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New Perturbation Analysis of Generalized Singular Values of Grassmann Matrix Pairs with Arbitrary Permutation

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Abstract. In this paper we give new perturbation analysis of generalized singular values of Grassmann matrix pairs with arbitrary permutation from a new perspective. The new explicit expressions of the chordal metric between generalized singular values of Grassmann matrix pairs with arbitrary permutation are presented, which result in only two small-size singular value decompositions to evaluate. The proposed results are generalizations of several results on bounds on perturbation of generalized singular values.

AMS subject classifications: 15A18, 15A23, 65F15

Key words: Perturbation analysis, Chordal metric, Generalized singular value, Grassman matrix pair, Arbitrary permutation.

1 Introduction

The generalized singular value decomposition (GSVD) for a matrix pair of two matrices with the same number of columns was proposed by Van Loan in 1976 [1]. Numerical methods and perturbation analysis of GSVD have been well developed, see [2–7]. Zha [8] proposed a generalized SVD for matrix triplets. Stewart [9] and Van Loan [10] proposed two algorithms for computing the GSVD. Sun [11] presented the Hoffman-Wielandt theorem for the generalized singular values (GSVs) of Grassman matrix pairs and gave bounds on perturbations of GSVs, which generalized several well-known results for the standard singular value problem. Li [12] presented several perturbation bounds of GSVs

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and its associated subspace. Xu et al. [13] provided the explicit expression and sharper bounds of the chordal metric between generalized singular values of Grassmann matrix pairs. We note that in these papers theoretical existence of chordal metric between generalized singular values of Grassmann matrix pairs with the permutation are discussed more.

In this paper, from a computable and practical perspective, we will analyze the calculation formula and model of the chordal metric between the generalized singular values of Grassmann matrix pairs with arbitrary permutations, and subsequently design an efficient algorithm. The proposed results can theoretically be considered generalizations of several findings regarding bounds on the perturbation of generalized singular values presented in [12, 14]. They can be further well applied to comparative analysis of gene mRNA expression data [15] and other aspects in applications.

1.1 Notation

Throughout this paper we always use the following notations and definitions. i denotes imaginary unit $\sqrt{-1}$. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^n$, $\mathbb{C}^{m\times n}$ and $\mathbb{U}_n$ are the sets of real numbers, complex numbers, n-dimensional real vectors, $m\times n$ complex matrices and $n\times n$ unitary matrices, respectively. $|\cdot|$, $\mathfrak{Im}(\cdot)$ and $\mathfrak{Re}(\cdot)$ stand for absolute value, imaginary part and real part of a complex number, respectively. The symbols I_n and $O_{m\times n}$ stand for the identity matrix of order n and $m\times n$ zero matrix, respectively. For a square matrix $A\in\mathbb{C}^{n\times n}$, $\bar{A}$, A^T, A^H, A^{-1}, $\det(A)$, $\mathrm{tr}(A)$ denote the conjugate, transpose, conjugate transpose, inverse, determinant and trace, respectively. By $\|\cdot\|_2$ we denote the spectral norm of a matrix, and by $\sin(\cdot)$ and $\cos(\cdot)$ we denote sin and cos functions, respectively. $e^{\imath\kappa}$ stands for $\cos\kappa+\imath\sin\kappa$ for angle κ. The singular value set of A is denoted by $\sigma(A)$. For given matrices A, $B\in\mathbb{C}^{n\times n}$, the notation $A<(\leq)B$ indicates that $B-A$ is a positive (semi-)definite matrix. The conjugate of a number $c\in\mathbb{C}$ is denoted by c^*. $diag(0, \ldots, 0, 1_k, 0, \ldots, 0)\in\mathbb{C}^{m\times m}$ denotes m-order diagonal matrices with the kth row and kth column element being 1 and the remaining main diagonal elements being zero. For a matrix $A\in\mathbb{C}^{n\times n}$, we denote the singular values by $\sigma_1(A)\geq\sigma_2(A)\geq\cdots\geq\sigma_n(A)\geq 0$, arranged in decreasing order. We denote $a(b)$ by a or b. Let $X, Y\in\mathbb{C}^{m\times n}(m>n)$ both have full column rank n, and define the angle $\Theta(X, Y)$ between X and Y as [16]

$$\Theta(X,Y)\equiv arccos((X^HX)^{-1/2}X^HY(Y^HY)^{-1}Y^HX(X^HX)^{-1/2})^{-1/2}\geq 0.$$

Definition 1.1. [17] Let $A\in\mathbb{C}^{m\times n}$ and $B\in\mathbb{C}^{p\times n}$. A matrix pair $\{A, B\}$ is an (m, p, n) Grassman matrix pair (GMP) if rank $(A^T, B^T)^T=n$.

Definition 1.2. [17] Let $\{A, B\}$ be an (m, p, n)-GMP. A nonnegative number-pair (α, β) is a GSV of the GMP $\{A, B\}$ if

$$(\alpha, \beta)=(\sqrt{\lambda}, \sqrt{\mu}),\ where\ (\lambda, \mu)\in\lambda(A^HA, B^HB)\ and\ \lambda, \mu\geq 0.$$

The set of GSV of $\{A, B\}$ is denoted by $\sigma\{A, B\}$, where

$$\sigma\{A, B\}=\{(\alpha, \beta)\neq(0, 0)\,|\,\det(\beta^2A^HA-\alpha^2B^HB)=0,\ \alpha,\ \beta\geq 0\}.$$

Evidently, for GSVD it has several formulations in the literature. In this paper we adopt the following form as in [10,15,18].

Definition 1.3. [17] Let {A, B} be an (m, p, n)-GMP. Then there exists unitary matrices $U\in\mathbb{C}^{m\times m}$, $V\in\mathbb{C}^{p\times p}$, and a nonsingular matrix $R\in\mathbb{C}^{n\times n}$ such that

$$U^HAR=\Sigma_A,\ V^HBR=\Sigma_B, \tag{1.1}$$

$$\Sigma_A=\begin{pmatrix}\Lambda & \\ & O_{(m-r-s)\times(n-r-s)}\end{pmatrix},\ \Sigma_B=\begin{pmatrix}O_{(p+r-n)\times r} & \\ & \Omega\end{pmatrix}, \tag{1.2}$$

where $O_{(m-r-s)\times(n-r-s)}$ and $O_{(p+r-n)\times r}$ are zero matrices, and

$$\Lambda=\text{diag}(\alpha_1,\ \dots,\ \alpha_{r+s}),\ \Omega=\text{diag}(\beta_{r+1},\ \dots,\ \beta_n),$$

with

$$1=\alpha_1=\cdots=\alpha_r>\alpha_{r+1}\geq\cdots\geq\alpha_{r+s}>\alpha_{r+s+1}=\cdots=\alpha_n=0,$$
$$0=\beta_1=\cdots=\beta_r<\beta_{r+1}\leq\cdots\leq\beta_{r+s}<\beta_{r+s+1}=\cdots=\beta_n=1,$$

and

$$\alpha_i^2+\beta_i^2=1,\ 1\leq i\leq n.$$

According to [10–12,19], we use the chordal metric on the Riemann sphere to measure the difference between two GSVs. The pair (α, β) can be regarded as a point in Gauss plane $\mathfrak{P}(1, 1)$. For two points $(\alpha, \beta)\neq(0, 0)$, $(\gamma, \delta)\neq(0, 0)$ in Gauss plane $\mathfrak{P}(1, 1)$, their chordal distance is defined by

$$\rho((\alpha, \beta), (\gamma, \delta))=\frac{|\alpha\delta-\beta\gamma|}{\sqrt{(|\alpha|^2+|\beta|^2)(|\gamma|^2+|\delta|^2)}}, \tag{1.3}$$

which measures the difference between two points (α, β) and (γ, δ). Next, we outline some definitions related to GSVD. Readers may refer to [2,12,19] for these definitions. In order to give the perturbation bound of GSVs of GMPs, in [19] the authors use the distance of two points Z and $\tilde{Z}$ in the Grassmann manifold $\mathfrak{P}(N, n)$ $(N>n)$, as follows:

$$d(Z, \widetilde{Z})=\left\{1-\frac{|\det(Z^H\widetilde{Z})|^2}{\det(Z^HZ)\det(\widetilde{Z}^H\widetilde{Z})}\right\}^{\frac{1}{2}}. \tag{1.4}$$

The following theorems are basic and known in [2,19]. Throughout this paper, $\{A, B\}$ and $\{C, D\}$ are always reserved two (m, p, n)-GMPs. The following results [19] provide perturbation bounds based on the chordal metric of GSVs.

Theorem 1.1. [12] *Let* $\{A, B\}$, $\{C, D\}$ *be two* (m, p, n)*-GMPs and let* $\sigma\{A, B\}=\{(\alpha_i, \beta_i)\}_{i=1}^{n}$ *and* $\sigma\{C, D\}=\{(\tilde{\alpha}_i, \tilde{\beta}_i)\}_{i=1}^{n}$. *Then there exists a permutation* π *of* $\{1, 2, \ldots, n\}$ *such that*

$$\max_{1\le i\le n}\rho((\alpha_i, \beta_i), (\tilde{\alpha}_{\pi(i)}, \tilde{\beta}_{\pi(i)}))\le \|\sin\Theta(Z, \tilde{Z})\|_2,$$

where

$$Z=\begin{pmatrix} A \\ B \end{pmatrix}, \widetilde{Z}=\begin{pmatrix} C \\ D \end{pmatrix}$$

and $\rho((\alpha_i, \beta_i),(\tilde{\alpha}_i, \tilde{\beta}_i))$ *is given by (1.3) and* $d(Z, \widetilde{Z})$ *is defined by (1.4).*

Theorem 1.2. [12] *Let* $\{A, B\}$, $\{C, D\}$ *be two* (m, p, n)*-GMPs and let* $\sigma\{A, B\}=\{(\alpha_i, \beta_i)\}_{i=1}^{n}$ *and* $\sigma\{C, D\}=\{(\tilde{\alpha}_i, \tilde{\beta}_i)\}_{i=1}^{n}$. *Then there exists a permutation* μ *of* $\{1, 2, \ldots, n\}$ *such that*

$$\sqrt{\sum_{i=1}^{n}[\rho((\alpha_i, \beta_i), (\tilde{\alpha}_{\mu(i)}, \tilde{\beta}_{\mu(i)}))]^2}\le \|\sin\Theta(Z, \tilde{Z})\|_F,$$

where

$$Z=\begin{pmatrix} A \\ B \end{pmatrix}, \widetilde{Z}=\begin{pmatrix} C \\ D \end{pmatrix}$$

and $\rho((\alpha_i, \beta_i), (\tilde{\alpha}_i, \tilde{\beta}_i))$ *is given by (1.3) and* $d(Z, \widetilde{Z})$ *is defined by (1.4).*

1.2 Organization

The rest of this paper is organized as follows. In Section 2, we will give some lemmas, which are useful to deduce the main results. In Sections 3 and 4, we will provide new formula model for computing generalized singular values of Grassman matrix pair.

2 Preliminaries

In this section, we propose some useful lemmas in order to deduce the main results.

