

Convergence of Stochastic Gradient Descent Schemes for Łojasiewicz-Landscapes

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Abstract. In this article, we consider convergence of stochastic gradient descent schemes (SGD), including momentum stochastic gradient descent (MSGD), under weak assumptions on the underlying landscape. More explicitly, we show that on the event that the SGD stays bounded we have convergence of the SGD if there is only a countable number of critical points or if the objective function satisfies Łojasiewicz-inequalities around all critical levels as all analytic functions do. In particular, we show that for neural networks with analytic activation function such as softplus, sigmoid and the hyperbolic tangent, SGD converges on the event of staying bounded, if the random variables modelling the signal and response in the training are compactly supported.

Keywords:

Stochastic gradient descent,
Stochastic approximation,
Robbins-Monro,
Almost sure convergence,
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1 Introduction

In this article, we analyse stochastic gradient descent schemes for C^1 -objective functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ with $d \in \mathbb{N} := \{1, 2, \dots\}$ being an arbitrary dimension. We denote by $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \in \mathbb{N}_0}, \mathbb{P})$ a filtered probability space and consider an \mathbb{R}^d -valued stochastic process $(X_n)_{n \in \mathbb{N}_0}$ that admits a representation

$$X_n = X_{n-1} + \gamma_n(\Gamma_n + D_n) \quad (1.1)$$

for $n \in \mathbb{N}$, where

- $(\gamma_n)_{n \in \mathbb{N}}$ is a sequence of strictly positive reals, the step-sizes or learning rates.
- $(\Gamma_n)_{n \in \mathbb{N}}$ is an $(\mathcal{F}_n)_{n \in \mathbb{N}_0}$ -predictable sequence of random variables, the drift.
- $(D_n)_{n \in \mathbb{N}}$ is an $(\mathcal{F}_n)_{n \in \mathbb{N}}$ -adapted sequence of random variables, the perturbation.
- X_0 is an \mathcal{F}_0 -measurable random variable, the initial value.

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The choice $(\Gamma_n)_{n \in \mathbb{N}} = (-\nabla f(X_{n-1}))_{n \in \mathbb{N}}$ leads to a standard representation of stochastic gradient descent. However, our results hold for a more general class of dynamical systems, including momentum stochastic gradient descent (see Section 2), assuming the drift is comparable in size and direction to the gradient vector field (the precise condition is given in Definition 1.1). Additional assumptions will be imposed in the theorems below.

Stochastic gradient descent schemes form a subclass of Robbins-Monro schemes which were introduced in 1951 [63] and have been highly influential since then. Their relevance stems from their applicability of finding zeros of functions $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$ in the case where one has only simulations at hand which give approximations to the value of F in mean. Following the original papers a variety of results were derived, and we refer the reader to the mathematical accounts [9, 23, 41] on stochastic approximation methods.

In this article, we analyse convergence of $(X_n)_{n \in \mathbb{N}_0}$. This problem is intimately related to understanding the asymptotic behaviour of $(\nabla f(X_n))_{n \in \mathbb{N}_0}$. The classical analysis of Polyak and Tsytkin [60] yields existence of the limit $\lim_{n \rightarrow \infty} f(X_n)$ and proves $\liminf_{n \rightarrow \infty} |\nabla f(X_n)| = 0$ under appropriate assumptions. Later, Walk [70] showed that in an appropriate setting one has almost sure convergence $\lim_{n \rightarrow \infty} \nabla f(X_n) = 0$. Similar results were established in various settings, see [10, 26, 30, 43, 47, 49].

We will provide a short proof for $\lim_{n \rightarrow \infty} \nabla f(X_n) = 0$ under weak assumptions. Here, we include the case where ∇f is only Hölder continuous and where $(D_n)_{n \in \mathbb{N}}$ is an L^p -martingale difference sequence for a $p \in (1, 2]$. Then, we will conclude that we have almost sure convergence of $(X_n)_{n \in \mathbb{N}_0}$ on the event that $(X_n)_{n \in \mathbb{N}_0}$ stays bounded in the case where the set of critical points of f does not contain a continuum, see Theorem 1.1 below.

In the case where the set of critical points of f contains a continuum of points the situation is more subtle. In that case, Tadic [68] showed that under a Łojasiewicz-inequality stochastic gradient descent schemes converge under appropriate additional assumptions. In this article, our considerations are also based on the validity of certain Łojasiewicz-inequalities. However, here we allow the drift to be more general so that the new convergence theorem is applicable for a bigger class of optimisation methods such as momentum stochastic gradient descent (MSGD). We stress that the proofs developed in this article are significantly different from the ones in [68]. We mention that the asymptotic behaviour of a stochastic gradient descent scheme is tightly related to that of the first order differential equation

$$\dot{x}_t = -\nabla f(x_t) \quad (1.2)$$

(though many approximation results only hold for a finite time-horizon), see e.g. [50, Proposition 1]. Convergence for the latter differential equation is a non-trivial issue even in the case where the solution stays on a compact set: One can find C^∞ -functions f together with solutions $(x_t)_{t \geq 0}$ that stay on compact sets but do not converge, see [54, Example 3, page 14]. Counterexamples of this structure have been known for a long time (see e.g. [16]) and include the famous Mexican hat function [1]. To guarantee convergence (at least in the case where the solution stays on a compact set), one needs to impose additional assumptions. An appropriate assumption is the validity of a Łojasiewicz-inequality, see Definition 1.2 below. This assumption has the appeal that it is satisfied by analytic functions, see [45, 46].

