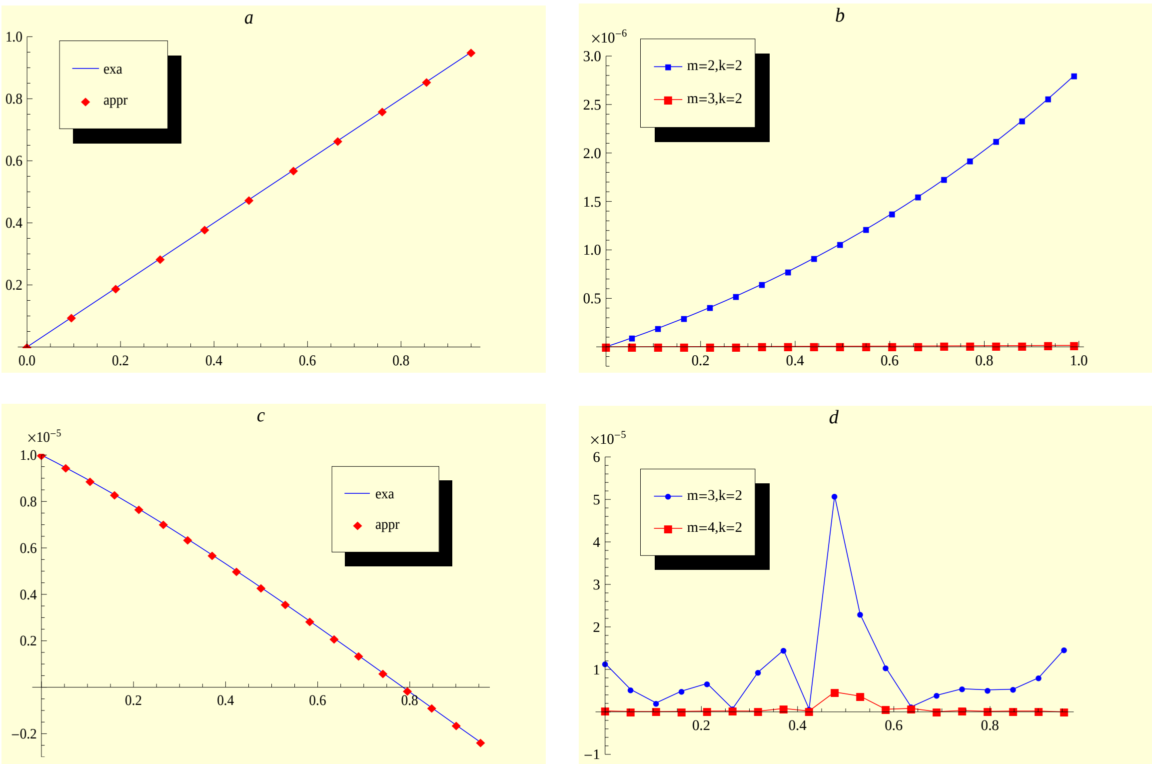

Figure 1: (a ,c) The exact and the approximate solution of Example 4.4 and 4.5, respectively; (b, d) The absolute difference errors of Example 4.4 and 4.5 for variant value of m, k, respectively;

Table 3. Exact solutions and approximate solutions of Example 4.5

t	Exact solution	Present method		Method of [4]	
		m=3	m=4	m=8	
		k=2		k=4	k=8
0.1	0.895170	0.895170	0.895170	0.894912	0.895186
0.3	0.659816	0.659818	0.659816	0.659114	0.659732
0.5	0.398157	0.398172	0.398152	0.397870	0.398169
0.7	0.120624	0.120621	0.120624	0.120360	0.120671
0.9	-0.161716	-0.161723	-0.161716	-0.161466	-0.161638

Example 4.5. Consider the first-order nonlinear VFHIDE [4]

$$u'(t)-\int_0^t \cos(t-s)u^2(s)ds=-2\sin(t)-\frac{\cos(t)}{3}-\frac{2\cos(2t)}{3},\quad 0\le t\le 1,$$

with the initial condition $u(0)=1$. The exact solution is $u(t)=cos(t)-sin(t)$.

