The Hesitant Fuzzy Weighted OWA Operator and Its Application In Multiple Attribute Decision Making

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Abstract. The weighted ordered weighted averaging(WOWA) operator introduced by Torra is an important aggregation technique, which includes the famous weighted averaging(WA) operator and the ordered weighted averaging(OWA) operator as special cases. In this paper, we introduce a new hesitant fuzzy decision making technique called the hesitant fuzzy weighted OWA(HFWOWA) operator. It is an extension of the WOWA operator with the uncertain information represented as hesitant fuzzy numbers.It is shown that many existing hesitant fuzzy aggregation operators are the special cases of our proposed operator. We also study some of its main properties, such as commutativity, monotonicity and boundary.Moreover, an approach is proposed for multiple attribute decision making based on the proposed operator. Finally, an example is given to illustrate the developed method.

Keywords: hesitant fuzzy sets, multiple attribute decision making, HFWOWA operator.

1. Introduction

As an important extension of fuzzy set(FS), hesitant fuzzy set(HFS)[1,2] proposed by Torra and Narukawa, which allows the membership of an element to a set represent by several possible values, is a powerful tool to express people's hesitancy in real applications.

Since its appearance, the HFS has received more and more attention from many researchers in applied mathematics, computing science, management science and many others, and fruitful research works have been published about the HFS theory. For example, Xu and Xia[3] presented a lot of distance measures and similarity measures for HFSs. Based on the intuitionistic fuzzy entropy measures, Xu[4] proposed a variety of entropy measures for HFSs. Xia and Xu[5] developed a series of aggregation operators for hesitant fuzzy information, and discussed the relationship between the intuitionistic fuzzy set and the hesitant fuzzy set. Zhu et al.[6] developed some hesitant fuzzy geometric Bonferroni means to aggregate the hesitant fuzzy information.

On the other hand, the weighted averaging(WA) operator and the ordered weighted averaging(OWA) operator are two well-known aggregation techniques. The difference of the two operators is that the WA operator allows to weight each information source in relation to their reliability whereas the OWA operator allows to weight the values according to their ordering[7]. Therefore, weights represents different aspects in both the WA and OWA operators. However, both the operators consider only one of them. To solve this drawback, Torra[8] introduced the weighted ordered weighted averaging(WOWA) operator which can deal with the situation where both the importance of information sources and the importance of values have to be taken into account. Usually, when using the WOWA operator, it is assumed that the available information is clearly known and can be assessed with exact numbers. However, due to the increasing complexity of economic environment, it is very difficult for a decision maker to express his/her preferences over alternatives with exact numbers. In such cases, the fuzzy numbers may be more suitable. Therefore, in this paper, we will extend the famous WOWA operator to the hesitant fuzzy environment, and propose the hesitant fuzzy WOWA(HFWOWA) operator. It is a new hesitant fuzzy aggregation operator that uses the main characteristics of the WOWA operator and uncertain information represented in the form of hesitant

fuzzy numbers. We also present a generalization of the HFWOWA operator by using the generalized mean. The generalization includes the HFWOWA operator and many other famous operators as special cases.

In order to do so, the remainder of this paper is organized as follows: In Section 2, we introduce some basic concepts related to HFSs and some well-known aggregation operators. In Section 3, we develop the hesitant fuzzy weighted OWA(IFWOWA) operator, and study its desirable properties and its special cases. We also give the two generalizations of the IFWOWA operator in this section. In Section 4, we shall apply the proposed operator to deal with MADM under hesitant fuzzy environments. In Section 5, an illustrative example is given to verify the proposed method, and some conclusions are given in the last section.

2. Preliminaries

This section briefly reviews the hesitant fuzzy set(HFS), the weighted averaging(WA) operator, the ordered weighted averaging(OWA) operator, the weighted ordered weighted averaging(WOWA) operator. Torra and Narukawa[1,2] gave the definition of HFS as follows.

Definition 1 [1,2]. Let X be a fixed set, and a hesitant fuzzy set(HFS) on X is in term of a function that when applied to X returns a subset of [0,1], which can be represented as the following mathematical symbol:

$$E = \{\langle x, h(x)\rangle \mid x \in X\} \tag{1}$$

where $h(x)$ is a set of some values in [0,1], denoting the possible membership degrees of the element $x \in X$ to the set E. For convenience, we call $h(x)$ a hesitant fuzzy element(HFE) and H the set of all the HFEs.