Recently, there is a growing interest in the case where a Łojasiewicz-inequality holds with exponent $\beta = 1/2$, often referred to as Polyak-Łojasiewicz-inequality (or PŁ-inequality). The idea first appeared in 1963 in an article of Polyak [57] where linear convergence in the objective function is shown for non-perturbed gradient descent. In machine learning, this inequality turns out to be quite effective as a tool to weaken strict convexity assumptions and allow multiple global minima, see e.g. [7, 27, 29, 37, 69, 72, 76] and the references therein. We stress that still the respective assumptions are significantly stricter than the ones we will impose. In particular, the existence of a global PŁ-inequality implies that the objective function f is quasi-strongly convex, see e.g. [4, 62]. In contrast, our analysis allows the landscape to contain local minima, maxima or saddle points.

In the main result of this article, we prove convergence of $(X_n)_{n \in \mathbb{N}_0}$ for objective functions that are Łojasiewicz-functions (i.e. locally satisfy a general Łojasiewicz-inequality) on the event that the scheme stays bounded, see Theorem 1.2 below. In particular, our result applies if the objective function is analytic. This generalises the analysis in [1] where convergence of gradient descent without stochastic noise is considered. We stress that the limit points may be local minima as well as saddle points. After a preprint version of this article appeared, there has been a lot of interest in the interplay between stochastic optimisation algorithms and loss landscapes satisfying locally Kurdyka-Łojasiewicz-inequalities (a slight generalisation of the Łojasiewicz-inequality) and we point to [14, 24, 34, 44, 51] for recent developments in this direction.

As an example we provide a machine learning application: In the case where in a deep learning representation the activation function is analytic (for instance, softplus, hyperbolic tangent, or sigmoid) and the random variables modelling the signal and the response are compactly supported, the respective objective function is analytic, see Theorem 5.1. Hence, in that case our results show that a SGD scheme associated with the training of the network converges almost surely on the event of staying bounded. Concerning the training of neural networks via SGD we also refer the reader to [5, 6, 11, 19, 22, 25, 28, 35, 36, 42]. Related objective functions (loss landscapes) are analysed in [15, 17, 20, 52, 55, 61].

Our analysis is based on non-asymptotic inequalities and if one has additional information about the explicit Łojasiewicz-type inequalities for the objective function to hold our results entail moment estimates for the objective function value, see Proposition 4.2, and the distance that can be overcome by the SGD when observing convergence to a particular critical level, see Proposition 4.4.

Let us summarise the contribution of the present work:

- We show almost sure convergence of $(f(X_n))_{n \in \mathbb{N}_0}$ and $(\nabla f(X_n))_{n \in \mathbb{N}_0}$ under the assumption that ∇f is locally Hölder continuous and $(D_n)_{n \in \mathbb{N}}$ is an L^p -martingale difference sequence for a $p \in (1, 2]$, see Theorem 1.1.
- We generalise the result in [68] showing almost sure convergence of $(X_n)_{n \in \mathbb{N}_0}$ under the assumption that f is a Łojasiewicz-function, see Definition 1.2, where we weaken the assumptions on the step-sizes $(\gamma_n)_{n \in \mathbb{N}}$ and stochastic noise $(D_n)_{n \in \mathbb{N}}$, see Theorem 1.2.
- Both results are shown for a general class of dynamical systems assuming a certain angle condition between the drift term $(\Gamma_n)_{n \in \mathbb{N}}$ and the gradient of the objective

function $(\nabla f(X_{n-1}))_{n \in \mathbb{N}}$, see Definition 1.1. Therefore, our results hold for other optimisation methods including momentum stochastic gradient descent, see Theorem 2.1.

- We verify the validity of a Łojasiewicz-inequality in the supervised learning setting using neural networks with analytic activation function, see Theorem 5.1.

We proceed with the central definitions and statements.

Definition 1.1. 1. We denote by \mathbb{L} the event that $(X_n)_{n \in \mathbb{N}_0}$ stays bounded, i.e.

$$\mathbb{L} = \left\{ \limsup_{n \rightarrow \infty} |X_n| < \infty \right\}.$$

2. We denote by \mathbb{G} the event

$$\mathbb{G} = \left\{ \liminf_{n \rightarrow \infty} \frac{\langle -\nabla f(X_{n-1}), \Gamma_n \rangle}{|\nabla f(X_{n-1})|^2} > 0 \text{ and } \liminf_{n \rightarrow \infty} \frac{|\nabla f(X_{n-1})|}{|\Gamma_n|} > 0 \right\},$$

where we interpret $0/0$ as ∞ .

3. For $q \geq 1$ and a sequence $(\sigma_n)_{n \in \mathbb{N}}$ of strictly positive reals, we denote by $\mathbb{M}^{\sigma, q}$ the event¹

$$\mathbb{M}^{\sigma, q} = \left\{ \limsup_{n \rightarrow \infty} \sigma_n^{-1} \mathbb{E}[|D_n|^q | \mathcal{F}_{n-1}]^{\frac{1}{q}} < \infty, \right. \\ \left. \text{and } \mathbb{E}[D_n | \mathcal{F}_{n-1}] = 0 \text{ for all but finitely many } n \right\}.$$

Remark 1.1. Let us discuss the satisfiability of the latter events. First, for continuously differentiable objective functions f with Lipschitz continuous differential satisfying $f(x) \rightarrow \infty$ as $|x| \rightarrow \infty$ the SGD process almost surely stays bounded (see [19, Lemma D.1]). In the case that f does not necessarily satisfy the latter property, one can add an L^2 -regularisation term, i.e. replace f by \tilde{f} given by $\tilde{f}(x) = f(x) + a|x|^2/2$ for a sufficiently large $a > 0$ (see [19, Remark 2.8]). Next, when considering stochastic gradient descent, i.e. $(\Gamma_n)_{n \in \mathbb{N}} = (-\nabla f(X_{n-1}))_{n \in \mathbb{N}}$, we clearly have $\mathbb{P}(\mathbb{G}) = 1$. However, the class of optimisation methods satisfying $\mathbb{P}(\mathbb{G}) = 1$ is much broader and include, e.g. line-search methods (see [53, Chapter 3] and [1, Chapter 4.1]), inexact gradient methods (see, e.g. [39]) and momentum stochastic gradient descent (see Section 2). The two conditions defining the set \mathbb{G} guarantee that the size of the drift Γ_n and the negative gradient $-\nabla f(X_{n-1})$ are asymptotically comparable and the angle between the two vectors stays uniformly below 90° at late times. See [12] for more examples of gradient-based optimisation dynamics satisfying this angle condition. Last, in stochastic optimisation a common assumption on the perturbation is the so-called ABC condition, stating that there exist constants $A, B, C \geq 0$ such that

$$\mathbb{E}[|D_n|^2 | \mathcal{F}_{n-1}] \leq 2A(f(X_{n-1}) - f(x^*)) + B|\nabla f(X_{n-1})|^2 + C,$$

