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Numerical solution of the nonlinear Fredholm-Volterra-Hammerstein integral equations via Bessel functions

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Abstract. In this paper, a collocation method based on the Bessel polynomials are used for the solution of nonlinear Fredholm-Volterra-Hammerstein integral equations (FVHIEs). This method transforms the nonlinear (FVHIEs) in to matrix equations with the help of Bessel polynomials of the first kind and collocation points. The matrix equations corresponds to a system of nonlinear algebraic equations with the unknown Bessel coefficients. Present results demonstrate proposed method in comparison with other methods is more accurate, efficiency and reliability.

Keywords: Bessel polynomials, Integral equations, Collocation, Fredholm, Volterra, Hammerestein.

1. Introduction

In recent years, many different method have estimated the solution of integral equations. Also, for solution of these equations many analytical and numerical methods have been exited but most of the time numerical methods have been used to solve these equations. Ordokhani [1] has used Walsh-hybrid functions with Newton-Cotes nodes for solving of Fredholm-Hemmerstein integral equations. Authors [2] have solved nonlinear integral equations of Hammerstein type by Chebyshev polynomials. Maleknejad in [3], has used computational method based on Bernstein operational matrics for nonlinear Volterra-Fredholm-Hammerstein integral equations. Babolian and Shahsavaran in [4] have solved the nonlinear Fredholm integral equations of the second kind using Haar wavelets. Yousefi and Razzaghi in [5] have solved nonlinear Volterra-Fredholm integral equations by Legendre wavelers method. Yuzbasi et al. [6], Yuzbasi and Sezer [7], Yuzbasi et al. [8] have worked on the Bessel matrix and collocation methods for the numerical solutions of the neutral delay differential equations, the pantograph equations and the Lane-Emden differential equations. Also, readers who are interested to learn more about this topic could refer to [10 - 15].
Recently, Yazbasi in [16] used Bessel polynomials and Bessel collocation method [8] for solving high-order linear Fredholm-Volterra integro-differential equations. In this article, by Bessel polynomials and Bessel collocation method estimate solution of nonlinear (FVHIEs) to form:

$$y(x) = g(x) + \lambda_1 \int_a^b k_1(x,t)\psi_1(t, y(t))dt + \lambda_2 \int_a^x k_2(x,t)\psi_2(t, y(t))dt, \quad 0 \le a \le x, t \le b, \tag{1}$$

where $y(x)$ is an unknown function, the known functions $g(x)$, $k_1(x,t)$, $k_2(x,t)$, $\psi_1(t, y(t))$ and $\psi_2(t, y(t))$. Also, λ_1 and λ_2 are real or complex constants.

2. Bessel polynomial of first kind

The m-th degree truncated Bessel polynomial of first kind are defined by [16]

$$J_m(x) = \sum_{k=0}^{[\frac{N-m}{2}]} \frac{(-1)^k}{k!(K+m)!}(\frac{x}{2})^{2k+m}, \qquad 0 \le x < \infty, \quad m \in \mathrm{N}, \tag{2}$$

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where N is chosen positive integer so that $N \geq m$ and $m = 0,1,\cdots,N.$ We are transform the Bessel polynomials of first kind to N-th degree Taylor basis functions. In matrix form as

$$J(x) = DX(x), \tag{3}$$

where

$$J(x) = [J_0(x), J_1(x), \cdots, J_N(x)]^T, \qquad X(x) = [1, x, x^2, \cdots, x^N]^T. \tag{4}$$

If N is odd

$$D = \begin{bmatrix} \frac{1}{0!0!2^0} & 0 & \frac{-1}{1!1!2^2} & \cdots & \frac{(-1)^{\frac{N-1}{2}}}{(\frac{N-1}{2})!(\frac{N-1}{2})!2^{N-1}} & 0 \\ 0 & \frac{1}{0!1!2^1} & 0 & \cdots & 0 & \frac{(-1)^{\frac{N-1}{2}}}{(\frac{N-1}{2})!(\frac{N-1}{2})!2^N} \\ 0 & 0 & \frac{1}{0!2!2^2} & \cdots & \frac{(-1)^{\frac{N-3}{2}}}{(\frac{N-3}{2})!(\frac{N+1}{2})!2^{N-1}} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{0!(N-1)!2^{N-1}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{0!N!2^N} \end{bmatrix}_{(N+1)\times(N+1)}.$$