Definition 2 [5]. For a HFE h, $s(h) = \dfrac{1}{\#h}\sum_{\gamma \in h}\gamma$ is called the score function of h, where $\#h$ is the number of the elements in h. Moreover, for two HEFs h_1 and h_2, if $s(h_1) > s(h_2)$, then $h_1 > h_2$; if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

Let h, h_1 and h_2 be three HFEs, then the operational laws on the HFEs are given as follows.

(1) $h^\lambda = \cup_{\gamma \in h}\{\gamma^\lambda\}$.

(2) $\lambda h = \cup_{\gamma \in h}\{1 - (1-\gamma)^\lambda\}$.

(3) $h_1 \oplus h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$.

(4) $h_1 \otimes h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\gamma_1\gamma_2\}$.

The weighted averaging(WA) operator is a basic aggregation technique, which is defined as follows:

Definition 3. Let $a_i\,(i=1,2,\ldots,n)$ be a collection of real numbers. If

$$\mathrm{WA}(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} p_i a_i, \tag{2}$$

where $p = (p_1, p_2, \ldots, p_n)^\bullet$ is the weighting vector of $a_i\,(i=1,2,\ldots,n)$, such that $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0,1]$, then the mapping WA is called the weighted averaging(WA) operator.

Since its appearance, the ordered weighted averaging(OWA) operator, which was introduced by Yager[9] in 1988, has received more and more attention.

Definition 4. [9] An OWA operator of dimension n is a mapping OWA: $R^n \to R$, defined by an associated weighting vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^\bullet$, such that $\sum_{i=1}^{n} \omega_i = 1$ and $\omega_i \in [0,1]$, according to the following formula:

$$\mathrm{OWA}(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} \omega_i a_{\sigma(i)}, \tag{3}$$

where $a_{\sigma(i)}$ is the i th largest number in $(a_1, a_2, \ldots, a_n)$.

Note that the characteristic of the OWA operator is that it weights the ordered position of each datum, while the WA operator weights only the input datum. Therefore, weights represents different aspects in both the OWA and WA operators. However, both the operators consider only one of them. To solve this issue, Torra[8] proposed the following weighted ordered weighted averaging(WOWA) operator.

Definition 5. [8] Let $a_i(i=1,2,\ldots,n)$ be a collection of real numbers with the ordered position's weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ such that $\omega_i > 0(i=1,2,\ldots,n)$ and $\sum_{i=1}^{n} \omega_i = 1$, and the input datum's weight vector $p = (p_1, p_2, \ldots, p_n)$ such that $p_i > 0(i=1,2,\ldots,n)$ and $\sum_{i=1}^{n} p_i = 1$. Then the mapping WOWA: $R^n \rightarrow R$, defined by

$$\text{WOWA}(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} v_i a_{\sigma(i)}, \qquad (4)$$

is called the weighted OWA(WOWA) operator, where $a_{\sigma(i)}$ is the i th largest element in $(a_1, a_2, \ldots, a_n)$, and the weight v_i is defined by

$$v_i = f(\sum_{j \le i} p_{\sigma(j)}) - f(\sum_{j \le i-1} p_{\sigma(j)}), \qquad (5)$$

where f is a nondecreasing function that interpolates the points $(i/n, \sum_{j \le i} \omega_j)$ together with the point $(0,0)$.

Remark 1. If $p = (1/n, 1/n, \ldots, 1/n)^{\centerdot}$, then from (5), we have that the WOWA operator is reduced to the ordered weighted averaging(OWA) operator.

Remark 2. If $\omega = (1/n, 1/n, \ldots, 1/n)^{\centerdot}$, then from (5) again, we have that the WOWA operator is reduced to the weighted averaging(WA) operator.

3. The hesitant fuzzy weighted OWA operator

The WOWA operator fulfills many interesting properties, and is applicable in a wide range of situations such as in economics, statistics, decision making et al. However, the WOWA operator is mainly used to the data taking the form of exact numbers. In the following, we shall extend the WOWA operator to accommodate the situations in which the input data is represented in the form of HFEs.