Lemma 2.1. [14] *Let* $A_1, \ldots, A_m\in\mathbb{C}^{n\times n}$ *with* $\sigma_1(A_j)\ge\sigma_2(A_j)\ge\cdots\ge\sigma_n(A_j)\ge 0$ *the singular* *values arranged in decreasing order and* $c \in \mathbb{R}$, $j=1, \ldots, m$. *We have the following extreme values:*

$$\max_{U_1,\ldots,U_m \in \mathbb{U}_n} \left|\det(cI_n \pm \prod_{j=1}^{m} A_j U_j)\right| = \prod_{i=1}^{n}(|c| + \prod_{j=1}^{m}\sigma_i(A_j)),$$

$$\max_{U_1,\ldots,U_m \in \mathbb{U}_n} \left|\mathrm{tr}(cI_n \pm \prod_{j=1}^{m} U_j A_j)\right| = n|c| + \sum_{i=1}^{n}\prod_{j=1}^{m}\sigma_i(A_j),$$

$$\max_{U_1,\ldots,U_m \in \mathbb{U}_n} \mathfrak{Re}\big(\mathrm{tr}(cI_n \pm \prod_{j=1}^{m} U_j A_j)\big) = nc + \sum_{i=1}^{n}\prod_{j=1}^{m}\sigma_i(A_j),$$

$$\min_{U_1,\ldots,U_m \in \mathbb{U}_n} \left|\det(cI_n \pm \prod_{j=1}^{m} A_j U_j)\right| = \begin{cases} \prod_{i=1}^{n}(|c| - \prod_{j=1}^{m}\sigma_i(A_j)), & \prod_{j=1}^{m}\sigma_1(A_j) \le |c|, \\ \prod_{i=1}^{n}(\prod_{j=1}^{m}\sigma_i(A_j) - |c|), & \prod_{j=1}^{m}\sigma_n(A_j) \ge |c|, \\ 0, & \textit{otherwise}. \end{cases}$$

Lemma 2.2. [20] *If A and B are* $n \times n$ *positive semidefinite Hermitian matrices with eigenvalues* $\lambda_1(A) \ge \lambda_2(A) \ge \cdots \ge \lambda_n(A)$ and $\lambda_1(B) \ge \lambda_2(B) \ge \cdots \ge \lambda_n(B)$, *then*

$$\mathrm{tr}(AB) = \sum_{i=1}^{n}\lambda_i(AB) \le \sum_{i=1}^{n}\lambda_i(A)\lambda_i(B).$$

If A and B are $n \times n$ *positive semidefinite Hermitian matrices, then*

$$\mathrm{tr}(AB) = \sum_{i=1}^{n}\lambda_i(AB) \ge \sum_{i=1}^{n}\lambda_i(A)\lambda_{n-i+1}(B).$$

Lemma 2.3. *Let*

$$\Delta = diag(\delta_1, \delta_2, \ldots, \delta_n),\ \Gamma = diag(\gamma_1, \gamma_2, \ldots, \gamma_n)$$

with $\delta_1 \ge \delta_2 \ge \cdots \ge \delta_n \ge 0$, $0 \le \gamma_1 \le \gamma_2 \le \cdots \le \gamma_n$. *Then*

$$\min_{U \in \mathbb{U}_n} \mathrm{tr}(U\Delta U^H \Gamma) = \sum_{i=1}^{n}\delta_i\gamma_i,$$

$$\max_{U \in \mathbb{U}_n} \mathrm{tr}(U\Delta U^H \Gamma) = \sum_{i=1}^{n}\delta_i\gamma_{n-i+1}.$$

Proof. Let the eigenvalues of Δ and Γ have descending sorts, then $\lambda_i(\Delta) = \delta_i$, $\lambda_i(\Gamma) = \gamma_{n-i+1}$, $i = 1, \ldots, n$. By Lemma 2.2 we have for any $U \in \mathbb{U}_n$

$$\mathrm{tr}(U\Delta U^H \Gamma) \ge \sum_{i=1}^{n}\lambda_i(U\Delta U^H)\lambda_{n-i+1}(\Gamma) = \sum_{1}^{n}\delta_i\gamma_i,$$

which implies that

$$\min_{U\in\mathbb{U}_n}\operatorname{tr}(U\Delta U^H\Gamma)=\sum_{i=1}^{n}\delta_i\gamma_i.$$

By Lemma 2.2, we have for any $U\in\mathbb{U}_n$

$$\operatorname{tr}(UV)=\sum_{1}^{n}\lambda_i(U\Delta U^H\Gamma)\leq\sum_{1}^{n}\lambda_i(U\Delta U^H)\lambda_i(\Gamma)=\sum_{i=1}^{n}\delta_i\gamma_{n-i+1},$$

which implies that

$$\max_{U\in\mathbb{U}_n}\operatorname{tr}(U\Delta U^H\Gamma)=\sum_{i=1}^{n}\delta_i\gamma_{n-i+1}.$$

The proof is complete. □

Lemma 2.4. *For every permutation $\pi:\pi(i)=\pi_i$, $1\leq i\leq n$, there exists the corresponding $n\times n$ permutation matrix Π such that $\Pi diag(1,2,\cdots,n)\Pi^H=diag(\pi_1,\pi_2,\cdots,\pi_n)$.*

Proof. For every permutation $\pi:\pi(i)=\pi_i$, $1\leq i\leq n$ and let $\Pi=(e_{\Pi_1},e_{\Pi_2},\cdots,e_{\Pi_n})$, where e_{Π_i} is a column vector with the ith entry is 1, and all other entries are 0, $i=1,2,\ldots,n$, then

$$\Pi diag(1,2,\cdots,n)\Pi^H=diag(\pi_1,\pi_2,\cdots,\pi_n).$$

The proof is complete. □

Let $\{A,B\}$ and $\{\tilde{A},\tilde{B}\}$ be two (m,p,n)-GMPs and

$$S_1=(A^HA+B^HB)^{-1/2}B^H,\ S_2=(\tilde{A}^H\tilde{A}+\tilde{B}^H\tilde{B})^{-1/2}\tilde{B}^H$$

and the SVDs of S_1, S_2 be

$$S_1=P_1\Lambda_1F_1^H,\ S_2=P_2\Lambda_2F_2^H, \tag{2.1}$$

where $P_1\in\mathbb{U}_n$, $F_1\in\mathbb{U}_p$, $P_2\in\mathbb{U}_n$, $F_2\in\mathbb{U}_p$ and diagonal elements of Λ_1, Λ_2 have ascending sorts.

Lemma 2.5. *Let $\{A,B\}$, $\{\tilde{A},\tilde{B}\}$ be two (m,p,n)-GMPs. $\sigma\{A,B\}=\{(\alpha_i,\beta_i)\}_{i=1}^n$ and*

$\sigma\{\tilde{A}, \tilde{B}\} = \{(\tilde{\alpha}_i, \tilde{\beta}_i)\}_{i=1}^n$. *Then*

$$\begin{aligned}
\beta_1^2 &= \min_{\Phi_1 \in \mathbb{U}_p} \operatorname{tr}((A^HA+B^HB)^{-1/2}B^H\Phi_1^HK_1\Phi_1B(A^HA+B^HB)^{-1/2}) \\
&= \operatorname{tr}((A^HA+B^HB)^{-1/2}B^HF_1K_1F_1^HB(A^HA+B^HB)^{-1/2}) \\
&\equiv \theta_1, \\
\beta_n^2 &= \max_{\Phi_1 \in \mathbb{U}_p} \operatorname{tr}((A^HA+B^HB)^{-1/2}B^H\Phi_1^HK_2\Phi_1B(A^HA+B^HB)^{-1/2}) \\
&= \operatorname{tr}((A^HA+B^HB)^{-1/2}B^HF_1K_2F_1^HB(A^HA+B^HB)^{-1/2}) \\
&\equiv \theta_n, \\
\tilde{\beta}_1^2 &= \min_{\Phi_1 \in \mathbb{U}_p} \operatorname{tr}((\tilde{A}^H\tilde{A}+\tilde{B}^H\tilde{B})^{-1/2}\tilde{B}^H\Phi_1^HK_1\Phi_1\tilde{B}(\tilde{A}^H\tilde{A}+\tilde{B}^H\tilde{B})^{-1/2}) \\
&= \operatorname{tr}((\tilde{A}^H\tilde{A}+\tilde{B}^H\tilde{B})^{-1/2}\tilde{B}^HF_2K_1F_2^H\tilde{B}(\tilde{A}^H\tilde{A}+\tilde{B}^H\tilde{B})^{-1/2}) \\
&\equiv \tilde{\theta}_1, \\
\tilde{\beta}_n^2 &= \max_{\Phi_1 \in \mathbb{U}_p} \operatorname{tr}((\tilde{A}^H\tilde{A}+\tilde{B}^H\tilde{B})^{-1/2}\tilde{B}^H\Phi_1^HK_2\Phi_1\tilde{B}(\tilde{A}^H\tilde{A}+\tilde{B}^H\tilde{B})^{-1/2}) \\
&= \operatorname{tr}((\tilde{A}^H\tilde{A}+\tilde{B}^H\tilde{B})^{-1/2}\tilde{B}^HF_2K_2F_2^H\tilde{B}(\tilde{A}^H\tilde{A}+\tilde{B}^H\tilde{B})^{-1/2}) \\
&\equiv \tilde{\theta}_n,
\end{aligned}$$

where

$$K_1 = diag(1, 0, \dots, 0) \in \mathbb{C}^{p\times p},\ K_2 = diag(0, \dots, 0, 1) \in \mathbb{C}^{p\times p}.$$

Proof. Let

$$G_1 = diag(1, 0, \dots, 0) \in \mathbb{C}^{n\times n},\ G_2 = diag(0, 0, \dots, 1) \in \mathbb{C}^{n\times n},$$

if $n \le p$ and let $\Sigma_B = \begin{pmatrix} 0_{(p-n)\times n} \\ \hat{\Sigma}_B \end{pmatrix}$, then

$$\begin{aligned}
&\min_{\Phi_1 \in \mathbb{U}_p} \operatorname{tr}((A^HA+B^HB)^{-1/2}B^H\Phi_1^HK_1\Phi_1B(A^HA+B^HB)^{-1/2}) \\
&= \min_{\Phi_1 \in \mathbb{U}_p} \operatorname{tr}(\Sigma_B^HV^H\Phi_1^HK_1\Phi_1V\Sigma_B) \\
&= \min_{\phi_{24}\phi_{24}^H \le I_n} \operatorname{tr}(\hat{\Sigma}_B^H\phi_{24}^HG_1\phi_{24}\hat{\Sigma}_B),
\end{aligned}$$

which can attain its minimum modulus on the characteristic manifold $\{\phi_{24} \in \mathbb{C}^{n\times n} : \phi_{24}\phi_{24}^H = I_n\}$. Here $\Phi_1V = \begin{pmatrix} \phi_{21} & \phi_{22} \\ \phi_{23} & \phi_{24} \end{pmatrix}$ and $\phi_{24} \in \mathbb{C}^{n\times n}$. By Lemma 2.3, we have

$$\begin{aligned}
&\min_{\Phi_1 \in \mathbb{U}_p} \operatorname{tr}((A^HA+B^HB)^{-1/2}B^H\Phi_1^HK_1\Phi_1B(A^HA+B^HB)^{-1/2}) \\
&= \min_{\phi_{24}\phi_{24}^H = I_n} \operatorname{tr}(\hat{\Sigma}_B^H\phi_{24}^HG_1\phi_{24}\hat{\Sigma}_B) \\
&= \beta_1^2.
\end{aligned} \tag{2.2}$$

Similarly, by Lemma 2.3, we have

$$\begin{aligned}\beta_n^2 &= \max_{\Phi_1\in\mathbb{U}_p} \operatorname{tr}((A^HA+B^HB)^{-1/2}B^H\Phi_1^HK_2\Phi_1B(A^HA+B^HB)^{-1/2})\\ &= \max_{\phi_{24}\phi_{24}^H\leq I_n} \operatorname{tr}(\hat{\Sigma}_B^H\phi_{24}^HG_2\phi_{24}\hat{\Sigma}_B),\end{aligned}$$

which can attain its maximum modulus on the characteristic manifold $\{\phi_{24}\in\mathbb{C}^{n\times n}: \phi_{24}\phi_{24}^H=I_n\}$. Then by Lemma 2.3, we have

$$\begin{aligned}&\max_{\Phi_1\in\mathbb{U}_p} \operatorname{tr}((A^HA+B^HB)^{-1/2}B^H\Phi_1^HK_2\Phi_1B(A^HA+B^HB)^{-1/2})\\ &=\max_{\phi_{24}\phi_{24}^H=I_n} \operatorname{tr}(\hat{\Sigma}_B^H\phi_{24}^HG_2\phi_{24}\hat{\Sigma}_B)\\ &=\beta_n^2.\end{aligned}\tag{2.3}$$