If N is even

$$D = \begin{bmatrix} \frac{1}{0!0!2^0} & 0 & \frac{-1}{1!1!2^2} & \cdots & 0 & \frac{(-1)^{\frac{N}{2}}}{(\frac{N}{2})!(\frac{N}{2})!2^N} \\ 0 & \frac{1}{0!1!2^1} & 0 & \cdots & \frac{(-1)^{\frac{N-2}{2}}}{(\frac{N-2}{2})!(\frac{N}{2})!2^{N-1}} & 0 \\ 0 & 0 & \frac{1}{0!2!2^2} & \cdots & 0 & \frac{(-1)^{\frac{N-2}{2}}}{(\frac{N-2}{2})!(\frac{N+2}{2})!2^N} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{0!(N-1)!2^{N-1}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{0!N!2^N} \end{bmatrix}_{(N+1)\times(N+1)}.$$

3. Fundamental relations

3.1. Matrix relation for the Fredholm integral part

In this section we can approximate the kernel function $k_1(x,t)$ by the truncated Maclaurin series and truncated Bessel series [16], respectively

$$k_1(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} {}_t k_{mn}^1 x^m t^n,$$

$$k_1(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} {}_b k_{mn}^1 J_m(x) J_n(t), \tag{5}$$

Where

$$ {}_t k^1_{mn} = \frac{1}{m!n!}\frac{\partial^{m+n} k_1(0,0)}{\partial x^m \partial t^n}, \qquad m,n = 0,1,\cdots,N. $$

We can write Eqs. (5) to matrix form as

$$ k_1(x,t) = X^T(x) k^1_t X(t), \qquad k^1_t = [{}_t k^1_{mn}], \qquad m,n = 0,1,\cdots,N. \tag{6} $$

$$ k_1(x,t) = J^T(x) k^1_b J(t), \qquad k^1_b = [{}_b k^1_{mn}], \qquad m,n = 0,1,\cdots,N. \tag{7} $$

By substituting Eq. (3) in Eq. (7) and putting equal to Eq. (6), we obtain

$$ k^1_t = D^T k^1_b D, \qquad k^1_b = (D^T)^{-1} k^1_t (D)^{-1}. \tag{8} $$

Now, for solving these equations we need to define $Z_1(t)$ and $Z_2(t)$ as

$$ Z_1(t) = \psi_1(t, y(t)), \tag{9} $$
$$ Z_2(t) = \psi_2(t, y(t)). $$

Also, we approximate $Z_1(t)$, $Z_2(t)$ by Bessel polynomials of first kind and with the help of Eq. (3), we have

$$ Z_1(t) = J^T(t) A_1 = X^T D^T A_1, \tag{10} $$
$$ Z_2(t) = J^T(t) A_2 = X^T D^T A_2, $$

where

$$ A_1 = [a_{10}, a_{11}, \cdots, a_{1N}]^T, \qquad A_2 = [a_{20}, a_{21}, \cdots, a_{2N}]^T. $$

By substituting the matrix forms of Eqs. (7) and (10) in Fredholm integral part of Eq. (1), we get

$$ \int_a^b k_1(x,t)\psi_1(t,y(t))dt = \int_a^b J^T(x) k^1_b J(t) J^T(t) A_1 dt = J^T(x) k^1_b Q_1 A_1, \tag{11} $$

so that

$$ Q_1 = \int_a^b J(t) J^T(t) dt = \int_a^b D X(t) X^T(t) D^T dt = D H_1 D^T, $$

where H_1 the integration of dual operational matrix of Taylor polynomials [17] is define

$$ X(x) = [1, x, x^2, \cdots, x^N]^T, $$

$$ H_1 = \int_a^b X(t) X^T(t) dt, \qquad H_1 = [h^1_{ij}], \qquad i,j = 0,1,\cdots,N, \tag{12} $$

where

$$ h^1_{ij} = \frac{b^{i+j+1} - a^{i+j+1}}{i+j+1}, \qquad i,j = 0,1,\cdots,N, \tag{13} $$