Definition 6. Let $h_i(i=1,2,\ldots,n)$ be a collection of HFEs with the ordered position's weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ such that $\omega_i > 0(i=1,2,\ldots,n)$ and $\sum_{i=1}^{n} \omega_i = 1$, and the input datum's weight vector $p = (p_1, p_2, \ldots, p_n)$ such that $p_i > 0(i=1,2,\ldots,n)$ and $\sum_{i=1}^{n} p_i = 1$. Then the mapping HFWOWA: $H^n \rightarrow H$, defined by

$$\text{HFWOWA}(h_1, h_2, \ldots, h_n) = \overset{n}{\underset{i=1}{\oplus}} v_i h_{\sigma(i)}, \qquad (6)$$

is called the hesitant fuzzy weighted OWA(HFWOWA) operator, where $h_{\sigma(i)}$ is the i th largest element in $(h_1, h_2, \ldots, h_n)$, and the weight v_i is defined by Eq.(5).

On the basis of the operational laws of the HFEs, we can derive the following theorem.

Theorem 1. Let $h_i(i=1,2,\ldots,n)$ be a collection of HFEs, then the aggregated value by using the HFWOWA operator is also an HFE, and

$$\text{HFWOWA}(h_1, h_2, \ldots, h_n) = \bigcup_{\gamma_{\sigma(i)} \in h_{\sigma(i)}(i=1,2,\ldots,n)} \left\{ 1 - \prod_{i=1}^{n} \left(1 - \gamma_{\sigma(i)}\right)^{v_i} \right\}. \qquad (7)$$

Proof. The proof is straightforward.

Now we give an example about the HFWOWA operator.

Example 1. Let $h_1 = \{0.2, 0.3, 0.5\}, h_2 = \{0.4, 0.6\}$ be two HFEs, and the weight vectors $\omega = (0.7, 0.3)$ and $p = (0.4, 0.6)^\bullet$. Then, from (5), the definition of v_i, we have $v = (0.76, 0.24)^\bullet$. From Eq.(7), we get

$$HFWOWA(h_1, h_2) = \{0.3571, 0.3774, 0.4257, 0.5276, 0.5425, 0.5780\}.$$

The HFWOWA operator is commutative, monotone and bounded.

Theorem 2. (Commutativity) Let $h_i (i = 1, 2, \ldots, n)$ be a collection of HFEs, then

$$HFWOWA(h_1, h_2, \ldots, h_n) = HFWOWA(h_{1'}, h_{2'}, \ldots, h_{n'}),$$

where $h_{1'}, h_{2'}, \ldots, h_{n'}$ is any permutation of $h_1, h_2, \ldots, h_n$.

Proof. Because $h_{\sigma(i)} = h_{\sigma(i)'} (i = 1, 2, \ldots, n)$, then from Definition 6, the above conclusion is right.

Theorem 3. (Monotonicity)

Let $h_i (i = 1, 2, \ldots, n)$ and $f_i (i = 1, 2, \ldots, n)$ be two collections of HFEs. If for all i, $\zeta_i \le \gamma_i$, where ζ_i is any element of hesitant fuzzy set h_i, and γ_i is any element of hesitant fuzzy set f_i, then

$$HFWOWA(h_1, h_2, \ldots, h_n) \le HFWOWA(f_1, f_2, \ldots, f_n).$$

Proof. According to Definition 6 and Eq. (7), we get

$$HFWOWA(h_1, h_2, \ldots, h_n) = \bigcup_{\zeta_{\sigma(i)} \in h_{\sigma(i)} (i=1,2,\ldots,n)} \left\{ 1 - \prod_{i=1}^{n} \left(1 - \zeta_{\sigma(i)} \right)^{v_i} \right\}$$

$$\le \bigcup_{\gamma_{\sigma(i)} \in f_{\sigma(i)} (i=1,2,\ldots,n)} \left\{ 1 - \prod_{i=1}^{n} \left(1 - \gamma_{\sigma(i)} \right)^{v_i} \right\} := HFWOWA(f_1, f_2, \ldots, f_n)$$

This completes the proof.