If $p\leq n$ and let $\Sigma_B=(0_{p\times(n-p)},\hat{\Sigma}_B)$, then

$$\begin{aligned}&\min_{\Phi_1\in\mathbb{U}_p} \operatorname{tr}((A^HA+B^HB)^{-1/2}B^H\Phi_1^HK_1\Phi_1B(A^HA+B^HB)^{-1/2})\\ &=\min_{\Phi_1\in\mathbb{U}_p} \operatorname{tr}(\Sigma_B^HV^H\Phi_1^HK_1\Phi_1V\Sigma_B)\\ &=\min_{\Phi_1\in\mathbb{U}_p} \operatorname{tr}\left(\begin{pmatrix}\hat{\Sigma}_B^HV^H\Phi_1^HK_1\Phi_1V\hat{\Sigma}_B & O_{p\times(n-p)}\\ O_{(n-p)\times p} & O_{(n-p)\times(n-p)}\end{pmatrix}\right)\\ &=\min_{\Phi_1\in\mathbb{U}_p} \operatorname{tr}(\hat{\Sigma}_B^HV^H\Phi_1^HK_1\Phi_1V\hat{\Sigma}_B)\\ &=\beta_1^2.\end{aligned}\tag{2.4}$$

Similarly, by Lemma 2.3 we have

$$\begin{aligned}&\max_{\Phi_1\in\mathbb{U}_p} \operatorname{tr}((A^HA+B^HB)^{-1/2}B^H\Phi_1^HK_2\Phi_1B(A^HA+B^HB)^{-1/2})\\ =&\max_{\Phi_1\in\mathbb{U}_p} \operatorname{tr}\left(\begin{pmatrix}\hat{\Sigma}_B^HV^H\Phi_1^HK_2\Phi_1V\hat{\Sigma}_B & O_{p\times(n-p)}\\ O_{(n-p)\times p} & O_{(n-p)\times(n-p)}\end{pmatrix}\right)\\ =&\max_{\Phi_1\in\mathbb{U}_p} \operatorname{tr}(\hat{\Sigma}_B^HV^H\Phi_1^HK_2\Phi_1V\hat{\Sigma}_B)\\ =&\ \beta_n^2.\end{aligned}\tag{2.5}$$

Therefore, by (2.5)-(2.8), we have

$$\beta_1^2=\min_{\Phi_2\in\mathbb{U}_p} \operatorname{tr}((A^HA+B^HB)^{-1/2}B^H\Phi_1^HK_1\Phi_1B(A^HA+B^HB)^{-1/2}),$$

$$\beta_n^2=\max_{\Phi_2\in\mathbb{U}_p} \operatorname{tr}((A^HA+B^HB)^{-1/2}B^H\Phi_1^HK_2\Phi_1B(A^HA+B^HB)^{-1/2}).$$

By (2.1), we have

$$S_1=(A^HA+B^HB)^{-1/2}B^H,\ S_2=(\tilde{A}^H\tilde{A}+\tilde{B}^H\tilde{B})^{-1/2}\tilde{B}^H$$

and the SVDs of S_1, S_2 are

$$S_1=P_1\Lambda_1F_1^H,\ S_2=P_2\Lambda_2F_2^H,$$

where $P_1\in\mathbb{U}_n$, $F_1\in\mathbb{U}_p$, $P_2\in\mathbb{U}_n$, $F_2\in\mathbb{U}_p$ and diagonal elements of Λ_1, Λ_2 have ascending sorts. Then by Lemma 2.1, we have

$$\begin{aligned}\beta_1^2 &= \min_{\Phi_2\in\mathbb{U}_p}\operatorname{tr}((A^HA+B^HB)^{-1/2}B^H\Phi_1^HK_1\Phi_1B(A^HA+B^HB)^{-1/2})\\ &= \operatorname{tr}((A^HA+B^HB)^{-1/2}B^HF_1K_1F_1^HB(A^HA+B^HB)^{-1/2})\\ &\equiv \theta_1\end{aligned}$$

and

$$\begin{aligned}\beta_n^2 &= \max_{\Phi_2\in\mathbb{U}_p}\operatorname{tr}((A^HA+B^HB)^{-1/2}B^H\Phi_1^HK_2\Phi_1B(A^HA+B^HB)^{-1/2})\\ &= \operatorname{tr}((A^HA+B^HB)^{-1/2}B^HF_1K_2F_1^HB(A^HA+B^HB)^{-1/2})\\ &\equiv \theta_n.\end{aligned}$$

Similarly, by the above similar methods, we have

$$\begin{aligned}\tilde{\beta}_1^2 &= \min_{\Phi_2\in\mathbb{U}_p}\operatorname{tr}((\tilde{A}^H\tilde{A}+\tilde{B}^H\tilde{B})^{-1/2}\tilde{B}^H\Phi_1^HK_1\Phi_1\tilde{B}(\tilde{A}^H\tilde{A}+\tilde{B}^H\tilde{B})^{-1/2})\\ &= \operatorname{tr}((\tilde{A}^H\tilde{A}+\tilde{B}^H\tilde{B})^{-1/2}\tilde{B}^HF_2K_1F_2^H\tilde{B}(\tilde{A}^H\tilde{A}+\tilde{B}^H\tilde{B})^{-1/2})\\ &\equiv \tilde{\theta}_1,\end{aligned}$$

$$\begin{aligned}\tilde{\beta}_n^2 &= \max_{\Phi_2\in\mathbb{U}_p}\operatorname{tr}((\tilde{A}^H\tilde{A}+\tilde{B}^H\tilde{B})^{-1/2}\tilde{B}^H\Phi_1^HK_2\Phi_1\tilde{B}(\tilde{A}^H\tilde{A}+\tilde{B}^H\tilde{B})^{-1/2})\\ &= \operatorname{tr}((\tilde{A}^H\tilde{A}+\tilde{B}^H\tilde{B})^{-1/2}\tilde{B}^HF_2K_2F_2^H\tilde{B}(\tilde{A}^H\tilde{A}+\tilde{B}^H\tilde{B})^{-1/2})\\ &\equiv \tilde{\theta}_n.\end{aligned}$$

This completes the proof. □

3 Explicit Formula of $\sum_{i=1}^{n}\rho^2((\alpha_i,\ \beta_i),\ (\tilde{\alpha}_{\pi(i)},\ \tilde{\beta}_{\pi(i)}))$ for Arbitrary Permutation π

In this section, we will give explicit formula of $\sum_{i=1}^{n}\rho^2((\alpha_i,\beta_i),(\tilde{\alpha}_{\pi(i)},\tilde{\beta}_{\pi(i)}))$ for every given permutation π of $\{1, 2, \ldots, n\}$. We first give some necessary symbols as follows. Let Π be an $n\times n$ permutation matrix corresponding to the given permutation π and S_j, F_j, $j=1, 2$ be given by (2.1) and

$$\mathcal{S}_j=\begin{cases} S_j\,(if\, n\leq p) \\ (0_{n\times(n-p)},S_j)\,(if\, p\leq n),\ j=1,2 \end{cases};\mathbf{\Pi}=\begin{cases} diag(\Pi,I_{p-n})\,(if\, n\leq p) \\ \Pi\,(if\, p\leq n) \end{cases},$$

$$\mathcal{F}_j=\begin{cases} F_j\,(if\, n\leq p) \\ diag(I_{n-p},F_j)\,(if\, p\leq n),\ j=1,2, \end{cases};Q_i=\begin{cases} diag(0_{(p-n)\times(p-n)},T_i)\,(if\, n\leq p) \\ T_i\,(if\, p\leq n) \end{cases},$$

$$T_i=\mathrm{diag}(1,\ldots,1_i,0_{i+1},0,\ldots,0_n)\in\mathbb{C}^{n\times n},\ 1\leq i\leq n,$$

$$\Pi_0=\begin{pmatrix} 0 & \cdots & 0 & \cdots & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix},$$

$$\begin{aligned} \varphi(i)&=\mathrm{tr}(\mathcal{S}_1\mathcal{F}_1Q_i\mathcal{F}_1^H\mathcal{S}_1^H)-\mathrm{tr}(\mathcal{S}_1\mathcal{F}_1Q_{i-1}\mathcal{F}_1^H\mathcal{S}_1^H),\\ \phi(i)&=\mathrm{tr}(\mathcal{S}_2\mathcal{F}_2Q_i\mathcal{F}_2^H\mathcal{S}_2^H)-\mathrm{tr}(\mathcal{S}_2\mathcal{F}_2Q_{i-1}\mathcal{F}_2^H\mathcal{S}_2^H). \end{aligned} \tag{3.1}$$

Theorem 3.1. *Let $\{A, B\}$, $\{\tilde{A}, \tilde{B}\}$ be two (m, p, n)-GMPs and let $\sigma\{A, B\}=\{(\alpha_i, \beta_i)\}_{i=1}^n$ and $\sigma\{\tilde{A}, \tilde{B}\}=\{(\tilde{\alpha}_i, \tilde{\beta}_i)\}_{i=1}^n$. Then for every given permutation π of $\{1, 2, \ldots, n\}$ we have*

$$\begin{aligned} &\sum_{i=1}^{n}\rho^2((\alpha_i,\beta_i),(\tilde{\alpha}_{\pi(i)},\tilde{\beta}_{\pi(i)}))\\ =\ &\mathrm{tr}(S_1S_1^H+S_2S_2^H)-2\mathrm{tr}(\mathcal{S}_1\mathcal{F}_1\mathbf{\Pi}\mathcal{F}_2^H\mathcal{S}_2^H\mathcal{S}_2\mathcal{F}_2\mathbf{\Pi}^H\mathcal{F}_1^H\mathcal{S}_1^H)\\ &-2\sum_{i=1}^{n}\sqrt{\varphi(i)}\sqrt{1-\varphi(i)^2}\sqrt{\phi(\pi(i))}\sqrt{1-\phi(\pi(i))^2}, \end{aligned}$$

where $\mathbf{\Pi}$ is given by (3.1).

Proof. Let $T_i=\mathrm{diag}(1, \ldots, 1_i, 0_{i+1}, 0, \ldots, 0_n)\in\mathbb{C}^{n\times n}$. We will consider two cases as follows. Case 1: If $n\leq p$, then for $1\leq i\leq n$ we have $Q_i=diag(0_{(p-n)\times(p-n)}, T_i)$. Let $\Sigma_B=\begin{pmatrix} O_{(p-n)\times n} \\ \hat{\Sigma}_B \end{pmatrix}$, $\Phi_1V=\begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{13} & \phi_{14} \end{pmatrix}\in\mathbb{U}_p$ with $\phi_{14}\in\mathbb{C}^{n\times n}$, then by the GSVD of $\{A, B\}$ and Lemma 2.3, we have

$$\begin{aligned} &\min_{\Phi_1\in\mathbb{U}_p}\mathrm{tr}(\mathcal{S}_1\Phi_1^HQ_i\Phi_1\mathcal{S}_1^H)=\min_{\Phi_1\in\mathbb{U}_p}\mathrm{tr}(S_1\Phi_1^HQ_i\Phi_1S_1^H)\\ &=\min_{\Phi_1\in\mathbb{U}_p}\mathrm{tr}((A^HA+B^HB)^{-1/2}B^H\Phi_1^HQ_i\Phi_1B(A^HA+B^HB)^{-1/2})\\ &=\min_{\Phi_1\in\mathbb{U}_p}\mathrm{tr}(\Sigma_B^HV^H\Phi_1^HQ_i\Phi_1V\Sigma_B)\\ &=\min_{\phi_{14}\phi_{14}^H\leq I_n}\mathrm{tr}(\hat{\Sigma}_B^H\phi_{14}^HT_i\phi_{14}\hat{\Sigma}_B), \end{aligned}$$

which can attain its minimum modulus on the characteristic manifold $\{\phi_{14}\in\mathbb{C}^{n\times n}:\phi_{14}\phi_{14}^H=I_n\}$. Then

$$\min_{\Phi_1\in\mathbb{U}_p}\operatorname{tr}(\mathcal{S}_1\Phi_1^HQ_i\Phi_1\mathcal{S}_1^H)=\min_{\phi_{14}\phi_{14}^H=I_n}\operatorname{tr}(\hat{\Sigma}_B^H\phi_{14}^HT_i\phi_{14}\hat{\Sigma}_B)$$
$$=\beta_1^2+\cdots+\beta_i^2. \tag{3.2}$$

Similarly, we have

$$\min_{\Phi_1\in\mathbb{U}_p}\operatorname{tr}(\mathcal{S}_1\Phi_1^HQ_{i-1}\Phi_1\mathcal{S}_1^H)=\beta_1^2+\cdots+\beta_{i-1}^2. \tag{3.3}$$