Finally, by substituting Eq. (3) in Eq. (11), we have matrix form of Fredholm integral part

$$ \int_a^b k_1(x,t)\psi_1(t,y(t)) = X^T(x) D^T k^1_b Q_1 A_1. \tag{14} $$

3.2. Matrix relation for the Volterra integral part

We can write kernel function $k_2(x,t)$ such as $k_1(x,t)$ and approximated by truncated Maclaurin series and truncated Bessel series [16]

$$k_2(x,t)=\sum_{m=0}^{N}\sum_{n=0}^{N}{}_t k_{mn}^2 x^m t^n, \tag{15}$$

$$k_2(x,t)=\sum_{m=0}^{N}\sum_{n=0}^{N}{}_b k_{mn}^2 J_m(x)J_n(t),$$

where

$${}_t k_{mn}^2=\frac{1}{m!n!}\frac{\partial^{m+n}k_2(0,0)}{\partial x^m\partial t^n}, \qquad m,n=0,1,\cdots,N.$$

Matrix form as

$$k_2(x,t)=X^T(x)k_t^2X(t), \qquad k_t^2=[{}_t k_{mn}^2], \qquad m,n=0,1,\cdots,N. \tag{16}$$

$$k_2(x,t)=J^T(x)k_b^2J(t), \qquad k_b^2=[{}_b k_{mn}^2], \qquad m,n=0,1,\cdots,N. \tag{17}$$

By substituting Eq. (3) in Eq. (17) and putting equal to Eq. (16), we obtain

$$k_t^2=D^Tk_b^2D, \qquad k_b^2=(D^T)^{-1}k_t^2(D)^{-1}. \tag{18}$$

By using the matrix form of Eqs. (10) and (17) in Volterra integral part of Eq. (1), we have

$$\int_a^x k_2(x,t)\psi_2(t,y(t))dt=\int_a^x J^T(x)k_b^2J(t)J^T(t)A_2dt=J^T(x)k_b^2Q_2(x)A_2, \tag{19}$$

so that

$$Q_2(x)=\int_a^x J(t)J^T(t)dt=\int_a^x DX(t)X^T(t)D^Tdt=DH_2(x)D^T.$$

where $H_2(x)$ the integration of dual operational matrix of Taylor polynomials is defined [17] as

$$H_2(x)=\int_a^x X(t)X^T(t)dt=[h_{ij}^2(x)], \qquad i,j=0,1,\cdots,N,$$

$$h_{ij}^2(x)=\frac{x^{i+j+1}-a^{i+j+1}}{i+j+1}, \qquad i,j=0,1,\cdots,N.$$

By substituting Eq. (3) in Eq. (19), we have matrix form of Volterra integral part

$$\int_a^x k_2(x,t)\psi_2(t,y(t))=X^T(x)MH_2(x)D^TA_2, \qquad M=D^Tk_b^2D. \tag{20}$$

3.3. Method of solution

To solve Eq. (1), we have used Eqs. (14) and (20) as

$$y(x)=g(x)+\lambda_1X^T(x)D^Tk_b^1Q_1A_1+\lambda_2X^T(x)MH_2(x)D^TA_2. \tag{21}$$

Now, by substituting Eqs. (21) in Eq. (9), we get

$$\begin{cases}Z_1(x)=\psi_1(x,g(x)+\lambda_1X^T(x)D^Tk_b^1Q_1A_1+\lambda_2X^T(x)MH_2(x)D^TA_2),\\ Z_2(x)=\psi_2(x,g(x)+\lambda_1X^T(x)D^Tk_b^1Q_1A_1+\lambda_2X^T(x)MH_2(x)D^TA_2).\end{cases} \tag{22}$$

