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An Improvement on the Hardy-Hilbert Integral Type Inequality

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Abstract. In this paper, a new Hardy-Hilbert integral inequality is obtained by proving an inequality on weight coefficients.

Keywords: Hardy-Hilbert inequality, Hölder inequality, weight coefficient.

1. Introduction

In this paper, we study the famous Hardy Hilbert inequality. $f \in L^p$ denotes a measurable function f satisfying $\int_0^{+\infty} |f(x)|^p dx < \infty$,where $p > 0$. If $f \in L^p$, $g \in L^p$, $f \geq 0$, $g \geq 0$, $p^{-1} + q^{-1} = 1$, $p > 1$, $q > 1$, then we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{x+y+1} dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^{+\infty} f^p(x)dx\right)^{1/p} \left(\int_0^{+\infty} g^q(y)dy\right)^{1/q}. \quad (1)$$

Inequality (1) is called the Hardy Hilbert integral inequality, that is , as follows:

$$\int_0^{+\infty}\left(\int_0^{+\infty} \frac{f(x)}{x+y+1} dx\right)^p dy < \left(\frac{\pi}{\sin(\pi/p)}\right)^p \int_0^{+\infty} f^p(x)dx. \quad (2)$$

In 2003, Yang[1] obtained the following result.

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{x+y+1} dxdy$$

$$< \left(\int_0^{+\infty}\left[\frac{\pi}{\sin(\pi/p)} - \frac{1}{13(x+1)(2x+1)^{1/p}}\right] f^p(x)dx\right)^{1/p} \left(\int_0^{+\infty}\left[\frac{\pi}{\sin(\pi/q)} - \frac{1}{13(y+1)(2y+1)^{1/p}}\right] g^q(y)dy\right)^{1/q}. \quad (3)$$

In the special case $p = q = 2$,

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{x+y+1} dxdy$$

$$< \left(\int_0^{+\infty}\left[\pi - \frac{1}{13(x+1)(2x+1)^{1/2}}\right] f^2(x)dx\right)^{1/2} \left(\int_0^{+\infty}\left[\pi - \frac{1}{13(y+1)(2y+1)^{1/2}}\right] g^2(y)dy\right)^{1/2}. \quad (4)$$

In the paper, we obtain an improvement form of the above by means of the Hölder inequality and related lemmas.

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2. New Results and their Proofs

2.1. Some lemmas

Lemma 1 Let $r>1$, $y>0$, then we have the foloeing inequality on weight coefficients:

$$\omega(y,r)\overset{def}{=}\int_0^{+\infty}\frac{1}{x+y+1}\left(\frac{2y+1}{2x+1}\right)^{1/r}dx<\frac{\pi}{\sin(\pi/r)}-\frac{2}{3(r-1)(2y+1)^{1-1/r}}. \tag{5}$$

Proof: Let $f_y(x)=\frac{1}{x+y+1}\left(\frac{2y+1}{2x+1}\right)^{1/r}$, then

$$\int_0^{+\infty}f_y(x)dx=\int_{\frac{1}{2y+1}}^{+\infty}(1+u)^{-1}u^{-1/r}du=\frac{\pi}{\sin(\pi/r)}-\int_0^{\frac{1}{2y+1}}(1+u)^{-1}u^{-1/r}du,$$

$$\int_0^{\frac{1}{2y+1}}(1+u)^{-1}u^{-1/r}du>\frac{1}{(2y+1)^{1-1/r}}\left[\frac{r}{r-1}\frac{2y+1}{2(y+1)}+\frac{r^2}{(r-1)(2r-)}\frac{2y+1}{4(y+1)^2}\right],$$

$$\omega(y,r)<\frac{\pi}{\sin(\pi/r)}-\frac{1}{(2y+1)^{1-1/r}}\left[\left(\frac{r}{r-1}-1-\frac{1}{3r}\right)\frac{2y+1}{2(y+1)}+\frac{r^2}{(r-1)(2r-1)}\frac{2y+1}{4(y+1)^2}-\frac{2y+1}{12(y+1)^2}\right],$$

where,

$$\left(\frac{r}{r-1}-1-\frac{1}{3r}\right)\frac{2y+1}{2(y+1)}=\frac{2}{3(r-1)}+\frac{2y+1}{6r(r-1)(y+1)}-\frac{1}{3(r-1)(y+1)},$$

$$\frac{r^2}{(r-1)(2r-1)}\frac{2y+1}{4(y+1)^2}>\frac{1}{4(y+1)}+\frac{3}{8(r-1)(y+1)}-\frac{1}{8(y+1)^2}-\frac{3}{16(r-1)(y+1)^2},$$

$$\frac{2y+1}{12(y+1)^2}=\frac{1}{6(y+1)}-\frac{1}{12(y+1)^2},$$

Hence,

$$\omega(y,r)<\frac{\pi}{\sin(\pi/r)}-\frac{1}{(2y+1)^{1-1/r}}\left[\frac{2}{3(r-1)}+\frac{2y+1}{6r(r-1)(y+1)}+\frac{2r-1}{24(r-1)(y+1)}-\frac{2r+7}{48(r-1)(y+1)^2}\right],$$

by computation, we obtain

$$\frac{2y+1}{6r(r-1)(y+1)}+\frac{2r-1}{24(r-1)(y+1)}-\frac{2r+7}{48(r-1)(y+1)^2}>\frac{6r^2-11r+48}{48r(r-1)(y+1)^2}>0.$$

So we complete the proof of lemma 1.