By (3.2) and (3.3), we have

$$\beta_i^2=\min_{\Phi_1\in\mathbb{U}_p}\operatorname{tr}(\mathcal{S}_1\Phi_1^HQ_i\Phi_1\mathcal{S}_1^H)-\min_{\Phi_1\in\mathbb{U}_p}\operatorname{tr}(\mathcal{S}_1\Phi_1^HQ_{i-1}\Phi_1\mathcal{S}_1^H). \tag{3.4}$$

Let $\Phi_1F_1=\begin{pmatrix}F_{11} & F_{12}\\ F_{21} & F_{22}\end{pmatrix}\in\mathbb{U}_p$, where $\Phi_1\in\mathbb{U}_p$ and $F_{22}\in\mathbb{C}^{n\times n}$. $\Lambda_1=(0_{n\times(p-n)},\ \Lambda_{11})$, $\Lambda_{11}=diag(\gamma_i)\in\mathbb{C}^{n\times n}$, $\Lambda_2=(0_{n\times(p-n)},\ \Lambda_{22})$, $\Lambda_{22}=diag(\tilde{\gamma}_i)\in\mathbb{C}^{n\times n}$. Based on the SVDs of S_1 and S_2 given in (2.1), it follows that

$$\min_{\Phi_1\in\mathbb{U}_p}\operatorname{tr}(\mathcal{S}_1\Phi_1^HQ_i\Phi_1\mathcal{S}_1^H)=\min_{\Phi_1\in\mathbb{U}_p}\operatorname{tr}(S_1\Phi_1^HQ_i\Phi_1S_1^H)$$
$$=\min_{\Phi_1\in\mathbb{U}_p}\operatorname{tr}(P_1\Lambda_1F_1^H\Phi_1^HQ_i\Phi_1F_1\Lambda_1^HP_1^H)$$
$$=\min_{F_{22}^HF_{22}\leq I_n}\operatorname{tr}(\Lambda_{11}F_{22}^HT_iF_{22}\Lambda_{11}^H)=\min_{F_{22}^HF_{22}=I_n}\operatorname{tr}(\Lambda_{11}F_{22}^HT_iF_{22}\Lambda_{11}^H)$$
$$=\gamma_1^2+\cdots+\gamma_i^2 \tag{3.5}$$

holds when $\Phi_1=F_1^H=\mathcal{F}_1^H$. Similarly, we have

$$\min_{\Phi_1\in\mathbb{U}_p}\operatorname{tr}(\mathcal{S}_1\Phi_1^HQ_{i-1}\Phi_1\mathcal{S}_1^H)=\gamma_1^2+\cdots+\gamma_{i-1}^2 \tag{3.6}$$

holds when $\Phi_1=F_1^H=\mathcal{F}_1^H$. By (3.5) and (3.6), we have for $i=1,\ \ldots,\ n$,

$$\beta_i^2=\min_{\Phi_1\in\mathbb{U}_p}\operatorname{tr}(\mathcal{S}_1\Phi_1^HQ_i\Phi_1\mathcal{S}_1^H)-\min_{\Phi_1\in\mathbb{U}_p}\operatorname{tr}(\mathcal{S}_1\Phi_1^HQ_{i-1}\Phi_1\mathcal{S}_1^H)$$
$$=\operatorname{tr}(\mathcal{S}_1\mathcal{F}_1Q_i\mathcal{F}_1^H\mathcal{S}_1^H)-\operatorname{tr}(\mathcal{S}_1\mathcal{F}_1Q_{i-1}\mathcal{F}_1^H\mathcal{S}_1^H)$$
$$=\gamma_i^2\equiv\varphi(i). \tag{3.7}$$

Similarly, for $i=1,\ \ldots,\ n$, we have

$$\tilde{\gamma}_i^2=\min_{\Phi_1\in\mathbb{U}_p}\operatorname{tr}(\mathcal{S}_2\Phi_1^HQ_i\Phi_1\mathcal{S}_2^H)-\min_{\Phi_1\in\mathbb{U}_p}\operatorname{tr}(\mathcal{S}_2\Phi_1^HQ_{i-1}\Phi_1\mathcal{S}_2^H)$$
$$=\operatorname{tr}(\mathcal{S}_2\mathcal{F}_2Q_i\mathcal{F}_2^H\mathcal{S}_2^H)-\operatorname{tr}(\mathcal{S}_2\mathcal{F}_2Q_{i-1}\mathcal{F}_2^H\mathcal{S}_2^H)$$
$$=\tilde{\beta}_i^2\equiv\phi(i). \tag{3.8}$$

By Lemma 2.4, for every permutation $\pi:\pi(i)=\pi_i$, $1\le i\le n$, there exists the corresponding $n\times n$ permutation matrix Π such that $\Pi diag(1,\ 2,\ \cdots,\ n)\Pi^H = diag(\pi_1,\ \pi_2,\ \cdots,\ \pi_n)$. Let $\mathbf{\Pi}=diag(\Pi,\ I_{p-n})$. Using the SVDs of S_1 and S_2 in (2.1) and (3.7), (3.8), we have

$$\begin{aligned}
&\operatorname{tr}(\mathcal{S}_1\mathcal{F}_1\mathbf{\Pi}\mathcal{F}_2^H\mathcal{S}_2^H\mathcal{S}_2\mathcal{F}_2\mathbf{\Pi}^H\mathcal{F}_1^H\mathcal{S}_1^H)\\
&=\operatorname{tr}(S_1F_1\mathbf{\Pi}F_2^HS_2^HS_2F_2\mathbf{\Pi}^HF_1^HS_1^H)\\
&=\operatorname{tr}(P_1\Lambda_1F_1^HF_1\mathbf{\Pi}F_2^HF_2\Lambda_2^H\Lambda_2F_2^HF_2\mathbf{\Pi}^HF_1^HF_1\Lambda_1^HP_1^H)\\
&=\operatorname{tr}(\Lambda_{11}\Pi\Lambda_{22}^H\Lambda_{22}\Pi^H\Lambda_{11}^H)=\sum_{i=1}^n\gamma_i^2\tilde{\gamma}_{\pi_i}^2=\sum_{i=1}^n\beta_i^2\tilde{\beta}_{\pi_i}^2.
\end{aligned}\tag{3.9}$$

Note that

$$\begin{aligned}
\operatorname{tr}(S_1S_1^H+S_2S_2^H)&=\operatorname{tr}(P_1\Lambda_1F_1^H\Psi F_1\Lambda_1^HP_1^H+P_2\Lambda_2F_2^HF_2\Lambda_2^HP_2^H)\\
&=\operatorname{tr}(\Lambda_1F_1^HF_1\Lambda_1^H+\Lambda_2F_2^HF_2\Lambda_2^H)\\
&=\operatorname{tr}(\Lambda_{11}\Lambda_{11}^H+\Lambda_{22}\Lambda_{22}^H)\\
&=\operatorname{tr}(\Lambda_{11}\Psi_{11}\Lambda_{11}^H+\Lambda_{22}\tilde{\Psi}_{11}\Lambda_{22}^H)\\
&=\sum_{i=1}^n(\gamma_i^2+\tilde{\gamma}_i^2)=\sum_{i=1}^n(\gamma_i^2+\tilde{\gamma}_{\pi_i}^2)\\
&=\sum_{i=1}^n(\beta_i^2+\tilde{\beta}_i^2)=\sum_{i=1}^n(\beta_i^2+\tilde{\beta}_{\pi_i}^2).
\end{aligned}\tag{3.10}$$

Case 2: if $p<n$, let $\bar{B}=\begin{pmatrix}O_{(n-p)\times n}\\ B\end{pmatrix}$, $\bar{\Sigma}_B=\begin{pmatrix}O_{(n-p)\times n}\\ \Sigma_B\end{pmatrix}=diag(\beta_i)\in\mathbb{C}^{n\times n}$, $i=1,\ \dots,\ n$, $\tilde{V}=diag(I_{n-p},\ V)$, then by Lemma 2.2, we have for $i=1,\ \dots,\ n$,

$$\begin{aligned}
&\min_{\Phi_1\in\mathbb{U}_n}\operatorname{tr}(\mathcal{S}_1\Phi_1^HT_i\Phi_1\mathcal{S}_1^H)\\
=&\min_{\Phi_1\in\mathbb{U}_n}\operatorname{tr}((A^HA+B^HB)^{-\frac{1}{2}}\bar{B}^H\Phi_1^HT_i\Phi_1\bar{B}(A^HA+B^HB)^{-\frac{1}{2}})\\
=&\min_{\Phi_1\in\mathbb{U}_n}\operatorname{tr}(\bar{\Sigma}_B^H\tilde{V}^H\Phi_1^HT_i\Phi_1\tilde{V}\bar{\Sigma}_B)\\
=&\beta_1^2+\cdots+\beta_i^2.
\end{aligned}\tag{3.11}$$

Similarly, for $i=1,\ \dots,\ n$,

$$\min_{\Phi_1\in\mathbb{U}_n}\operatorname{tr}(\mathcal{S}_1\Phi_1^HT_{i-1}\Phi_1\mathcal{S}_1^H)=\beta_1^2+\cdots+\beta_{i-1}^2.\tag{3.12}$$

Let $\tilde{\Lambda}_1=(0_{n\times(n-p)},\ \Lambda_1)$, $\tilde{\Lambda}_2=(0_{n\times(n-p)},\ \Lambda_2)$, $\tilde{\Lambda}_1=diag(\gamma_i)\in\mathbb{C}^{n\times n}$, $\tilde{\Lambda}_2=diag(\tilde{\gamma}_i)\in\mathbb{C}^{n\times n}$, then

$$\begin{aligned}
&\min_{\Phi_1\in\mathbb{U}_n} \mathrm{tr}(\mathcal{S}_1\Phi_1^H T_i\Phi_1\mathcal{S}_1^H)\\
=&\min_{\Phi_1\in\mathbb{U}_n} \mathrm{tr}((A^HA+B^HB)^{-1/2}\bar{B}^H\Phi_1^H T_i\Phi_1\bar{B}(A^HA+B^HB)^{-1/2})\\
=&\min_{\Phi_1\in\mathbb{U}_n} \mathrm{tr}(P_1\tilde{\Lambda}_1\mathcal{F}_1^H\Phi_1^H T_i\Phi_1\mathcal{F}_1\tilde{\Lambda}_1^H P_1^H)\\
=&\min_{\Phi_1\in\mathbb{U}_n} \mathrm{tr}(\tilde{\Lambda}_1\mathcal{F}_1^H\Phi_1^H T_i\Phi_1\mathcal{F}_1\tilde{\Lambda}_1^H)\\
=&\gamma_1^2+\cdots+\gamma_i^2
\end{aligned}\tag{3.13}$$

holds when $\Phi_1=\mathcal{F}_1^H$. Similarly, $\min\limits_{\Phi_1\in\mathbb{U}_n} tr(\mathcal{S}_1\Phi_1^H T_{i-1}\Phi_1\mathcal{S}_1^H)=\gamma_1^2+\cdots+\gamma_{i-1}^2$ holds when $\Phi_1=\mathcal{F}_1^H$. By (3.11)-(3.13), we have for $i=1,\ldots,n$,

$$\begin{aligned}
\beta_i^2=&\min_{\Phi_1\in\mathbb{U}_n} \mathrm{tr}(\mathcal{S}_1\Phi_1^H T_i\Phi_1\mathcal{S}_1^H)-\min_{\Phi_1\in\mathbb{U}_n} \mathrm{tr}(\mathcal{S}_1\Phi_1^H T_{i-1}\Phi_1\mathcal{S}_1^H)\\
=&\mathrm{tr}(\mathcal{S}_1\mathcal{F}_1T_i\mathcal{F}_1^H\mathcal{S}_1^H)-\mathrm{tr}(\mathcal{S}_1\mathcal{F}_1T_{i-1}\mathcal{F}_1^H\mathcal{S}_1^H)\\
=&\gamma_i^2\equiv\varphi(i).
\end{aligned}\tag{3.14}$$