By using Eq. (10) and substituting it in Eq. (22), we obtain

$$\begin{cases} X^T(x)D^T A_1 = \psi_1(x, g(x) + \lambda_1 X^T(x)D^T k_b^1 Q_1 A_1 + \lambda_2 X^T(x)MH_2(x)D^T A_2), \\ X^T(x)D^T A_2 = \psi_2(x, g(x) + \lambda_1 X^T(x)D^T k_b^1 Q_1 A_1 + \lambda_2 X^T(x)MH_2(x)D^T A_2). \end{cases} \quad (23)$$

With the help of Eqs. (23) and collocation points [16] defined by

$$x_i = a + \frac{b-a}{N} i, \qquad i = 0,1,\cdots,N.$$

We have

$$\begin{cases} X^T(x_i)D^T A_1 = \psi_1(x_i, g(x_i) + \lambda_1 X^T(x_i)D^T k_b^1 Q_1 A_1 + \lambda_2 X^T(x_i)MH_2(x_i)D^T A_2), \\ X^T(x_i)D^T A_2 = \psi_2(x_i, g(x_i) + \lambda_1 X^T(x_i)D^T k_b^1 Q_1 A_1 + \lambda_2 X^T(x_i)MH_2(x_i)D^T A_2), \end{cases} \quad (24)$$

where $i = 0,1,\cdots,N.$ Or briefly the fundamental matrix system is

$$\begin{cases} XD^T A_1 = \psi_1(x_i, G + \lambda_1 XD^T k_b^1 Q_1 A_1 + \lambda_2 \overline{X}\,\overline{M}\,\overline{H}\,\overline{D} A_2), \\ XD^T A_2 = \psi_2(x_i, G + \lambda_1 XD^T k_b^1 Q_1 A_1 + \lambda_2 \overline{X}\,\overline{M}\,\overline{H}\,\overline{D} A_2). \end{cases} \quad (25)$$

where $i = 0,1,\cdots,N$ and

$$G = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}, \quad X = \begin{bmatrix} X^T(x_0) \\ X^T(x_1) \\ \vdots \\ X^T(x_N) \end{bmatrix}, \quad D = \begin{bmatrix} D^T \\ D^T \\ \vdots \\ D^T \end{bmatrix}, \quad \overline{M} = \begin{bmatrix} M & 0 & 0 & 0 \\ 0 & M & 0 & 0 \\ 0 & 0 & M & 0 \\ 0 & 0 & 0 & M \end{bmatrix},$$

$$\overline{X} = \begin{bmatrix} X^T(x_0) & 0 & \cdots & 0 \\ 0 & X^T(x_1) & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & X^T(x_N) \end{bmatrix}, \quad \overline{H} = \begin{bmatrix} H_2(x_0) & 0 & \cdots & 0 \\ 0 & H_2(x_1) & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & H_2(x_N) \end{bmatrix}.$$

We can obtain A_1 and A_2 from system of Eq. (25) and with substituting A_1 and A_2 in Eq. (21). Ultimately, we get approximate solution of Eq. (1).

4. Illustrative examples

In this section, we report the results of approximation solution with some examples where given in the different papers. In addition, we have expressed absolute error function which are define as $|y(x) - y_N(x)|$, where $y(x)$ is the exact solution of Eq. (1) and $y_N(x)$ is the approximate of $y(x)$. All the examples were performed on the computer by using a program written in MATLAB.

Example 1. Let us first consider the nonlinear FVHIE [18]

$$y(x) = x^3 + \frac{\cos(1)-1}{3} + \int_0^1 t^2 \sin(y(t))dt, \qquad 0 \le x < 1, \quad (26)$$

The exact solution to Eq. (26) is $y(x) = x^3$. Now, we obtain approximate solutions of this example for