¹Formally, $\mathbb{E}[D_n | \mathcal{F}_{n-1}] = 0$ is to be understood as condition $\mathbb{E}[D_n^+ | \mathcal{F}_{n-1}] = \mathbb{E}[D_n^- | \mathcal{F}_{n-1}] < \infty$, where D_n^+ and D_n^- denote the positive and negative part of D_n . Note that for general non-negative random variables conditional expectations are always well-defined and, in particular, we do not assume integrability of D_n in this article.

where x^* denotes a critical point of f (see, e.g. [29, 38, 43]). Note that, after establishing boundedness of $(X_n)_{n \in \mathbb{N}}$, the first and second terms on the right-hand side in the latter inequality can be bounded by a constant depending on the bounds of the objective function and its differential on the relevant domain. In this work, we allow the size of the stochastic noise to be comparable to a (possibly unbounded) sequence $(\sigma_n)_{n \in \mathbb{N}}$.

Let us state our first main result concerning the convergence of $(f(X_n))_{n \in \mathbb{N}_0}$ as well as $(\nabla f(X_n))_{n \in \mathbb{N}_0}$. We stress that, comparing to classical results, we weaken the assumptions on the objective function (allowing functions with Hölder continuous differential) and on the stochastic perturbation (allowing martingale differences in $L^{1+\alpha_1}$ for an $\alpha_1 > 0$). See [32, 65] for heavy-tailed noises in SGD that are in $L^{1+\alpha_1}$ for an $\alpha_1 < 1$ but have an infinite second moment. Various sequences $(\sigma_n)_{n \in \mathbb{N}}$ appear in multilevel stochastic gradient descent algorithms, see [21].

Theorem 1.1. *Let $0 < \alpha_1 \leq \alpha_2 \leq 1$ and $(\sigma_n)_{n \in \mathbb{N}}$ be a sequence of strictly positive reals. Suppose that ∇f is locally α_2 -Hölder continuous, that $\gamma_n \rightarrow 0$,*

$$\sum_{n=1}^{\infty} (\gamma_n \sigma_n)^{1+\alpha_1} < \infty,$$

and in the case where $\alpha_2 < 1$ that, additionally,

$$\sum_{n=1}^{\infty} \gamma_n^{\frac{1+\alpha_2}{1-\alpha_2}} < \infty.$$

Then, almost surely, on $\mathbb{C} := \mathbb{L} \cap \mathbb{G} \cap \mathbb{M}^{\sigma, 1+\alpha_1}$, the limit $\lim_{n \rightarrow \infty} f(X_n)$ exists and, if additionally, $\sum_{n=1}^{\infty} \gamma_n = \infty$, one has almost surely, on \mathbb{C} , that

$$\lim_{n \rightarrow \infty} \nabla f(X_n) = 0.$$

Moreover, in the case where the set of critical points of f , $\mathcal{C} = \{x : \nabla f(x) = 0\}$, does not contain a continuum of elements, i.e. there does not exist an injective mapping taking $[0, 1]$ to \mathcal{C} , then we have almost sure convergence of $(X_n)_{n \in \mathbb{N}_0}$ on \mathbb{C} .

Remark 1.2. Note that in the case where \mathcal{C} is countable the set \mathcal{C} does not contain a continuum of elements.

Next we prepare the main result of convergence under Łojasiewicz-type assumptions.

Definition 1.2. *We call a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ Łojasiewicz-function, if f is continuously differentiable with locally Lipschitz continuous differential and if for every $x \in \mathcal{C} = \nabla f^{-1}(\{0\})$, the Łojasiewicz-inequality is true on a neighbourhood U_x of x with parameters $L > 0$ and $\beta \in [1/2, 1)$, i.e. for all $y \in U_x$,*

$$|\nabla f(y)| \geq L |f(y) - f(x)|^\beta.$$

Remark 1.3. Note that for all $x \notin \mathcal{C}$ and $\beta > 0$ there trivially exist a neighbourhood U_x and constant $L > 0$ such that the Łojasiewicz-inequality with parameter L and β hold on U_x .

Theorem 1.2. Let f be a Łojasiewicz-function and $q \geq 2$. Suppose that for $n \in \mathbb{N}$,

$$\gamma_n = C_\gamma n^{-\gamma}, \quad \sigma_n = n^\sigma,$$

where $C_\gamma > 0, \gamma \in (1/2, 1]$ and $\sigma \in \mathbb{R}$. If

$$\frac{2}{3}(\sigma + 1) < \gamma, \quad \frac{1}{2\gamma - \sigma - 1} < q,$$

then, on $\mathbb{C} := \mathbb{L} \cap \mathbb{G} \cap \mathbb{M}^{\sigma, q}$, the process $(X_n)_{n \in \mathbb{N}_0}$ converges, almost surely, to a critical point of f .