Similarly, for $i=1,\ldots,n$, let $\hat{B}=\begin{pmatrix}O_{(n-p)\times n}\\ \tilde{B}\end{pmatrix}$ and we have

$$\begin{aligned}
\tilde{\beta}_i^2=&\min_{\Phi_1\in\mathbb{U}_n} \mathrm{tr}(\mathcal{S}_2\Phi_1^H T_i\Phi_1\mathcal{S}_2^H)-\min_{\Phi_1\in\mathbb{U}_n} \mathrm{tr}(\mathcal{S}_2\Phi_1^H T_{i-1}\Phi_1\mathcal{S}_2^H)\\
=&\min_{\Phi_1\in\mathbb{U}_n} \mathrm{tr}(\hat{B}^H\Phi_1^H T_i\Phi_1\hat{B}(\tilde{A}^H\tilde{A}+\tilde{B}^H\tilde{B})^{-1})-\min_{\Phi_1\in\mathbb{U}_p} \mathrm{tr}(\hat{B}^H\Phi_1^H T_{i-1}\Phi_1\hat{B}(\tilde{A}^H\tilde{A}+\tilde{B}^H\tilde{B})^{-1})\\
=&\mathrm{tr}(\hat{B}^H\mathcal{F}_2T_i\mathcal{F}_2^H\hat{B}(\tilde{A}^H\tilde{A}+\tilde{B}^H\tilde{B})^{-1})-\mathrm{tr}(\hat{B}^H\mathcal{F}_2T_{i-1}\mathcal{F}_2^H\hat{B}(\tilde{A}^H\tilde{A}+\tilde{B}^H\tilde{B})^{-1})\\
=&\tilde{\gamma}_i^2\equiv\phi(i).
\end{aligned}\tag{3.15}$$

By Lemma 2.4, for every permutation $\pi:\pi(i)=\pi_i$, $1\le i\le n$, there exists the corresponding $n\times n$ permutation matrix Π such that $\Pi diag(1,2,\cdots,n)\Pi^H=diag(\pi_1,\pi_2,\cdots,\pi_n)$. According to the SVDs of S_1 and S_2 in (2.1), we have

$$\begin{aligned}
\mathrm{tr}(\mathcal{S}_1\mathcal{F}_1\Pi\mathcal{F}_2^H\mathcal{S}_2^H\mathcal{S}_2\mathcal{F}_2\Pi^H\mathcal{F}_1^H\mathcal{S}_1^H)=&\mathrm{tr}(P_1\tilde{\Lambda}_1\mathcal{F}_1^H\mathcal{F}_1\Pi\mathcal{F}_2^H\mathcal{F}_2\tilde{\Lambda}_2^H\tilde{\Lambda}_2\tilde{F}_2^H\tilde{F}_2\Pi^H\tilde{F}_1^H\tilde{F}_1\tilde{\Lambda}_1^HP_1^H)\\
=&\mathrm{tr}(\tilde{\Lambda}_1\Pi\tilde{\Lambda}_2^H\tilde{\Lambda}_2\Pi^H\tilde{\Lambda}_1^H)\\
=&\sum_{i=1}^n\gamma_i^2\tilde{\gamma}_{\pi(i)}^2=\sum_{i=1}^n\beta_i^2\tilde{\beta}_{\pi(i)}^2.
\end{aligned}\tag{3.16}$$

According to the SVDs of S_1 and S_2 in (2.1), (3.14), (3.15), we get

$$\begin{aligned}
\operatorname{tr}(S_1S_1^H+S_2S_2^H) &= \operatorname{tr}(\mathcal{S}_1\mathcal{S}_1^H+\mathcal{S}_2\mathcal{S}_2^H)=\operatorname{tr}(P_1\tilde{\Lambda}_1\mathcal{F}_1^H\mathcal{F}_1\tilde{\Lambda}_1^HP_1^H+P_2\tilde{\Lambda}_2\mathcal{F}_2^H\mathcal{F}_2\tilde{\Lambda}_2^HP_2^H)\\
&= \sum_{i=1}^n(\gamma_i^2+\tilde{\gamma}_i^2)=\sum_{i=1}^n(\gamma_i^2+\tilde{\gamma}_{\pi_i}^2)\\
&= \sum_{i=1}^n(\beta_i^2+\tilde{\beta}_i^2)=\sum_{i=1}^n(\beta_i^2+\tilde{\beta}_{\pi_i}^2). \qquad (3.17)
\end{aligned}$$

By Definition (1.3), we have $\beta_i^2=1-\alpha_i^2$, $\tilde{\beta}_i^2=1-\tilde{\alpha}_i^2$ together with (3.7)-(3.10), (3.14)-(3.17). For every permutation π of $\{1, 2, \ldots, n\}$, it holds that

$$\begin{aligned}
&\sum_{i=1}^n\rho^2((\alpha_i,\beta_i),(\tilde{\alpha}_{\pi(i)},\tilde{\beta}_{\pi(i)}))=n-\sum_{i=1}^n(\alpha_i^2\tilde{\alpha}_{\pi(i)}^2+\beta_i^2\tilde{\beta}_{\pi(i)}^2)-\sum_{i=1}^n2\alpha_i\tilde{\alpha}_{\pi(i)}\beta_i\tilde{\beta}_{\pi(i)}\\
&=n-\sum_{i=1}^n\Big(2\beta_i^2\tilde{\beta}_{\pi(i)}^2-(\beta_i^2+\tilde{\beta}_{\pi(i)}^2)+1\Big)-\sum_{i=1}^n2\alpha_i\tilde{\alpha}_{\pi(i)}\beta_i\tilde{\beta}_{\pi(i)}\\
&=\sum_{i=1}^n(\beta_i^2+\tilde{\beta}_{\pi(i)}^2)-\sum_{i=1}^n2\beta_i^2\tilde{\beta}_{\pi(i)}^2-\sum_{i=1}^n2\alpha_i\tilde{\alpha}_{\pi(i)}\beta_i\tilde{\beta}_{\pi(i)}\\
&=\operatorname{tr}(S_1S_1^H+S_2S_2^H)-2\operatorname{tr}(\mathcal{S}_1\mathcal{F}_1\Pi\mathcal{F}_2^H\mathcal{S}_2^H\mathcal{S}_2\mathcal{F}_2\Pi^H\mathcal{F}_1^H\mathcal{S}_1^H)\\
&-2\sum_{i=1}^n\sqrt{\varphi_i}\sqrt{1-\varphi_i^2}\sqrt{\phi_{\pi(i)}}\sqrt{1-\phi_{\pi(i)}^2}.
\end{aligned}$$

By Cases 1 and 2, we deduce the desired conclusions. This completes the proof. □

Remark 3.1. We give explicit formula of $\sum_{i=1}^n\rho^2((\alpha_i,\beta_i),(\tilde{\alpha}_{\pi(i)},\tilde{\beta}_{\pi(i)}))$ for every given permutation π of $\{1, 2, \ldots, n\}$. When permutation π reduces to $\pi(i)=i$, Theorem 3.1 reduces to

$$\begin{aligned}
\rho^2((\alpha_i,\beta_i),(\tilde{\alpha}_{\pi(i)},\tilde{\beta}_{\pi(i)}))&=1-(\alpha_i\tilde{\alpha}_{\pi(i)}+\beta_i\tilde{\beta}_{\pi(i)})^2\\
&=1-(\sqrt{\varphi_i}\sqrt{\phi_{\pi(i)}}+\sqrt{1-\varphi_i^2}\sqrt{1-\phi_{\pi(i)}^2})^2.
\end{aligned}$$

When permutation π reduces to $\pi(i)=n-i+1$, Theorem 3.1 reduces to

$$\begin{aligned}
\rho^2((\alpha_i,\beta_i),(\tilde{\alpha}_{\pi(i)},\tilde{\beta}_{\pi(i)}))&=1-(\alpha_i\tilde{\alpha}_{\pi(i)}+\beta_i\tilde{\beta}_{\pi(i)})^2\\
&=1-(\sqrt{\varphi_i}\sqrt{\phi_{\pi(i)}}+\sqrt{1-\varphi_i^2}\sqrt{1-\phi_{\pi(i)}^2})^2.
\end{aligned}$$

4 Interval Estimates of $\sum_{i=1}^{n}\rho^2\big((\alpha_i,\ \beta_i),\ (\tilde{\alpha}_{\pi(i)},\ \tilde{\beta}_{\pi(i)})\big)$ for Any Permutation

In this section, we will give the upper and lower bounds of $\min_{\pi}\sum_{i=1}^{n}\rho^2((\alpha_i,\beta_i),(\tilde{\alpha}_{\pi(i)},\tilde{\beta}_{\pi(i)}))$ and $\max_{\pi}\sum_{i=1}^{n}\rho^2((\alpha_i,\beta_i),(\tilde{\alpha}_{\pi(i)},\tilde{\beta}_{\pi(i)}))$ for any permutation π of $\{1,2,\dots,n\}$.

Lemma 4.1. *Let $\{A,B\}$, $\{C,D\}$ be two (m,p,n)-GMPs and let $\sigma\{A,B\}=\{(\alpha_i,\beta_i)\}_{i=1}^n$ and $\sigma\{C,D\}=\{(\tilde{\alpha}_i,\tilde{\beta}_i)\}_{i=1}^n$. Then for any permutation π of $\{1,2,\dots,n\}$ the following issues hold true.*

$$(i)\quad \min_{\pi}\sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{\pi(i)}^2+\beta_i^2\tilde{\beta}_{\pi(i)}^2)=\sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{n-i+1}^2+\beta_i^2\tilde{\beta}_{n-i+1}^2),$$

$$(ii)\quad \max_{\pi}\sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{\pi(i)}^2+\beta_i^2\tilde{\beta}_{\pi(i)}^2)=\sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{i}^2+\beta_i^2\tilde{\beta}_{i}^2).$$

Proof. (i) It is easy to check that

$$\min_{\pi}\sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{\pi(i)}^2+\beta_i^2\tilde{\beta}_{\pi(i)}^2)\ge\min_{\pi}\sum_{i=1}^{n}\alpha_i^2\tilde{\alpha}_{\pi(i)}^2+\min_{\pi}\sum_{i=1}^{n}\beta_i^2\tilde{\beta}_{\pi(i)}^2. \tag{4.1}$$

By (3.9) and (3.16), we have

$$\min_{\pi}\sum_{i=1}^{n}\beta_i^2\tilde{\beta}_{\pi(i)}^2=\min_{\mathbf{\Pi}\in\mathbb{P}_{\max\{p,n\}}}\mathrm{tr}(\mathcal{S}_1\mathcal{F}_1\mathbf{\Pi}\mathcal{F}_2^H\mathcal{S}_2^H\mathcal{S}_2\mathcal{F}_2\mathbf{\Pi}^H\mathcal{F}_1^H\mathcal{S}_1^H), \tag{4.2}$$

where $\mathbf{\Pi}$ is given by (3.1) and $\Pi\in\mathbb{U}_{\max\{p,n\}}$. Then by Lemma 2.3 and (4.2), we have if $n\le p$,

$$\begin{aligned}
&\min_{\mathbf{\Pi}\in\mathbb{P}_{\max\{p,n\}}}\mathrm{tr}(\mathcal{S}_1\mathcal{F}_1\mathbf{\Pi}\mathcal{F}_2^H\mathcal{S}_2^H\mathcal{S}_2\mathcal{F}_2\mathbf{\Pi}^H\mathcal{F}_1^H\mathcal{S}_1^H)\\
=\ &\min_{\mathbf{\Pi}\in\mathbb{P}_p}\mathrm{tr}(\mathcal{S}_1\mathcal{F}_1\mathbf{\Pi}\mathcal{F}_2^H\mathcal{S}_2^H\mathcal{S}_2\mathcal{F}_2\mathbf{\Pi}^H\mathcal{F}_1^H\mathcal{S}_1^H)\\
=\ &\mathrm{tr}(S_1F_1\mathbf{\Pi}F_2^HS_2^HS_2F_2\mathbf{\Pi}^HF_1^HS_1^H)\\
\ge\ &\min_{\Phi\in\mathbb{U}_p}\mathrm{tr}(S_1\Phi S_2^HS_2\Phi^HS_1^H)\\
=\ &\min_{\Phi\in\mathbb{U}_p}\mathrm{tr}(P_1\Lambda_1\Phi\Lambda_2^H\Lambda_2\Phi^H\Lambda_1^HP_1^H)\\
=\ &\sum_{i=1}^{n}\beta_i^2\tilde{\beta}_{n-i+1}^2.
\end{aligned} \tag{4.3}$$