N = 2, 4, 8 by Bessel polynomials. where $k_1(x,t)=t^2$, $g(x)=x^3+\frac{\cos(1)-1}{3}$ and $\lambda_1=1$. Also, the set of collocation points for N = 2 is

$$\left\{x_0=0, x_1=\frac{1}{2}, x_2=1\right\}$$

so that

$$X=\begin{bmatrix}1 & 0 & 0\\ 1 & \frac{1}{2} & \frac{1}{4}\\ 1 & 1 & 1\end{bmatrix},\quad D=\begin{bmatrix}1 & 0 & \frac{-1}{4}\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & \frac{1}{8}\end{bmatrix},\quad k_{mn}^1=\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\quad k_b^1=\begin{bmatrix}0 & 0 & 8\\ 0 & 0 & 0\\ 0 & 0 & 16\end{bmatrix},$$

$$H_1=\begin{bmatrix}1 & \frac{1}{2} & \frac{1}{3}\\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4}\\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5}\end{bmatrix},\quad G=\begin{bmatrix}-0.153232\\ -0.018232\\ 0.846767\end{bmatrix},\quad Q=\begin{bmatrix}0.8458 & 0.2187 & 0.03541\\ 0.2187 & 0.0833 & 0.01562\\ 0.0354 & 0.0156 & 0.00312\end{bmatrix}.$$

Hence, by substituting these matrix in system (25), we obtain Bessel coefficient matrix as

$$A=[-0.002815 \quad -0.687978 \quad 9.488377]^T.$$

Ultimately, by substituting A in Eq. (21) we have approximate solutions of this example for N = 2

$$y_2(x)=x^3-2.818144\times10^{-3}.$$

Similarly for N = 4, 8 we have

$$y_4(x)=x^3-1.31544842\times10^{-3},$$

and

$$y_8(x)=x^3-4.13279\times10^{-6}-(8.923511\times10^{-21})x^8,$$

The absolute error values are given for different values of N in Table 1.

Table 1:Absolute errors of example 1

x	Present method N=2	Present method N=4	Present method N=8	Method of [18] n=15
0.0	2.81×10^{-3}	1.31×10^{-3}	4.13×10^{-6}	1.3×10^{-4}
0.2	2.81×10^{-3}	1.31×10^{-3}	4.13×10^{-6}	1.3×10^{-4}
0.4	2.81×10^{-3}	1.31×10^{-3}	4.13×10^{-6}	1.3×10^{-4}
0.6	2.81×10^{-3}	1.31×10^{-3}	4.13×10^{-6}	1.3×10^{-4}
0.8	2.81×10^{-3}	1.31×10^{-3}	4.13×10^{-6}	1.3×10^{-4}
1.0	2.81×10^{-3}	1.31×10^{-3}	4.13×10^{-6}	1.3×10^{-4}

Example 2. Now Consider the nonlinear FVHIE [19]

$$y(x)=x\cos(x)+\int_0^x x\sin(y(t))dt, \qquad 0\le x<1,$$

The exact solution to this equation is $y(x) = x$. The values obtained in Table 2 show that if the accuracy increases, N will increase.

Table 2: Absolute errors of example 2

x	Present method			Method of [19]
	N=2	N=3	N=4	n=100
0.0	0	0	0	-
0.2	2.07×10^{-4}	7.01×10^{-6}	3.65×10^{-7}	6.85×10^{-7}
0.4	9.63×10^{-4}	1.96×10^{-5}	3.38×10^{-9}	1.46×10^{-6}
0.6	1.58×10^{-3}	1.76×10^{-5}	5.60×10^{-7}	2.44×10^{-6}
0.8	1.40×10^{-3}	3.62×10^{-5}	1.07×10^{-6}	3.57×10^{-6}
1.0	1.04×10^{-3}	9.62×10^{-5}	4.50×10^{-6}	4.76×10^{-6}

Example 3. Consider the nonlinear FVIE [20, 21, 22]

$$y(x) = g(x) + \int_0^1 (x+t)y(t)dt + \int_0^x (x-t)y^2(t)dt, \qquad 0 \le x < 1,$$

The exact solution to this example is $y(x) = x^2 - 2$ and $g(x) = \frac{-1}{30}x^6 + \frac{1}{3}x^4 - x^2 + \frac{5}{3}x - \frac{5}{4}$. The computational result of absolute error for $N = 2, 4$ with the result of other methods are given in Table 3.