Table 3 shows the approximate solutions and exact solutions of the present method and the methods of

[4].We display a plot of the approximate and exact solutions of this example for $m = 3, k = 2$, in Figure 1(c) and a plot of absolute difference errors of this example for the variant value of m, k in Figure 1(d).

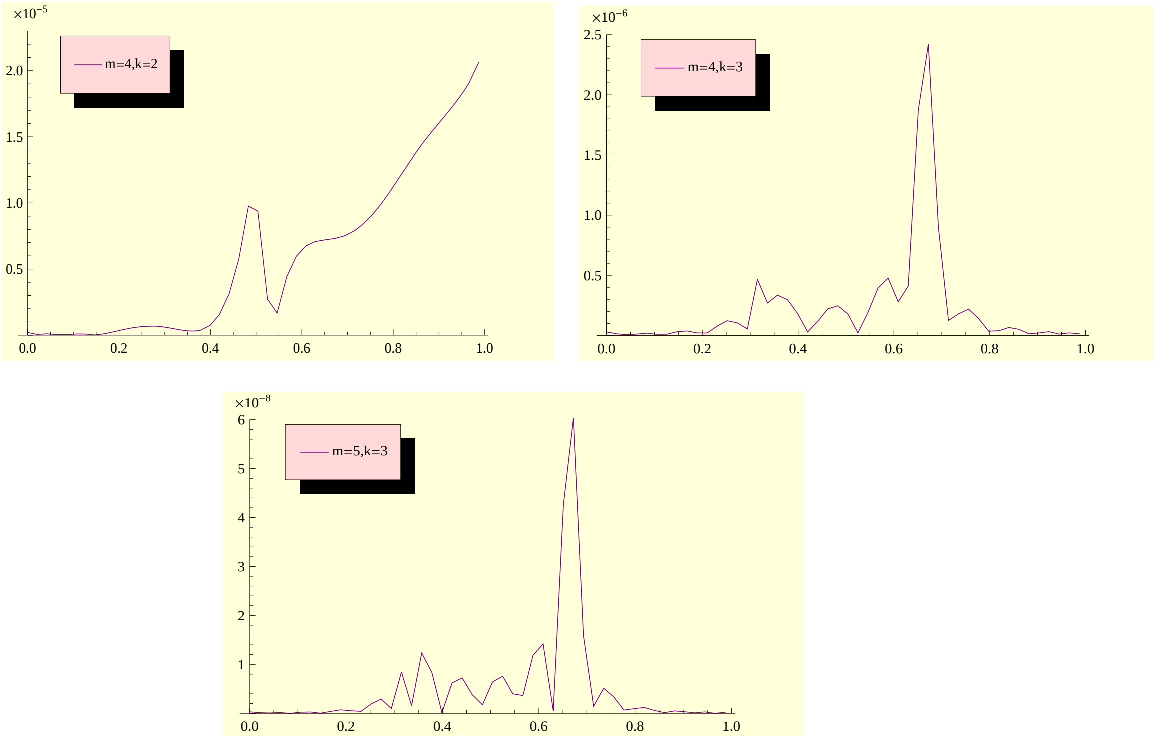

Figure 2: The comparison between absolute errors of Example 4.3 for some m and k;

5. Conclusion

Our results indicate that the Tau method with BMS basis functions can be regarded as a structurally simple algorithm that is conventionally applicable to the numerical solution of VFHIDEs. In fact in this method, by using the Tau operational matrices the mentioned equations was converted to a system of nonlinear algebraic equations. The advantages of this method are as (1) It solves nonlinear VFHIDEs without linearizing the nonlinear terms. (2) It improves accuracy by increasing m, k,(particular m) reasonably. (3) the time of calculations is short.

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