Remark 1.4. In the case where $\gamma > 1, q \in [1, 2]$ and $\sigma \in \mathbb{R}$ with $q(\gamma - \sigma) > 1$ we clearly have almost sure convergence on

$$\mathbb{C} := \mathbb{L} \cap \mathbb{G} \cap \mathbb{M}^{\sigma, q}.$$

However, in that case $\lim_{n \rightarrow \infty} X_n$ is not necessarily a critical point of f . Indeed, let K be a compact set and consider the events

$$\mathbb{A}_n^{N, C, K} = \bigcap_{\ell=N}^n \{X_\ell \in K, |\Gamma_{\ell+1}| \leq C|\nabla f(X_\ell)|, \mathbb{E}[|D_{\ell+1}|^q | \mathcal{F}_\ell] \leq C\sigma_{\ell+1}^q, \mathbb{E}[D_{\ell+1} | \mathcal{F}_n] = 0\}.$$

Then, on $\mathbb{A}_\infty^{N, C, K} = \bigcap_{n \geq N} \mathbb{A}_n^{N, C, K}$ we have for all $n > N$,

$$|X_n - X_N| \leq C\|\nabla f\|_{L^\infty(K)} \sum_{\ell=N+1}^n \gamma_\ell + \left| \sum_{\ell=N+1}^n \gamma_\ell D_\ell \right|.$$

Moreover,

$$(M_n)_{n > N} = \left(\sum_{\ell=N+1}^n \mathbb{1}_{\mathbb{A}_{\ell-1}^{N, C, K}} \gamma_\ell D_\ell \right)_{n > N}$$

is a martingale which converges almost surely on $\mathbb{A}_\infty^{N, C, K}$ due to Lemma 3.2. The result then follows by taking the countable union over such sets.

Remark 1.5. 1. The assumption $q > (2\gamma - \sigma - 1)^{-1}$ in Theorem 1.2 is only a technical condition to control the undershoots of the SGD scheme near a local maximum or saddle point. If, on $\mathbb{L} \cap \mathbb{M}^{\sigma, 2}$, we have almost surely that $d(X_n, \mathcal{M}) \rightarrow 0$, where \mathcal{M} denotes the set of all local minima of f , then $q = 2$ is sufficient for all choices of γ and σ satisfying $2(\sigma + 1)/3 < \gamma$.

2. We consider the case, where $(\gamma_n)_{n \in \mathbb{N}} = (C_\gamma n^{-\gamma})_{n \in \mathbb{N}}$ with $C_\gamma > 0$ and $\gamma \in (1/2, 1]$ and $(D_n)_{n \in \mathbb{N}}$ is a sequence of martingale differences with size of order $\mathcal{O}(1)$. In an earlier work, Tadic [68] showed almost sure convergence of stochastic gradient descent (see also [67] for an extended version) for analytic objective functions under the assumption that there exists an $r > 1$ such that, almost surely,

$$\limsup_{n \rightarrow \infty} \max_{k \geq n} \left\| \sum_{i=n}^k \gamma_i t_i^r D_i \right\| < \infty,$$

where $t_n = \sum_{i=1}^n \gamma_i$. While this assumption requires $\gamma > 3/4$ our analysis allows us to go as low as $\gamma > 2/3$.

3. The SGD scheme shows similar behaviour as the solution to an SDE with a drift term which is comparable in size and direction to $-\nabla f$ plus a diffusion term with vanishing diffusivity. Such SDEs are analysed in [18], where the diffusive term is assumed to have a bracket process whose magnitude is bounded by $t^{-2\rho} dt$ with $\rho \in (0, \infty)$. If $\rho > 1$, one can devise a convergence proof for the SDE under analogous assumptions as imposed here. Conversely, there exists a counterexample with $\rho = 1$ for which the SDE diverges and circles around a sphere infinitely often, see [18]. When translating the SGD problem (with exponent $\gamma < 1$) analysed here into the SDE problem the associated ρ is

$$\rho = \frac{\gamma - 2\sigma}{2 - 2\gamma}.$$

Note that $2(\sigma + 1)/3$ is the largest γ -value for which the respective ρ is smaller or equal to 1. This suggests that in Theorem 1.2 the assumption $\gamma > 2(\sigma + 1)/3$ is natural and that the result does in general not hold for smaller γ .

We give two important applications of the main statement above. As found by Ruppert [64] and Polyak [58, 59], in many scenarios the running average of a Robbins-Monro algorithm yields a better performance as the Robbins-Monro algorithm itself. This is even the case where the potential limit points are non-discrete and form a stable manifold, see [19]. Convergence of the algorithm seems to be (up to very particular examples) a necessity for the Ruppert-Polyak-averaging to have a positive effect. Our research suggests that Ruppert-Polyak-averaging may also have a positive effect in our setting. At least it entails that the average converges to the same parameter value and thus gives in the limit the same target value (loss) as the Robbins-Monro algorithm itself.

A promising line of recent research focusses on the implicit bias of gradient descent algorithms, see e.g. [2, 13, 31, 66, 74] and the references mentioned therein. Although the model is often trained using a fixed training set, stochastic gradient descent seems to pick an empirical risk minimiser that generalises well to unseen data. A common state-of-the-art hypothesis in the machine learning community is that implicit bias might be one of the reasons for the outstanding performance of stochastic gradient descent in practice. Our results further motivates the analysis of the implicit bias. In particular, we show that SGD converges to a limit point and one is left to investigate key properties of the limit that may be important to achieve a good generalisation error (such as flatness of the minima, e.g. [73, 75, 77]).

The article is arranged as follows. In Section 2, we apply our results to show convergence of momentum stochastic gradient descent. In Section 3, we provide the proof of Theorem 1.1. Section 4 proceeds with the proof of Theorem 1.2. Finally, Section 5 discusses analytic neural networks and, in particular, analyticity of a particular deep learning network is established in Theorem 5.1. Recall that analyticity of the objective function f implies that f is a Łojasiewicz-function. Therefore, our main results apply for the objective functions that arise from a regression task in supervised learning when using analytic activation functions and an analytic loss function, see Theorem 5.1.