Table 3: Absolute errors of example 3

x	Present method		Method of [20]	Method of [21]	Method of [22]
	N=2	N=4	k=16	k=16	n=8,m=8
0.0	6.67×10^{-3}	2.07×10^{-11}	5×10^{-3}	-	8.50×10^{-5}
0.1	7.91×10^{-3}	2.36×10^{-11}	1×10^{-3}	2.4×10^{-3}	1.04×10^{-4}
0.2	9.34×10^{-3}	2.69×10^{-11}	5×10^{-3}	6.8×10^{-3}	1.25×10^{-4}
0.3	1.11×10^{-2}	3.14×10^{-11}	2×10^{-3}	1.95×10^{-2}	1.52×10^{-4}
0.4	1.31×10^{-2}	3.78×10^{-11}	1×10^{-3}	2.78×10^{-2}	1.74×10^{-4}
0.5	1.50×10^{-2}	4.77×10^{-11}	2×10^{-3}	1.66×10^{-2}	1.95×10^{-4}
0.6	1.64×10^{-2}	6.33×10^{-11}	3×10^{-3}	1.89×10^{-2}	2.08×10^{-4}
0.7	1.68×10^{-2}	8.76×10^{-11}	1.2×10^{-2}	2.00×10^{-2}	2.04×10^{-4}
0.8	1.59×10^{-2}	1.24×10^{-10}	1×10^{-3}	2.58×10^{-2}	7.42×10^{-4}
0.9	1.34×10^{-2}	1.80×10^{-10}	5×10^{-3}	6.03×10^{-2}	1.49×10^{-4}
1.0	9.92×10^{-3}	2.60×10^{-10}	6×10^{-3}	-	8.60×10^{-4}

Example 4. Consider the nonlinear FVHIE [3]

$$y(x)=\ln(x)-\frac{x}{4}-\frac{x^3}{6}+\int_0^1 xty^2(t)dt+\int_0^x (x-t)e^{y(t)}dt, \qquad 0\le x<1,$$

The exact solution to this example is $y(x) = \ln(x)$. To solve this equation, we can not use the collocation points defined above, because at this points the $g(x)=\ln(x)-\frac{x}{4}-\frac{x^3}{6}$ function is undefined. So, to solve this equation we used the Newton-Cotes nodes

$$x_i=\frac{2i-1}{2N}, \qquad i=0,1,\cdots,N.$$

The values in Table 4 show that by increasing N, the accuracy of solution will increase.

Table 4: Absolute errors of example 4

x	Present method N=3	Present method N=6	Method of [3] m=16
0.1	1.18×10^{-3}	8.42×10^{-5}	9.79×10^{-4}
0.2	3.63×10^{-3}	1.68×10^{-4}	2.25×10^{-3}
0.3	5.45×10^{-3}	2.55×10^{-4}	4.63×10^{-3}
0.4	7.30×10^{-3}	3.44×10^{-4}	7.28×10^{-3}
0.5	9.17×10^{-3}	4.39×10^{-4}	9.83×10^{-3}
0.6	1.10×10^{-2}	5.43×10^{-4}	1.18×10^{-2}
0.7	1.30×10^{-2}	6.95×10^{-4}	1.30×10^{-2}
0.8	1.51×10^{-2}	7.93×10^{-4}	1.28×10^{-2}
0.9	1.73×10^{-2}	9.50×10^{-4}	1.10×10^{-2}
1.0	1.96×10^{-2}	1.13×10^{-3}	7.19×10^{-3}

5. Conclusions

In this paper, we have solved nonlinear FVHIEs by Bessel polynomials of the first kind and collocation method. One significant advantage of this method is that there is a direct relationship between increase of N and increase accuracy. According to the proposed method one of the major reasons for reduce errors, is produce the sparse matrix. In addition, these features make the better results compared with other methods such as the Bernstein polynomial in example (4) and Legendre-hybrid in example (3). Also, our compared with satisfactory results show the validity and efficiency of proposed method.

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