Theorem 4. (Boundary) Let $h_i (i = 1, 2, \ldots, n)$ be a collections of HFEs on X, and $\gamma_i^- = \min\{\gamma_i : \gamma_i \in h_i\}$, $\gamma_i^+ = \max\{\gamma_i : \gamma_i \in h_i\} (i = 1, 2, \ldots, n)$. Set $h^- = \{h_1^-, h_2^-, \ldots, h_n^-\}$, $h^- = \{h_1^+, h_2^+, \ldots, h_n^+\}$. Then

$$HFWOWA(h^-, h^-, \ldots, h^-) \le HFWOWA(h_1, h_2, \ldots, h_n) \le HFWOWA(h^+, h^+, \ldots, h^+).$$

Proof. Since $\gamma_i^- \le \gamma_i \le \gamma_i^+$ for all i, then by Theorem 3, we have that the above two inequalities hold.

Remark 3. If $p = (1/n, 1/n, \ldots, 1/n)^\bullet$, then from (6), we have that the HFWOWA operator is reduced to the following hesitant fuzzy ordered weighted averaging(HFOWA) operator:

$$HFOWA(h_1, h_2, \ldots, h_n) = \overset{n}{\underset{i=1}{\oplus}} \omega_i h_{\sigma(i)}.$$

Remark 4. If $\omega = (1/n, 1/n, \ldots, 1/n)^\bullet$, then from (6) again, we have that the HFWOWA operator is reduced to the following hesitant fuzzy weighted averaging(HFWA) operator in [5].

$$HFWA(h_1, h_2, \ldots, h_n) = \overset{n}{\underset{i=1}{\oplus}} p_i h_i.$$

In the following, we will generalize the HFWOWA operator using the generalized average.

Definition 7. Consider the same conditions in Definition 6 and let $\lambda > 0$. Then the mapping GHFWOWA: $H^n \rightarrow H$, defined by

$$GHFWOWA(h_1, h_2, \ldots, h_n) = \left(\overset{n}{\underset{i=1}{\oplus}} v_i h_{\sigma(i)}^\lambda \right)^{1/\lambda}, \tag{8}$$

is called the generalized hesitant fuzzy weighted OWA(GHFWOWA) operator, where $h_{\sigma(i)}$ is the i th largest element in $(h_1, h_2, \ldots, h_n)$.

Theorem 5. Let $h_i (i = 1, 2, \ldots, n)$ be a collection of HFEs, then the aggregated value by using the GHFWOWA operator is also an HFE, and

$$\text{GHFWOWA}(h_1, h_2, \ldots, h_n) = \bigcup_{\gamma_{\sigma(i)} \in h_{\sigma(i)}(i=1,2,\ldots,n)} \left\{ \left(1 - \prod_{i=1}^{n}\left(1 - \gamma_{\sigma(i)}^{\lambda}\right)^{v_i}\right)^{1/\lambda}\right\}.$$

Proof. The proof is omitted here.

Example 2. Consider the same hesitant fuzzy elements in Example 1. Here, we also adopt the same weight vectors ω and p in Example 1. Then if the parameter λ takes different values in $(0,10]$, then we get different aggregated results and different scores. Figure 1 shows the scores of the aggregated results as the parameter λ changes.
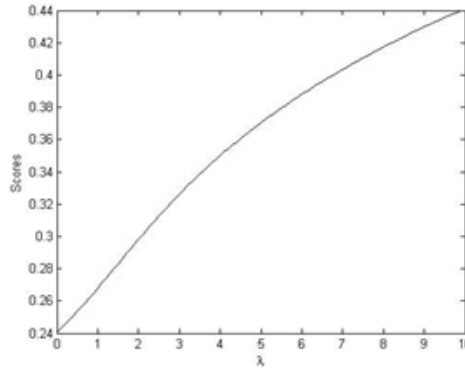

Fig. 1: Scores of aggregated results obtained by GHFWOWA operator Obviously, if $\lambda = 1$, then the GHFWOWA operator is reduced to the HFWOWA operator.

4. Multiple Attribute Decision Making Under Hesitant Fuzzy Environment

In this section, we propose a practical approach to multiple attribute decision making under hesitant fuzzy environment.

We suppose $A = \{A_1, A_2, \ldots, A_m\}$ is a set of m alternatives, and $G = \{G_1, G_2, \ldots, G_n\}$ be the set of n attributes. The ordered position's weigh vector is $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\cdot}$ such that $\omega_i \geq 0$ and $\sum_{i=1}^{n}\omega_i = 1$, and the input datum's weight vector $p = (p_1, p_2, \ldots, p_n)$ such that $p_i > 0(i=1,2,\ldots,n)$ and $\sum_{i=1}^{n}p_i = 1$. The decision makers provide all the possible values that the alternative A_i satisfies the criterion G_j represented by HFE $h_{ij} = \cup_{\gamma_{ij} \in h_{ij}}\{\gamma_{ij}\}$, and all $h_{ij}(i=1,2,\ldots,m; j=1,2,\ldots,n)$ construct the hesitant fuzzy decision matrix $H = (h_{ij})_{m \times n}$.