2 Convergence of momentum stochastic gradient descent

Consider the second order system satisfying

$$\begin{aligned} X_n &= X_{n-1} + \gamma_n(V_{n-1} + D_n^{(1)}), \\ V_n &= V_{n-1} - \gamma_n(\mu V_{n-1} + \nabla f(X_{n-1}) - D_n^{(2)}) \end{aligned} \quad (2.1)$$

for $n \in \mathbb{N}$, where $\mu > 0$, $(D_n^{(1)})_{n \in \mathbb{N}}$ and $(D_n^{(2)})_{n \in \mathbb{N}}$ are $(\mathcal{F}_n)_{n \in \mathbb{N}}$ -adapted sequences of random variables, $(\gamma_n)_{n \in \mathbb{N}}$ is a sequence of strictly positive reals and X_0, V_0 are \mathcal{F}_0 -measurable random variables. Set

$$\Gamma_n := \begin{pmatrix} V_{n-1} \\ -\mu V_{n-1} - \nabla f(X_{n-1}) \end{pmatrix}, \quad D_n := \begin{pmatrix} D_n^{(1)} \\ D_n^{(2)} \end{pmatrix},$$

and note that (2.1) can be written as the first order system (1.1).

Theorem 2.1. (i) Let $0 < \alpha_1 \leq \alpha_2 \leq 1$ and $(\sigma_n)_{n \in \mathbb{N}}$ be a sequence of strictly positive reals. Suppose that f is C^2 and $\text{Hess } f$ is locally α_2 -Hölder continuous, that $\gamma_n \rightarrow 0$,

$$\sum_{n=1}^{\infty} \gamma_n = \infty, \quad \sum_{n=1}^{\infty} (\gamma_n \sigma_n)^{1+\alpha_1} < \infty,$$

and in the case where $\alpha_2 < 1$ that, additionally,

$$\sum_{n=1}^{\infty} \gamma_n^{\frac{1+\alpha_2}{1-\alpha_2}} < \infty.$$

Then we have almost surely on the event $\mathbb{L} \cap \mathbb{M}^{\sigma, 1+\alpha_1}$ that $V_n \rightarrow 0$ and $\nabla f(X_n) \rightarrow 0$.

(ii) Let f be a C^2 -Łojasiewicz-function such that $\text{Hess } f$ is locally Lipschitz continuous and $q \geq 2$. Suppose that for $n \in \mathbb{N}$,

$$\gamma_n = C_\gamma n^{-\gamma}, \quad \sigma_n = n^\sigma,$$

where $C_\gamma > 0, \gamma \in (1/2, 1]$ and $\sigma \in \mathbb{R}$. If

$$\frac{2}{3}(\sigma + 1) < \gamma, \quad \frac{1}{2\gamma - \sigma - 1} < q,$$

then, on $\mathbb{L} \cap \mathbb{M}^{\sigma, q}$, the process $(X_n)_{n \in \mathbb{N}_0}$ converges, almost surely, to a critical point of f .

Proof. (i) Let $K \subset \mathbb{R}^d$ be a compact set, define $\mathbb{L}^K = \bigcap_{n \in \mathbb{N}_0} \{X_n \in K\}$ and note that it suffices to show the statement on the event $\mathbb{L}^K \cap \mathbb{M}^{\sigma, 1+\alpha_1}$ since $\mathbb{L} = \bigcup_{n \in \mathbb{N}} \mathbb{L}^{K_n}$, where $(K_n)_{n \in \mathbb{N}}$ denotes an increasing sequence of compact sets with $\bigcup_{n \in \mathbb{N}} K_n = \mathbb{R}^d$. For $a > 0$ define the energy function

$$\mathcal{E}(x, v) := \frac{1}{2} \|v\|^2 + f(x) + a \langle \nabla f(x), v \rangle,$$

so that

$$\nabla \mathcal{E}(x, v) = \begin{pmatrix} \nabla f(x) + a \operatorname{Hess} f(x) v \\ v + a \nabla f(x) \end{pmatrix}.$$

Set $C := \sup_{x \in K} \|\operatorname{Hess} f(x)\|$ and note that for all $n \in \mathbb{N}$ we have

$$|\Gamma_n|^2 \leq (1 + 2\mu^2)|V_{n-1}|^2 + 2|\nabla f(X_{n-1})|^2$$

as well as

$$\begin{aligned} |\nabla \mathcal{E}(X_{n-1}, V_{n-1})|^2 &\geq (1 - 2a + a^2)|\nabla f(X_{n-1})|^2 + (1 - (C^2 + 1)a)|V_{n-1}|^2, \\ |\nabla \mathcal{E}(X_{n-1}, V_{n-1})|^2 &\leq (1 + 2a + a^2)|\nabla f(X_{n-1})|^2 + (1 + (C^2 + 1)a + C^2 a^2)|V_{n-1}|^2, \end{aligned}$$

and

$$\langle -\nabla \mathcal{E}(X_{n-1}, V_{n-1}), \Gamma_n \rangle \geq \left(\mu - aC - \frac{a}{2}\mu^2 \right) |V_{n-1}|^2 + \frac{a}{2} |\nabla f(X_{n-1})|^2.$$

Thus, for sufficiently small a we have

$$\mathbb{P} \left(\mathbb{L}^K \cap \left\{ \liminf_{n \rightarrow \infty} \frac{\langle -\nabla \mathcal{E}(X_{n-1}, V_{n-1}), \Gamma_n \rangle}{|\nabla \mathcal{E}(X_{n-1}, V_{n-1})|^2} \leq 0 \text{ or } \liminf_{n \rightarrow \infty} \frac{|\nabla \mathcal{E}(X_{n-1}, V_{n-1})|}{|\Gamma_n|} \leq 0 \right\} \right) = 0.$$