If $p \le n$,

$$\begin{aligned}
&\min_{\mathbf{\Pi}\in\mathbb{P}_{\max\{p,n\}}} \operatorname{tr}(\mathcal{S}_1\mathcal{F}_1\mathbf{\Pi}\mathcal{F}_2^H\mathcal{S}_2^H\mathcal{S}_2\mathcal{F}_2\mathbf{\Pi}^H\mathcal{F}_1^H\mathcal{S}_1^H) \\
\ge\ & \min_{\Phi\in\mathbb{P}_n} \operatorname{tr}(\mathcal{S}_1\Phi\mathcal{S}_2^H\mathcal{S}_2\Phi^H\mathcal{S}_1^H) \\
=\ & \min_{\Phi\in\mathbb{P}_n} \operatorname{tr}(P_1\tilde{\Lambda}_1\Phi\tilde{\Lambda}_2^H\tilde{\Lambda}_2\Phi^H\tilde{\Lambda}_1^H P_1^H) \\
=\ & \sum_{i=1}^{n}\beta_i^2\tilde{\beta}_{n-i+1}^2. \qquad (4.4)
\end{aligned}$$

It can be concluded from (4.3) and (4.4) that

$$\min_{\pi}\sum_{i=1}^{n}\beta_i^2\tilde{\beta}_{\pi(i)}^2 \ge \sum_{i=1}^{n}\beta_i^2\tilde{\beta}_{n-i+1}^2, \qquad (4.5)$$

and it is easy to check that

$$\min_{\pi}\sum_{i=1}^{n}\beta_i^2\tilde{\beta}_{\pi(i)}^2 \le \sum_{i=1}^{n}\beta_i^2\tilde{\beta}_{\pi_0(i)}^2 = \sum_{i=1}^{n}\beta_i^2\tilde{\beta}_{n-i+1}^2, \qquad (4.6)$$

where $\pi_0(i)=n-i+1$. By (4.5) and (4.6), we have

$$\min_{\pi}\sum_{i=1}^{n}\beta_i^2\tilde{\beta}_{\pi(i)}^2 = \sum_{i=1}^{n}\beta_i^2\tilde{\beta}_{n-i+1}^2. \qquad (4.7)$$

Similarly, we have

$$\min_{\pi}\sum_{i=1}^{n}\alpha_i^2\tilde{\alpha}_{\pi(i)}^2 = \sum_{i=1}^{n}\alpha_i^2\tilde{\alpha}_{n-i+1}^2. \qquad (4.8)$$

Then it follows from (4.1), (4.7) and (4.8) that

$$\begin{aligned}
\min_{\pi}\sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{\pi(i)}^2+\beta_i^2\tilde{\beta}_{\pi(i)}^2) &\ge \min_{\pi}\sum_{i=1}^{n}\alpha_i^2\tilde{\alpha}_{\pi(i)}^2+\min_{\pi}\sum_{i=1}^{n}\beta_i^2\tilde{\beta}_{\pi(i)}^2 \\
&= \sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{n-i+1}^2+\beta_i^2\tilde{\beta}_{n-i+1}^2). \qquad (4.9)
\end{aligned}$$

Meanwhile,

$$\begin{aligned}
\min_{\pi}\sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{\pi(i)}^2+\beta_i^2\tilde{\beta}_{\pi(i)}^2) &\le \sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{\pi_0(i)}^2+\beta_i^2\tilde{\beta}_{\pi_0(i)}^2) \\
&= \sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{n-i+1}^2+\beta_i^2\tilde{\beta}_{n-i+1}^2), \qquad (4.10)
\end{aligned}$$

where $\pi_0(i)=n-i+1$. Combining (4.9) and (4.10) yields

$$\min_{\pi}\sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{\pi(i)}^2+\beta_i^2\tilde{\beta}_{\pi(i)}^2)=\sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{n-i+1}^2+\beta_i^2\tilde{\beta}_{n-i+1}^2).$$

(ii) It follows that

$$\max_{\pi}\sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{\pi(i)}^2+\beta_i^2\tilde{\beta}_{\pi(i)}^2)\leq\max_{\pi}\sum_{i=1}^{n}\alpha_i^2\tilde{\alpha}_{\pi(i)}^2+\max_{\pi}\sum_{i=1}^{n}\beta_i^2\tilde{\beta}_{\pi(i)}^2. \tag{4.11}$$

By (3.9) and (3.16), we have

$$\max_{\pi}\sum_{i=1}^{n}\beta_i^2\tilde{\beta}_{\pi(i)}^2=\max_{\mathbf{\Pi}\in\mathbb{P}_{\max\{p,n\}}}\operatorname{tr}(\mathcal{S}_1\mathcal{F}_1\mathbf{\Pi}\mathcal{F}_2^H\mathcal{S}_2^H\mathcal{S}_2\mathcal{F}_2\mathbf{\Pi}^H\mathcal{F}_1^H\mathcal{S}_1^H).$$

From Lemma 2.3, it follows that if $n\leq p$,

$$\begin{aligned}\max_{\mathbf{\Pi}\in\mathbb{P}_p}\operatorname{tr}(\mathcal{S}_1\mathcal{F}_1\mathbf{\Pi}\mathcal{F}_2^H\mathcal{S}_2^H\mathcal{S}_2\mathcal{F}_2\mathbf{\Pi}^H\mathcal{F}_1^H\mathcal{S}_1^H) &\leq \max_{\Phi\in\mathbb{U}_p}\operatorname{tr}(S_1\Phi S_2^H S_2\Phi^H S_1^H)\\ &= \max_{\Phi\in\mathbb{P}_p}\operatorname{tr}(P_1\Lambda_1\Phi\Lambda_2^H\Lambda_2\Phi^H\Lambda_1^H P_1^H)=\sum_{i=1}^{n}\beta_i^2\tilde{\beta}_i^2.\end{aligned} \tag{4.12}$$

If $p\leq n$,

$$\begin{aligned}\max_{\mathbf{\Pi}\in\mathbb{P}_n}\operatorname{tr}(\mathcal{S}_1\mathcal{F}_1\mathbf{\Pi}\mathcal{F}_2^H\mathcal{S}_2^H\mathcal{S}_2\mathcal{F}_2\mathbf{\Pi}^H\mathcal{F}_1^H\mathcal{S}_1^H) &\leq \max_{\Phi\in\mathbb{U}_n}\operatorname{tr}(\mathcal{S}_1\Phi\mathcal{S}_2^H\mathcal{S}_2\Phi^H\mathcal{S}_1^H)\\ &= \max_{\Phi\in\mathbb{P}_n}\operatorname{tr}(P_1\tilde{\Lambda}_1\Phi\tilde{\Lambda}_2^H\tilde{\Lambda}_2\Phi^H\tilde{\Lambda}_1^H P_1^H)=\sum_{i=1}^{n}\beta_i^2\tilde{\beta}_i^2.\end{aligned} \tag{4.13}$$

By (4.12) and (4.13), we have

$$\max_{\pi}\sum_{i=1}^{n}\beta_i^2\tilde{\beta}_{\pi(i)}^2\leq\sum_{i=1}^{n}\beta_i^2\tilde{\beta}_i^2, \tag{4.14}$$

and it is easy to check that

$$\max_{\pi}\sum_{i=1}^{n}\beta_i^2\tilde{\beta}_{\pi(i)}^2\geq\sum_{i=1}^{n}\beta_i^2\tilde{\beta}_{\pi_1(i)}^2=\sum_{i=1}^{n}\beta_i^2\tilde{\beta}_i^2, \tag{4.15}$$

where $\pi_1(i)=i$. By (4.14) and (4.15), we have

$$\max_{\pi}\sum_{i=1}^{n}\beta_i^2\tilde{\beta}_{\pi(i)}^2=\sum_{i=1}^{n}\beta_i^2\tilde{\beta}_i^2. \tag{4.16}$$

Similarly,

$$\max_{\pi}\sum_{i=1}^{n}\alpha_i^2\tilde{\alpha}_{\pi(i)}^2=\sum_{i=1}^{n}\alpha_i^2\tilde{\alpha}_i^2. \tag{4.17}$$

Then it follows from (4.11), (4.16) and (4.17) that

$$\begin{aligned}\max_{\pi}\sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{\pi(i)}^2+\beta_i^2\tilde{\beta}_{\pi(i)}^2) &\leq \max_{\pi}\sum_{i=1}^{n}\alpha_i^2\tilde{\alpha}_{\pi(i)}^2+\max_{\pi}\sum_{i=1}^{n}\beta_i^2\tilde{\beta}_{\pi(i)}^2\\ &= \sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_i^2+\beta_i^2\tilde{\beta}_i^2). \end{aligned}\tag{4.18}$$

Meanwhile,

$$\begin{aligned}\max_{\pi}\sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{\pi(i)}^2+\beta_i^2\tilde{\beta}_{\pi(i)}^2) &\geq \sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{\pi_0(i)}^2+\beta_i^2\tilde{\beta}_{\pi_0(i)}^2)\\ &= \sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_i^2+\beta_i^2\tilde{\beta}_i^2), \end{aligned}\tag{4.19}$$

where $\pi_0(i)=i$. Combining (4.18) and (4.19) yields

$$\max_{\pi}\sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{\pi(i)}^2+\beta_i^2\tilde{\beta}_{\pi(i)}^2)=\sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_i^2+\beta_i^2\tilde{\beta}_i^2).$$

The proof is complete. □

Theorem 4.1. *Let $\{A, B\}$, $\{C, D\}$ be two (m, p, n)-GMPs and let $\sigma\{A, B\}=\{(\alpha_i, \beta_i)\}_{i=1}^n$ and $\sigma\{C, D\}=\{(\tilde{\alpha}_i, \tilde{\beta}_i)\}_{i=1}^n$. Then for any permutation π of $\{1, 2, \dots, n\}$ we have*

$$\sum_{i=1}^{n}\rho^2((\alpha_i, \beta_i), (\tilde{\alpha}_{\pi(i)}, \tilde{\beta}_{\pi(i)}))\in[\zeta_2, \zeta_1],$$

where

$$\begin{aligned}\zeta_1 &= \mathrm{tr}(S_1S_1^H+S_2S_2^H)-2\mathrm{tr}(S_1F_1\Pi_0F_2^HS_2^HS_2F_2\Pi_0^HF_1^HS_1^H),\\ \zeta_2 &= \mathrm{tr}(S_1S_1^H+S_2S_2^H)-2\mathrm{tr}(S_1F_1F_2^HS_2^HS_2F_2F_1^HS_1^H)-\sum_{i=1}^{n}2n\sqrt{1-\theta_1}\sqrt{1-\tilde{\theta}_1}\sqrt{\theta_n}\sqrt{\tilde{\theta}_n}\end{aligned}$$

and θ_1, θ_n, $\tilde{\theta}_1$, $\tilde{\theta}_n$ are given by Lemma 2.5 and Π_0 are given by (3.1).