Lemma 2 (H ö lder inequality)[2] If $f\in L^p(E)$, $g\in L^q(E)$, E is measurable set, $p>1$, $p^{-1}+q^{-1}=1$, then $fg\in L^1(E)$ and

$$\int_E|f(x)g(x)|dx\le\left(\int_E|f(x)|^p dx\right)^{1/p}\left(\int_E|g(x)|^q dx\right)^{1/q}.$$

2.2. Main results

Theorem 1 If $f\in L^p$, $g\in L^p$, $f\ge 0$, $g\ge 0$, $p^{-1}+q^{-1}=1$, $p>1$, $q>1$, then we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{x+y+1}dxdy$$

$$<\left(\int_0^{+\infty}\left[\frac{\pi}{\sin(\pi/q)}-\frac{2}{3(q-1)(2x+1)^{1/q}}\right]f^{p}(x)dx\right)^{1/p}\left(\int_0^{+\infty}\left[\frac{\pi}{\sin(\pi/p)}-\frac{2}{3(p-1)(2y+1)^{1/p}}\right]g^{q}(y)dy\right)^{1/q}. \qquad (6)$$

Proof: By lemma 2, we get

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{x+y+1}dxdy$$

$$<\left(\int_0^{+\infty}\int_0^{+\infty}\left[\left(\frac{2x+1}{2y+1}\right)^{\frac{1}{pq}}\frac{f(x)}{(x+y+1)^{1/p}}\right]^{p}dxdy\right)^{1/p}\left(\int_0^{+\infty}\int_0^{+\infty}\left[\left(\frac{2y+1}{2x+1}\right)^{\frac{1}{pq}}\frac{g(y)}{(x+y+1)^{1/q}}\right]^{q}dxdy\right)^{1/q}$$

$$=\left(\int_0^{+\infty}\int_0^{+\infty}\left[\left(\frac{2x+1}{2y+1}\right)^{\frac{1}{q}}\frac{1}{(x+y+1)}\right]dyf^{p}(x)dx\right)^{1/p}\left(\int_0^{+\infty}\int_0^{+\infty}\left[\left(\frac{2y+1}{2x+1}\right)^{\frac{1}{p}}\frac{1}{(x+y+1)}\right]dxg(y)dy\right)^{1/q}$$

$$=\left(\int_0^{+\infty}\omega(x,q)f^{p}(x)dx\right)^{1/p}\left(\int_0^{+\infty}\omega(y,p)f^{q}(x)dx\right)^{1/q}.$$

Reminding that $\frac{\pi}{\sin(\pi/p)}=\frac{\pi}{\sin(\pi/q)}$ holds for $p^{-1}+q^{-1}=1$, and by lemma 1, we complete the proof of theorem 1.

Theorem 2 If $f\in L^{p}$, $f\geq 0$, $p^{-1}+q^{-1}=1$, $p>1$, $q>1$, then we have

$$\int_0^{+\infty}\left(\int_0^{+\infty}\frac{f(x)}{x+y+1}dx\right)^{p}dy<\left(\frac{\pi}{\sin(\pi/p)}\right)^{p-1}\int_0^{+\infty}\left[\frac{\pi}{\sin(\pi/p)}-\frac{2}{3(q-1)(2x+1)^{1/p}}\right]f^{p}(x)dx. \qquad (7)$$

Proof: By lemma 2, we get

$$\int_0^{+\infty}\frac{f(x)}{x+y+1}dx\leq\left(\int_0^{+\infty}\frac{f^{p}(x)}{x+y+1}\left(\frac{2x+1}{2y+1}\right)^{1/q}dx\right)^{1/p}\left(\int_0^{+\infty}\frac{1}{x+y+1}\left(\frac{2y+1}{2x+1}\right)^{1/p}dx\right)^{1/q}$$

$$=\left(\int_0^{+\infty}\frac{f^{p}(x)}{x+y+1}\left(\frac{2x+1}{2y+1}\right)^{1/q}dx\right)^{1/p}\omega^{\frac{1}{q}}(y,p),$$

where $\omega(y,p)<\frac{\pi}{\sin(\pi/p)}$, by lemma 1.

Thus we get

$$\int_0^{+\infty}\left(\int_0^{+\infty}\frac{f(x)}{x+y+1}dx\right)^{p}dy\leq\int_0^{+\infty}\int_0^{+\infty}\frac{f^{p}(x)}{x+y+1}\left(\frac{2x+1}{2y+1}\right)^{1/q}\omega^{\frac{p}{q}}(y,p)dxdy$$

$$<\int_0^{+\infty}\int_0^{+\infty}\frac{f^{p}(x)}{x+y+1}\left(\frac{2x+1}{2y+1}\right)^{1/q}\left[\frac{\pi}{\sin(\pi/p)}\right]^{p/q}dxdy$$

$$=\left[\frac{\pi}{\sin(\pi/p)}\right]^{p/q}\int_0^{+\infty}\int_0^{+\infty}\omega(x,q)dyf^{p}(x)dx,$$

by lemma 1 again, we complete the proof of theorem 2.

Remark: Taking $p = q = 2$ in (6), we obtain

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{x+y+1}dxdy$$

$$<\left(\int_0^{+\infty}\left[\pi-\frac{2}{3(2x+1)^{1/2}}\right]f^2(x)dx\right)^{1/2}\left(\int_0^{+\infty}\left[\pi-\frac{2}{3(2y+1)^{1/2}}\right]g^2(y)dy\right)^{1/2},$$

which is an improvement of (4).

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