Next, we show that on $\mathbb{L}^K \cap \mathbb{M}^{\sigma, 1+\alpha_1}$ we almost surely have $\limsup_{n \rightarrow \infty} |V_n| < \infty$. For arbitrary $C_\sigma, C_V > 0$ and $N \in \mathbb{N}$ such that $\sup_{n \geq N} \gamma_n < \mu$ consider the adapted sequence of events $(\mathbb{A}_n)_{n \geq N}$ given by

$$\begin{aligned} \mathbb{A}_n &= \{ |V_N| \leq C_V, X_m \in K, \mathbb{E}[D_{m+1} | \mathcal{F}_m] = 0, \\ &\quad \mathbb{E}[|D_{m+1}|^{1+\alpha_1} | \mathcal{F}_m] \leq C_\sigma \sigma_{m+1}^{1+\alpha_1}, \forall m = N, \dots, n \} \end{aligned}$$

for $n \geq N$ and set $C_f := \sup_{x \in K} |\nabla f(x)|$. It suffices to show that, for a constant $\kappa \geq 0$, the process $(\tilde{V}_n)_{n \geq N}$ given by

$$\tilde{V}_N := \mathbb{1}_{\mathbb{A}_N} \left(|V_N| \vee \frac{C_f}{\mu} \right)^{1+\alpha_1},$$

and

$$\tilde{V}_n := \mathbb{1}_{\mathbb{A}_{n-1}} \left(|V_n| \vee \frac{C_f}{\mu} \right)^{1+\alpha_1} - \kappa C_\sigma \sum_{i=N+1}^n (\gamma_i \sigma_i)^{1+\alpha_1}, \quad n > N$$

is almost surely bounded, since $\sum_{n \in \mathbb{N}} (\gamma_n \sigma_n)^{1+\alpha_1} < \infty$. Since $f(x) = x^{1+\alpha_1}$ has an α_1 -Hölder continuous gradient there exists a constant $\kappa \geq 0$ such that

$$|V_n|^{1+\alpha_1} \leq |(W_{n-1})|^{1+\alpha_1} + \langle (1 + \alpha_1) W_{n-1} | W_{n-1} |^{-1+\alpha_1}, \gamma_n D_n \rangle + \kappa |\gamma_n D_n|^{1+\alpha_1},$$

where $W_{n-1} = (1 - \gamma_n \mu) V_{n-1} - \gamma_n \nabla f(X_{n-1})$. Thus, \tilde{V}_n is integrable for all $n > N$ and

$$\begin{aligned} &\mathbb{E}[\mathbb{1}_{\mathbb{A}_n} |V_n|^{1+\alpha_1} | \mathcal{F}_{n-1}] \\ &\leq \mathbb{1}_{\mathbb{A}_{n-1}} |(1 - \gamma_n \mu) V_{n-1} - \gamma_n \nabla f(X_{n-1})|^{1+\alpha_1} + \kappa C_\sigma (\gamma_n \sigma_n)^{1+\alpha_1} \\ &\leq \mathbb{1}_{\mathbb{A}_{n-1}} |V_{n-1}|^{1+\alpha_1} + \kappa C_\sigma (\gamma_n \sigma_n)^{1+\alpha_1}, \end{aligned}$$

if $|V_{n-1}| > C_f/\mu$, as well as

$$\mathbb{1}_{\mathbb{A}_{n-1}} \mathbb{E}[|V_n|^{1+\alpha_1} | \mathcal{F}_{n-1}] \leq \mathbb{1}_{\mathbb{A}_{n-1}} \left(\frac{C_f}{\mu} \right)^{1+\alpha_1} + \kappa C_\sigma (\gamma_n \sigma_n)^{1+\alpha_1},$$

if $|V_{n-1}| \leq C_f/\mu$. Since $(\tilde{V}_n)_{n \geq \mathbb{N}}$ is almost surely bounded from below, we can apply Doob's martingale convergence theorem to deduce that $\limsup_{n \rightarrow \infty} |V_n| < \infty$ on $\mathbb{A}_\infty := \bigcap_{n \geq N} \mathbb{A}_n$ and, thus, on $\mathbb{L}^K \cap \mathbb{M}^{\sigma, 1+\alpha_1}$. Now, using Theorem 1.1 we get almost surely on $\mathbb{L}^K \cap \mathbb{M}^{\sigma, 1+\alpha_1}$ that $\nabla \mathcal{E}(X_n, V_n) \rightarrow 0$, which for sufficiently small a implies $V_n \rightarrow 0$ and $\nabla f(X_n) \rightarrow 0$.

(ii) In order to show convergence of $(X_n)_{n \in \mathbb{N}_0}$ we need to show that for sufficiently small a and every $x \in K$ there exists a neighbourhood $U \subset \mathbb{R}^{2d}$ of $(x, 0) \in \mathbb{R}^{2d}$ satisfying a Łojasiewicz-inequality for \mathcal{E} on U . The proof for the latter statement can be found in [12, Section 3.2]. \square

3 Proof of Theorem 1.1

In this section, we prove almost sure convergence of $(f(X_n))_{n \in \mathbb{N}_0}$ and $(\nabla f(X_n))_{n \in \mathbb{N}_0}$ under the assumption that the martingale noise is in $L^{1+\alpha_1}$ and ∇f is α_2 -Hölder continuous, for $0 < \alpha_1 \leq \alpha_2 \leq 1$, by analysing the evolution of $(f(X_n))_{n \in \mathbb{N}_0}$ step by step. We use the classical argument, Lemma 3.2, to show that the influence of the martingale noise is negligible at late times. Then, we prove that Hölder continuity of ∇f is sufficient for the remainder in the Taylor-approximation to have a negligible effect. We conclude that, for small step-sizes, the system $(f(X_n))_{n \in \mathbb{N}_0}$ has a tendency to decrease and using the boundedness of f on a compact domain we deduce convergence of $(f(X_n))_{n \in \mathbb{N}_0}$. The main technical tool that allows us to consider objective functions with Hölder continuous gradient is the application of a Hölder inequality, see (3.2), that in a similar form appears in the analysis of the ODE-method for SGD, see [8, Proposition 4.2]. The convergence of $(\nabla f(X_n))_{n \in \mathbb{N}_0}$ and $(X_n)_{n \in \mathbb{N}_0}$ (for functions having isolated critical points) follows from a path-wise analysis using the convergence of $(f(X_n))_{n \in \mathbb{N}_0}$.