Based on the HFWOWA operator, we rank the alternatives $A_i(i=1,2,\ldots,m)$ by the following steps.

Step 1. Utilize the HFWOWA operator to aggregate all the performance values $h_{ij}(j=1,2,\ldots,n)$ of the i th line and get the overall performance value h_i corresponding to the alternative $A_i(i=1,2,\ldots,m)$:

$$h_i = \text{HFWOWA}(h_{i1}, h_{i2}, \ldots, h_{in}).$$

Step 2. Utilize the method in Definition 2 to calculate the scores $s(h_i)$ of h_i and rank all the alternatives $A_i(i=1,2,\ldots,m)$ according to $s(h_i)$ in descending order.

Step 3. End.

In the following, we apply our method to a multiple attribute decision making problem.

Example 3. Let us consider a factory which intends to select a new site for new buildings. Three alternatives $A_i(i=1,2,3)$ are available, and the decision makers consider three criteria to decide which site to choose: G_1 (income), G_2 (location), G_3 (environment). The ordered position's weight vector $\omega = (0.2, 0.6, 0.2)^{\cdot}$, and the input datum's weight vector is $p = (0.5, 0.3, 0.2)^{\cdot}$. Assume that the characteristics of the alternatives $A_i(i=1,2,3)$ with respect to the criteria $G_j(j=1,2,3)$ are represented by

HFEs $h_{ij} = \cup_{\gamma_{ij} \in h_{ij}} \{\gamma_{ij}\} (i, j = 1, 2, 3)$, where γ_{ij} indicates the degree that the alternative A_i satisfies the criterion G_j. All $h_{ij}(i, j = 1, 2, 3)$ are contained in the hesitant fuzzy decision matrix $H = (h_{ij})_{3 \times 3}$ (see Table 1).

Table 1. The hesitant fuzzy decision matrix.

	G_1	G_2	G_3
A_1	{0.6, 0.7, 0.8}	{0.25}	{0.4, 0.5}
A_2	{0.4}	{0.4, 0.5}	{0.3, 0.55, 0.6}
A_3	{0.2, 0.4}	{0.5, 0.6}	{0.5, 0.7}

From Eq.(5), we obtain the function

$$f(x) = \begin{cases} 0.6x, 0 \le x < 1/3; \\ 1.2x - 0.2, 1/3 \le x \le 2/3; \\ 0.6x + 0.4, 2/3 < x \le 1. \end{cases}$$

We utilize the HFWOWA operator to aggregate all the performance values $h_{ij}(j = 1, 2, 3)$ of the i th line and get the overall performance value h_i corresponding to the alternative A_i:

$$h_1 = \{0.4689, 0.5081, 0.5266, 0.5615, 0.5975, 0.6272\}.$$
$$h_2 = \{0.3888, 0.4192, 0.4204, 0.4492, 0.4285, 0.4569\}.$$
$$h_3 = \{0.3371, 0.4422, 0.3773, 0.4760, 0.3765, 0.4754, 0.4143, 0.5071\}.$$

then we calculate the scores of all the alternatives according to $h_i (i = 1, 2, 3)$:

$$s(h_1) = 0.5483, s(h_2) = 0.4272, s(h_3) = 0.4257.$$

Since $s(h_1) > s(h_2) > s(h_3)$, then, by Definition 2, we get the ranking of the HFEs: $h_1 > h_2 > h_3$, and thus, the ranking of the alternatives $A_i (i = 1, 2, 3)$ is $A_1 \succ A_2 \succ A_3$. Hence, A_1 is the best alternative.

5. Conclusion

In this paper, we have defined an HFWOWA operator, which has the advantages of the HFWA operator and the HFOWA operator. We also have generalized the HFWOWA operator using the generalized average and utilize the HFWOWA operator to solve the multiple attribute decision making problem with hesitant fuzzy information. Finally, an example is given to illustrate the developed approach.

6.   References

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