Proof. Let

$$\Omega_1 = diag(\alpha_1, \ldots, \alpha_n),\ \Omega_2 = diag(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n),$$
$$\Lambda_1 = diag(\beta_1, \ldots, \beta_n),\ \Lambda_2 = diag(\tilde{\beta}_1, \ldots, \tilde{\beta}_n),$$

where α_i, $\tilde{\alpha}_i$, β_i and $\tilde{\beta}_i$ are defined by (1.1) and (1.2). According to Lemma 2.4, for any permutation π, there exists the corresponding permutation matrix Π such that

$$\begin{aligned}\min_{\forall \pi} \sum_{i=1}^{n} \beta_i \tilde{\beta}_{\pi(i)} &= \min_{\Pi \in \mathbb{P}_n} \mathrm{tr}(\Lambda_1 \Pi \Lambda_2 \Pi^H) \geq \min_{\Phi \in \mathbb{U}_n} \mathrm{tr}(\Lambda_1 \Phi \Lambda_2 \Phi^H) \\ &= \sum_{i=1}^{n} \beta_i \tilde{\beta}_{n-i+1}. \end{aligned} \tag{4.20}$$

Meanwhile,

$$\min_{\forall \pi} \sum_{i=1}^{n} \beta_i \tilde{\beta}_{\pi(i)} \leq \sum_{i=1}^{n} \beta_i \tilde{\beta}_{\pi_0(i)} = \sum_{i=1}^{n} \beta_i \tilde{\beta}_{n-i+1}, \tag{4.21}$$

where $\pi_0(i) = n - i + 1$. By (4.20) and (4.21), we have

$$\min_{\forall \pi} \sum_{i=1}^{n} \beta_i \tilde{\beta}_{\pi(i)} = \sum_{i=1}^{n} \beta_i \tilde{\beta}_{n-i+1}. \tag{4.22}$$

By Lemmas 2.4 and 2.1, for any permutation π there exists the corresponding permutation matrix Π such that

$$\begin{aligned}\max_{\forall \pi} \sum_{i=1}^{n} \alpha_i \tilde{\alpha}_{\pi(i)} \beta_i \tilde{\beta}_{\pi(i)} &= \max_{\Pi \in \mathbb{P}_n} |\mathrm{tr}(\Omega_1 \Pi \Omega_2 \Pi^H \Lambda_1 \Pi \Lambda_2 \Pi^H)| \\ &\leq \max_{\Psi_1, \Psi_2, \Psi_3, \Psi_4 \in \mathbb{U}_n} |\mathrm{tr}(\Omega_1 \Psi_1 \Omega_2 \Psi_2 \Lambda_1 \Psi_3 \Lambda_2 \Psi_4)| \\ &= \sum_{i=1}^{n} \alpha_i \tilde{\alpha}_i \beta_{n-i+1} \tilde{\beta}_{n-i+1}. \end{aligned} \tag{4.23}$$

It is easy to check that for any permutation π of $\{1, 2, \ldots, n\}$,

$$\sum_{i=1}^{n} \rho^2((\alpha_i, \beta_i), (\tilde{\alpha}_{\pi(i)}, \tilde{\beta}_{\pi(i)})) = n - \sum_{i=1}^{n} (\alpha_i^2 \tilde{\alpha}_{\pi(i)}^2 + \beta_i^2 \tilde{\beta}_{\pi(i)}^2) - \sum_{i=1}^{n} 2\alpha_i \tilde{\alpha}_{\pi(i)} \beta_i \tilde{\beta}_{\pi(i)},$$

which implies that

$$\begin{aligned} &\max_{\forall \pi} \sum_{i=1}^{n} \rho^2((\alpha_i, \beta_i), (\tilde{\alpha}_{\pi(i)}, \tilde{\beta}_{\pi(i)})) \\ = \ & n - \min_{\forall \pi} \left(\sum_{i=1}^{n} (\alpha_i^2 \tilde{\alpha}_{\pi(i)}^2 + \beta_i^2 \tilde{\beta}_{\pi(i)}^2) + \sum_{i=1}^{n} 2\alpha_i \tilde{\alpha}_{\pi(i)} \beta_i \tilde{\beta}_{\pi(i)} \right) \end{aligned} \tag{4.24}$$

and

$$\begin{aligned}
&\min_{\forall\pi}\sum_{i=1}^{n}\rho^2((\alpha_i,\beta_i),(\tilde{\alpha}_{\pi(i)},\tilde{\beta}_{\pi(i)}))\\
=\;& n-\max_{\forall\pi}\left(\sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{\pi(i)}^2+\beta_i^2\tilde{\beta}_{\pi(i)}^2)+\sum_{i=1}^{n}2\alpha_i\tilde{\alpha}_{\pi(i)}\beta_i\tilde{\beta}_{\pi(i)}\right). \qquad (4.25)
\end{aligned}$$

It follows from Lemma 4.1 that

$$\begin{aligned}
&\min_{\forall\pi}\left(\sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{\pi(i)}^2+\beta_i^2\tilde{\beta}_{\pi(i)}^2)+\sum_{i=1}^{n}2\alpha_i\tilde{\alpha}_{\pi(i)}\beta_i\tilde{\beta}_{\pi(i)}\right)\\
\geq\;& \min_{\forall\pi}\sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{\pi(i)}^2+\beta_i^2\tilde{\beta}_{\pi(i)}^2)\\
=\;& \sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{n-i+1}^2+\beta_i^2\tilde{\beta}_{n-i+1}^2).
\end{aligned}$$

Therefore, together with (4.3), (4.4), (4.12), (4.13) and Lemma 4.1 yields

$$\begin{aligned}
&\max_{\pi}\sum_{i=1}^{n}\rho^2((\alpha_i,\beta_i),(\tilde{\alpha}_{\pi(i)},\tilde{\beta}_{\pi(i)}))\\
\leq\;& n-\sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{n-i+1}^2+\beta_i^2\tilde{\beta}_{n-i+1}^2)\\
=\;& n-\sum_{i=1}^{n}(2\beta_i^2\tilde{\beta}_{n-i+1}^2-(\beta_i^2+\tilde{\beta}_{n-i+1}^2)+1)\\
=\;& \sum_{i=1}^{n}(\beta_i^2+\tilde{\beta}_{n-i+1}^2)-\sum_{i=1}^{n}2\beta_i^2\tilde{\beta}_{n-i+1}^2\\
\leq\;& \mathrm{tr}(S_1S_1^H+S_2S_2^H)-2\,\mathrm{tr}(S_1F_1\Pi_0F_2^HS_2^HS_2F_2\Pi_0^HF_1^HS_1^H), \qquad (4.26)
\end{aligned}$$

where Π_0 is the corresponding permutation matrix for $\pi_0(i)=n-i+1$ and θ_n, $\tilde{\theta}_n$, φ_i, ϕ_i

are given by Lemma 2.5 and (3.1). By (4.12), (4.13) and Lemma 4.1 we have

$$\begin{aligned}
&\max_{\pi}\left(\sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{\pi(i)}^2+\beta_i^2\tilde{\beta}_{\pi(i)}^2)+\sum_{i=1}^{n}2\alpha_i\tilde{\alpha}_{\pi(i)}\beta_i\tilde{\beta}_{\pi(i)}\right)\\
\le\ & \max_{\pi}(\sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{\pi(i)}^2+\beta_i^2\tilde{\beta}_{\pi(i)}^2))+\max_{\pi}(\sum_{i=1}^{n}2\alpha_i\tilde{\alpha}_{\pi(i)}\beta_i\tilde{\beta}_{\pi(i)})\\
\le\ & \max_{\pi}(\sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{\pi(i)}^2+\beta_i^2\tilde{\beta}_{\pi(i)}^2))+\sum_{i=1}^{n}2\alpha_i\tilde{\alpha}_i\beta_{n-i+1}\tilde{\beta}_{n-i+1}\\
\le\ & \sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_i^2+\beta_i^2\tilde{\beta}_i^2)+\sum_{i=1}^{n}2\sqrt{1-\varphi_i}\sqrt{1-\phi_i}\sqrt{\varphi_{n-i+1}}\sqrt{\tilde{\phi}_{n-i+1}},
\end{aligned}$$

which together with (4.25) leads to

$$\begin{aligned}
&\min_{\pi}\sum_{i=1}^{n}\rho^2((\alpha_i,\beta_i),(\tilde{\alpha}_{\pi(i)},\tilde{\beta}_{\pi(i)}))\\
\ge\ & n-\sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_i^2+\beta_i^2\tilde{\beta}_i^2)-\sum_{i=1}^{n}2\sqrt{1-\varphi_i}\sqrt{1-\phi_i}\sqrt{\varphi_{n-i+1}}\sqrt{\tilde{\phi}_{n-i+1}}\\
=\ & \sum_{i=1}^{n}(\beta_i^2+\tilde{\beta}_i^2)-2\sum_{i=1}^{n}\beta_i^2\tilde{\beta}_i^2-\sum_{i=1}^{n}2n\sqrt{1-\theta_1}\sqrt{1-\tilde{\theta}_1}\sqrt{\theta_n}\sqrt{\tilde{\theta}_n}\\
=\ & \mathrm{tr}(S_1S_1^H+S_2S_2^H)-2\mathrm{tr}(S_1F_1F_2^HS_2^HS_2F_2F_1^HS_1^H)-\sum_{i=1}^{n}2n\sqrt{1-\theta_1}\sqrt{1-\tilde{\theta}_1}\sqrt{\theta_n}\sqrt{\tilde{\theta}_n}\\
\ge\ & \mathrm{tr}(S_1S_1^H+S_2S_2^H)-2\mathrm{tr}(S_1F_1F_2^HS_2^HS_2F_2F_1^HS_1^H)-\sum_{i=1}^{n}2n\sqrt{1-\theta_1}\sqrt{1-\tilde{\theta}_1}\sqrt{\theta_n}\sqrt{\tilde{\theta}_n},
\end{aligned}$$

which together with (4.26) yields the desired results. This completes the proof. □

Remark 4.1. We give upper and lower bounds of $\sum_{i=1}^{n}\rho^2((\alpha_i,\beta_i),(\tilde{\alpha}_{\pi(i)},\tilde{\beta}_{\pi(i)}))$ for any permutation π of $\{1, 2, \ldots, n\}$.

Let $\{A, B\}$ and $\{\tilde{A}, \tilde{B}\}$ be two (m, p, n)-GMPs and let $\sigma\{A,B\}=\{(\alpha_i, \beta_i)\}_{i=1}^n$ and $\sigma\{\tilde{A}, \tilde{B}\}=\{(\tilde{\alpha}_i, \tilde{\beta}_i)\}_{i=1}^n$. If $\alpha_i\beta_i\le\alpha_j\beta_j$ and $\tilde{\alpha}_i\tilde{\beta}_i\le\tilde{\alpha}_j\tilde{\beta}_j$, $i\neq j$, $i, j=1, \ldots, n$. Then, for any permutation π of $\{1, 2, \ldots, n\}$, we have

$$\begin{aligned}
&\max_{\pi}\sum_{i=1}^{n}\rho^2((\alpha_i,\beta_i),(\tilde{\alpha}_{\pi(i)},\tilde{\beta}_{\pi(i)}))=\sum_{i=1}^{n}\rho^2((\alpha_i,\beta_i),(\tilde{\alpha}_i,\tilde{\beta}_i))\\
&=\mathrm{tr}(S_1S_1^H+S_2S_2^H)-2\mathrm{tr}(\mathcal{S}_1\mathcal{F}_1\mathcal{F}_2^H\mathcal{S}_2^H\mathcal{S}_2\mathcal{F}_2\mathcal{F}_1^H\mathcal{S}_1^H)-2\sum_{i=1}^{n}\sqrt{\varphi_i}\sqrt{1-\varphi_i^2}\sqrt{\phi_i}\sqrt{1-\phi_i^2}.
\end{aligned}$$

We will give some illustrations of $\alpha_i\beta_i \preceq \alpha_j\beta_j$ and $\tilde{\alpha}_i\tilde{\beta}_i \preceq \tilde{\alpha}_j\tilde{\beta}_j$, $i \neq j$, $i, j = 1, \ldots, n$.