Lemma 3.1. *Let $N \in \mathbb{N}, C > 0, K \subset \mathbb{R}^d$ be a compact and convex set and denote by $(\mathbb{A}_n)_{n \geq N}$ a decreasing sequence of events such that for every $n \geq N, \mathbb{A}_n \in \mathcal{F}_n$ and*

$$\mathbb{A}_n \subset \{X_n \in K, |\nabla f(X_n)|^2 \leq C \langle \nabla f(X_n), -\Gamma_{n+1} \rangle, |\Gamma_{n+1}| \leq C |\nabla f(X_n)|\}.$$

Further, suppose that $\gamma_n \rightarrow 0$, there exists $0 < \alpha_1 \leq \alpha_2 \leq 1$ such that ∇f is α_2 -Hölder continuous on K and there exists a sequence $(\sigma_n)_{n \in \mathbb{N}}$ of positive reals such that

1. $\mathbb{E}[\mathbb{1}_{\mathbb{A}_{n-1}} |D_n|^{1+\alpha_1}]^{1/(1+\alpha_1)} \leq \sigma_n$ for all $n = N+1, \dots$
2. $\mathbb{E}[\mathbb{1}_{\mathbb{A}_{n-1}} D_n | \mathcal{F}_{n-1}] = 0$ for all $n = N+1, \dots$
3. $\sum_{n > N} (\gamma_n \sigma_n)^{1+\alpha_1} < \infty$.

4. In the case where $\alpha_2 < 1, \sum_{n>N} \gamma_n^{(1+\alpha_2)/(1-\alpha_2)} < \infty$.

Then on $\mathbb{A}_\infty = \bigcap_{n \geq N} \mathbb{A}_n$, almost surely, $(f(X_n))_{n \in \mathbb{N}_0}$ converges and, if additionally,

5. $\sum_{n>N} \gamma_n = \infty$,

then on \mathbb{A}_∞ , almost surely, $\lim_{n \rightarrow \infty} \nabla f(X_n) = 0$.

For the proof of Lemma 3.1 we need the following result.

Lemma 3.2 ([48, Corollary 4.19]). *Let $(M_n)_{n \in \mathbb{N}_0}$ be a martingale and, for $n \in \mathbb{N}$, set $\Delta M_n = M_n - M_{n-1}$. If, for a $\beta \in (0, 2]$,*

$$\sum_{n \in \mathbb{N}} \mathbb{E}[|\Delta M_n|^\beta] < \infty,$$

then $(M_n)_{n \in \mathbb{N}_0}$ converges almost surely.

Proof of Lemma 3.1. In the following C_1, C_2, \dots , denote finite constants that only depend on $(\gamma_n)_{n \in \mathbb{N}}, \alpha_2, \|\nabla f\|_{L^\infty(K)}$ and the Hölder constants of ∇f on K for the exponents α_1 and α_2 .

Step 1. We prove almost sure convergence of $(f(X_n))_{n \in \mathbb{N}_0}$ on \mathbb{A}_∞ . For $n > N$ we use a Taylor approximation of first order and write

$$\begin{aligned} f(X_n) - f(X_{n-1}) &= f(X_{n-1} + \gamma_n(\Gamma_n + D_n)) - f(X_{n-1}) \\ &= \underbrace{\gamma_n \langle f(X_{n-1}), \Gamma_n \rangle}_{=: \Delta A_n} + \underbrace{\gamma_n \langle \nabla f(X_{n-1}), D_n \rangle}_{=: \Delta M_n} + \Delta R_n, \end{aligned}$$

where ΔR_n is the remainder term in the Taylor approximation. For $n \geq N$ we set

$$\begin{aligned} A_n &= \sum_{m=N+1}^n \mathbb{1}_{\mathbb{A}_{m-1}} \gamma_m \langle \nabla f(X_{m-1}), \Gamma_m \rangle, \\ M_n &= \sum_{m=N+1}^n \mathbb{1}_{\mathbb{A}_{m-1}} \gamma_m \langle \nabla f(X_{m-1}), D_m \rangle, \\ R_n &= \sum_{m=N+1}^n \mathbb{1}_{\mathbb{A}_{m-1}} \Delta R_m. \end{aligned}$$

Note that by the Taylor formula, for all $n > N$, on \mathbb{A}_n , we can write $\Delta R_n = \Delta R_n^{(1)} + \Delta R_n^{(2)}$ with

$$|\Delta R_n^{(1)}| \leq C_1 \gamma_n^{1+\alpha_2} |\Gamma_n|^{1+\alpha_2}, \quad |\Delta R_n^{(2)}| \leq C_1 \gamma_n^{1+\alpha_1} |D_n|^{1+\alpha_1},$$

where C_1 only depends on the Hölder-constants of ∇f on K with respect to the exponents α_1 and α_2 . We define $(R_n^{(1)})_{n \geq N}$ and $(R_n^{(2)})_{n \geq N}$ in analogy to $(R_n)_{n \geq N}$ and note that

$$\sum_{m=N+1}^{\infty} \mathbb{E}[\mathbb{1}_{\mathbb{A}_{m-1}} \gamma_m^{1+\alpha_1} |D_m|^{1+\alpha_1}] \leq \sum_{m=N+1}^{\infty} (\gamma_m \sigma_m)^{1+\alpha_1} < \infty,$$