Corollary 4.1. *Let $\{A, B\}$ and $\{\tilde{A}, \tilde{B}\}$ be two (m, p, n)-GMPs and let $\sigma\{A, B\} = \{(\alpha_i, \beta_i)\}_{i=1}^n$ and $\sigma\{\tilde{A}, \tilde{B}\} = \{(\tilde{\alpha}_i, \tilde{\beta}_i)\}_{i=1}^n$. If $\arccos(\beta_1) \in [0, \frac{\pi}{4}]$ and $\arccos(\tilde{\beta}_1) \in [0, \frac{\pi}{4}]$ or $\arccos(\beta_1) \in [\frac{\pi}{4}, \frac{\pi}{2}]$ and $\arccos(\tilde{\beta}_1) \in [\frac{\pi}{4}, \frac{\pi}{2}]$. Then for any permutation π of $\{1, 2, \ldots, n\}$, we have*

$$\begin{aligned}
&\max_{\pi} \sum_{i=1}^{n} \rho^2((\alpha_i, \beta_i), (\tilde{\alpha}_{\pi(i)}, \tilde{\beta}_{\pi(i)})) = \sum_{i=1}^{n} \rho^2((\alpha_i, \beta_i), (\tilde{\alpha}_i, \tilde{\beta}_i)) \\
&= \mathrm{tr}(S_1 S_1^H + S_2 S_2^H) - 2\mathrm{tr}(\mathcal{S}_1 \mathcal{F}_1 \mathcal{F}_2^H \mathcal{S}_2^H \mathcal{S}_2 \mathcal{F}_2 \mathcal{F}_1^H \mathcal{S}_1^H) \\
&\quad -2\sum_{i=1}^{n} \sqrt{\varphi_i}\sqrt{1-\varphi_i^2}\sqrt{\phi_i}\sqrt{1-\phi_i^2}, \\
&\min_{\pi} \sum_{i=1}^{n} \rho^2((\alpha_i, \beta_i), (\tilde{\alpha}_{\pi(i)}, \tilde{\beta}_{\pi(i)})) = \sum_{i=1}^{n} \rho^2((\alpha_i, \beta_i), (\tilde{\alpha}_{\pi_0(i)}, \tilde{\beta}_{\pi_0(i)})) \\
&= \mathrm{tr}(S_1 S_1^H + S_2 S_2^H) - 2\mathrm{tr}(\mathcal{S}_1 \mathcal{F}_1 \Pi_0 \mathcal{F}_2^H \mathcal{S}_2^H \mathcal{S}_2 \mathcal{F}_2^H \Pi_0^H \mathcal{F}_1^H \mathcal{S}_1^H) \\
&\quad -2\sum_{i=1}^{n} \sqrt{\varphi_i}\sqrt{1-\varphi_i^2}\sqrt{\phi_{n-i+1}}\sqrt{1-\phi_{n-i+1}^2},
\end{aligned}$$

where Π_0 is an $n \times n$ permutation matrix corresponding to permutation $\pi_0(i) = n-i+1$.

Proof. Since

$$\sum_{i=1}^{n} \rho^2((\alpha_i, \beta_i), (\tilde{\alpha}_{\pi(i)}, \tilde{\beta}_{\pi(i)})) = n - \sum_{i=1}^{n} (\alpha_i \tilde{\alpha}_{\pi(i)} + \beta_i \tilde{\beta}_{\pi(i)})^2,$$

then

$$\max_{\pi} \sum_{i=1}^{n} \rho^2((\alpha_i, \beta_i), (\tilde{\alpha}_{\pi(i)}, \tilde{\beta}_{\pi(i)})) = n - \min_{\pi} \sum_{i=1}^{n} (\alpha_i \tilde{\alpha}_{\pi(i)} + \beta_i \tilde{\beta}_{\pi(i)})^2,$$

$$\min_{\pi} \sum_{i=1}^{n} (\alpha_i \tilde{\alpha}_{\pi(i)} + \beta_i \tilde{\beta}_{\pi(i)})^2 = \min_{\pi} \sum_{i=1}^{n} (\alpha_i^2 \tilde{\alpha}_{\pi(i)}^2 + \beta_i^2 \tilde{\beta}_{\pi(i)}^2 + 2\alpha_i \tilde{\alpha}_{\pi(i)} \beta_i \tilde{\beta}_{\pi(i)}).$$

It follows from Lemma 4.1 that

$$\min_{\pi} \sum_{i=1}^{n} (\alpha_i^2 \tilde{\alpha}_{\pi(i)}^2 + \beta_i^2 \tilde{\beta}_{\pi(i)}^2) = \sum_{i=1}^{n} (\alpha_i^2 \tilde{\alpha}_{n-i+1}^2 + \beta_i^2 \tilde{\beta}_{n-i+1}^2).$$

If $\arccos(\beta_1) \in [0, \frac{\pi}{4}]$ and $\arccos(\tilde{\beta}_1) \in [0, \frac{\pi}{4}]$ or $\arccos(\beta_1) \in [\frac{\pi}{4}, \frac{\pi}{2}]$ and $\arccos(\tilde{\beta}_1) \in [\frac{\pi}{4}, \frac{\pi}{2}]$, then

$$\alpha_1\beta_1 \leq \cdots \leq \alpha_n\beta_n, \; \tilde{\alpha}_1\tilde{\beta}_1 \leq \cdots \leq \tilde{\alpha}_n\tilde{\beta}_n$$

or

$$\alpha_1\beta_1 \geq \cdots \geq \alpha_n\beta_n,\ \tilde{\alpha}_1\tilde{\beta}_1 \geq \cdots \geq \tilde{\alpha}_n\tilde{\beta}_n,$$

which implies

$$\min_{\pi}\sum_{i=1}^{n} 2\alpha_i\tilde{\alpha}_{\pi(i)}\beta_i\tilde{\beta}_{\pi(i)} = \sum_{i=1}^{n} 2\alpha_i\beta_i\tilde{\alpha}_{n-i+1}\tilde{\beta}_{n-i+1}.$$

Then

$$\min_{\pi}\sum_{i=1}^{n}(\alpha_i\tilde{\alpha}_{\pi(i)}+\beta_i\tilde{\beta}_{\pi(i)})^2 = \sum_{i=1}^{n}(\alpha_i\tilde{\alpha}_{n-i+1}+\beta_i\tilde{\beta}_{n-i+1})^2,$$

which implies

$$\max_{\pi}\sum_{i=1}^{n}\rho^2((\alpha_i,\ \beta_i),\ (\tilde{\alpha}_{\pi(i)},\ \tilde{\beta}_{\pi(i)})) = n - \sum_{i=1}^{n}(\alpha_i\tilde{\alpha}_{n-i+1}+\beta_i\tilde{\beta}_{n-i+1})^2.$$

Since

$$\min_{\pi}\sum_{i=1}^{n}\rho^2((\alpha_i,\ \beta_i),\ (\tilde{\alpha}_{\pi(i)},\ \tilde{\beta}_{\pi(i)})) = n - \max_{\pi}\sum_{i=1}^{n}(\alpha_i\tilde{\alpha}_{\pi(i)}+\beta_i\tilde{\beta}_{\pi(i)})^2,$$

$$\max_{\pi}\sum_{i=1}^{n}(\alpha_i\tilde{\alpha}_{\pi(i)}+\beta_i\tilde{\beta}_{\pi(i)})^2 = \max_{\pi}\sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{\pi(i)}^2+\beta_i^2\tilde{\beta}_{\pi(i)}^2+2\alpha_i\tilde{\alpha}_{\pi(i)}\beta_i\tilde{\beta}_{\pi(i)}),$$

it follows from Lemma 4.1 that

$$\max_{\pi}\sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_{\pi(i)}^2+\beta_i^2\tilde{\beta}_{\pi(i)}^2) = \sum_{i=1}^{n}(\alpha_i^2\tilde{\alpha}_i^2+\beta_i^2\tilde{\beta}_i^2).$$

If $\arccos(\beta_1)\in[0,\ \frac{\pi}{4}]$ and $\arccos(\tilde{\beta}_1)\in[0,\ \frac{\pi}{4}]$ or $\arccos(\beta_1)\in[\frac{\pi}{4},\ \frac{\pi}{2}]$ and $\arccos(\tilde{\beta}_1)\in[\frac{\pi}{4},\ \frac{\pi}{2}]$, then

$$\alpha_1\beta_1 \leq \cdots \leq \alpha_n\beta_n,\ \tilde{\alpha}_1\tilde{\beta}_1 \leq \cdots \leq \tilde{\alpha}_n\tilde{\beta}_n$$

or

$$\alpha_1\beta_1 \geq \cdots \geq \alpha_n\beta_n,\ \tilde{\alpha}_1\tilde{\beta}_1 \geq \cdots \geq \tilde{\alpha}_n\tilde{\beta}_n,$$

which implies

$$\max_{\pi}\sum_{i=1}^{n} 2\alpha_i\tilde{\alpha}_{\pi(i)}\beta_i\tilde{\beta}_{\pi(i)} = \sum_{i=1}^{n} 2\alpha_i\beta_i\tilde{\alpha}_i\tilde{\beta}_i.$$

Then

$$\max_{\pi}\sum_{i=1}^{n}(\alpha_i\tilde{\alpha}_{\pi(i)}+\beta_i\tilde{\beta}_{\pi(i)})^2 = \sum_{i=1}^{n}(\alpha_i\tilde{\alpha}_i+\beta_i\tilde{\beta}_i)^2,$$

which implies

$$\min_{\pi}\sum_{i=1}^{n}\rho^2((\alpha_i,\beta_i),(\tilde{\alpha}_{\pi(i)},\tilde{\beta}_{\pi(i)}))=n-\sum_{i=1}^{n}(\alpha_i\tilde{\alpha}_i+\beta_i\tilde{\beta}_i)^2.$$

This completes the proof. □

Remark 4.2. We give the upper and lower bounds of $\min_{\pi}\sum_{i=1}^{n}\rho^2((\alpha_i,\beta_i),(\tilde{\alpha}_{\pi(i)},\tilde{\beta}_{\pi(i)}))$ and $\max_{\pi}\sum_{i=1}^{n}\rho^2((\alpha_i,\beta_i),(\tilde{\alpha}_{\pi(i)},\tilde{\beta}_{\pi(i)}))$ for any permutation π of $\{1, 2, \ldots, n\}$.

5 Numerical Experiments

In this section, we give some numerical examples to illustrate the efficiency of the theoretical result by computing the chordal distance of two matrix pencils $\{A, B\}$ and $\{\tilde{A}, \tilde{B}\}$. All the experiments were carried out on an Apple MacBook Pro late 2016 using MATLAB R2018b with machine epsilon $\epsilon\approx 2.2204\times 10^{-16}$.

The result we used to compare is the upper bound obtained in [13]. In order to make fair comparisons, we use the economic version of the generalized singular value decomposition, i.e. MATLAB command `gsvd(A, B, 0)` and `gsvd(Ã, B̃, 0)`, to get the compact forms of U, V and $\tilde{U}$, $\tilde{V}$. Then we define

$$Y=\begin{pmatrix} U^H A \\ V^H B \end{pmatrix},\ \tilde{Y}=\begin{pmatrix} \tilde{U}^H \tilde{A} \\ \tilde{V}^H \tilde{B} \end{pmatrix},$$

and the upper bound in [13] is now calculated from

$$n\left(1-\sqrt[n]{\frac{\det(Y^H\tilde{Y})^2}{\det(Y^HY)\det(\tilde{Y}^H\tilde{Y})}}\right),$$

which saves a lot of calculations for large m and p.

Example 5.1. To test the efficiency of the new explicit formula, we randomly generated the matrix pencils $\{A, B\}$ and $\{\tilde{A}, \tilde{B}\}$ by MATLAB command `rand(m, n)` for A, $\tilde{A}$ and `rand(p, n)` for B, $\tilde{B}$, whereas $m=p=20000$ are fixed, n is firstly chosen as 300 and then repeatedly increased by 100 each time until 2000. The running times for calculating the exact values and the upper bounds of the chordal distances are shown in Fig. 1. In our experiments, the calculation of the exact value is always about 10 times faster than that of the upper bound. The speedup is mainly due to less computations in the SVD rather than the GSVD.

It is necessary to point out that we need to properly scale H and $\tilde{H}$ to avoid overflow or underflow in the calculation of the determinants in the upper bound when the matrix sizes are getting large. While the calculation of our new formula is more robust.

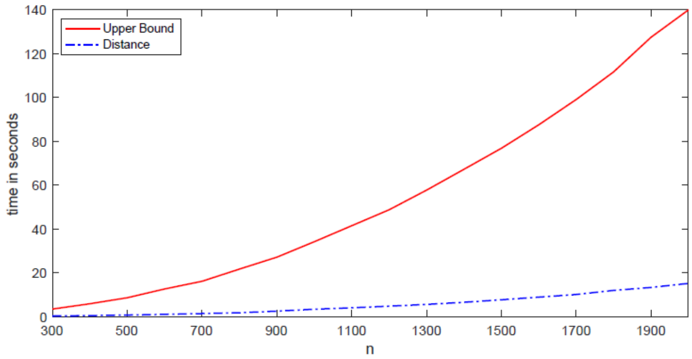

Figure 1: Performance comparison of calculating the exact values (by SVD) and the upper bounds (by GSVD) of the chordal distances. The matrices are randonly generated with $m = p = 20000$ and n increasing from 300 to 2000.

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Conflicts of Interest

The authors declare no conflict of interest.

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