so that $(R_n^{(2)})_{n \geq N}$ converges, almost surely, to a finite value. Moreover, $(M_n)_{n \geq N}$ is a martingale that converges, almost surely, to a finite value as consequence of Lemma 3.2 and

$$\sum_{m=N+1}^{\infty} \mathbb{E}[\mathbb{1}_{\mathbb{A}_{m-1}} |\gamma_m \langle \nabla f(X_{m-1}), D_m \rangle|^{1+\alpha_1}] \leq C_2^{1+\alpha_1} \sum_{m=N+1}^{\infty} \gamma_m^{1+\alpha_1} \mathbb{E}[\mathbb{1}_{\mathbb{A}_{m-1}} |D_m|^{1+\alpha_1}]$$

with $C_2 := \|f\|_{L^\infty(K)} < \infty$. Finally, we compare the contribution of the first remainder term $(R_n^{(1)})_{n \geq N}$ with $(A_n)_{n \geq N}$. In the case where $\alpha_2 = 1$, we have for all $n \geq N$ with $C_1 C^3 \gamma_n \leq 1/2$, on \mathbb{A}_{n-1} ,

$$|\Delta R_n^{(1)}| \leq C_1 C^2 \gamma_n^2 |\nabla f(X_{n-1})|^2 \leq -C_1 C^3 \gamma_n^2 \langle \nabla f(X_{n-1}), \Gamma_n \rangle \leq -\frac{1}{2} \Delta A_n. \quad (3.1)$$

This is the case for all but finitely many n 's and we conclude that by monotonicity of $(A_n)_{n \geq N}$ the random variables $(A_n + R_n^{(1)})_{n \geq N}$ converge almost surely, possibly to minus infinity. We note that on \mathbb{A}_∞

$$f(X_n) = f(X_N) + A_n + R_n^{(1)} + M_n + R_n^{(2)},$$

and by lower boundedness of f on K the latter limit needs to be finite and we proved convergence of $(f(X_n))_{n \in \mathbb{N}_0}$. Note that as consequence of (3.1), also $(A_n)_{n \geq N}$ converges to a finite value on \mathbb{A}_∞ .

It remains to consider the case where $\alpha_2 < 1$. We apply the Hölder inequality with the adjoint exponents $2/(1 - \alpha_2)$ and $2/(1 + \alpha_2)$ and get that

$$\begin{aligned} & \sum_{m=N+1}^n \gamma_m^{1+\alpha_2} \mathbb{1}_{\mathbb{A}_{m-1}} |\nabla f(X_{m-1})|^{1+\alpha_2} \\ & \leq \underbrace{\left(\sum_{m=N+1}^{\infty} \gamma_m^{\frac{1+\alpha_2}{1-\alpha_2}} \right)^{\frac{1-\alpha_2}{2}}}_{< \infty} \left(\sum_{m=N+1}^n \mathbb{1}_{\mathbb{A}_{m-1}} \gamma_m |\nabla f(X_{m-1})|^2 \right)^{\frac{1+\alpha_2}{2}}. \end{aligned} \quad (3.2)$$

Since $(1 + \alpha_2)/2$ is less than one we get with monotonicity of $(A_n)_{n \geq N}$ that $(A_n + R_n^{(1)})_{n \geq N}$ converges almost surely, again possibly to an infinite value. We proceed as above to conclude that the latter limit is finite and that the sequences $(f(X_n))_{n \geq N}$ and $(A_n)_{n \geq N}$ are almost surely convergent on \mathbb{A}_∞ with finite limits.

Step 2. We prove almost sure convergence of $(\nabla f(X_n))_{n \in \mathbb{N}_0}$ to 0 on \mathbb{A}_∞ in the case that $\sum \gamma_n = \infty$. First, note that in analogy to above the process $(\tilde{M}_n)_{n \geq N}$ given by

$$\tilde{M}_n = \sum_{m=N+1}^n \mathbb{1}_{\mathbb{A}_{m-1}} \gamma_m D_m$$

is a martingale that converges, almost surely, as consequence of Lemma 3.2. Consequently, we get with the first step that

$$\Omega_0 := \left\{ \inf_{n \geq N} A_n > -\infty \right\} \cap \left\{ \lim_{n \rightarrow \infty} \tilde{M}_n \text{ exists in } \mathbb{R}^d \right\}$$

is an almost sure event. Suppose now that there exists $\omega \in \Omega_0 \cap \mathbb{A}_\infty$ for which the sequence $(\nabla f(X_n(\omega)))_{n \in \mathbb{N}_0}$ does not converge to zero. Then there exist $\delta > 0$ and a strictly increasing sequence $(n_k)_{k \in \mathbb{N}}$ with $|\nabla f(X_{n_k}(\omega))| > \delta$ for all $k \in \mathbb{N}$. Note that $|\nabla f|$ is uniformly bounded over the set K by κ and choose $\varepsilon > 0$ such that

$$\sup_{x, y \in K: |x-y| \leq \varepsilon} |\nabla f(x) - \nabla f(y)| \leq \frac{\delta}{2}.$$

We let $(m_k)_{k \in \mathbb{N}}$ such that

$$\sum_{\ell=n_k+1}^{m_k} \gamma_\ell \leq \frac{\varepsilon}{2C\kappa} < \sum_{l=n_k+1}^{m_k+1} \gamma_l.$$

By thinning the original sequence $(n_k)_{k \in \mathbb{N}}$ we can ensure that $([n_k + 1, m_k] : k \in \mathbb{N})$ are disjoint intervals. Moreover, by discarding the first terms from the sequence $(n_k)_{k \in \mathbb{N}}$ we can ensure that for all $k \in \mathbb{N}$,

$$\sup_{m \geq n_k} |\tilde{M}_m(\omega) - \tilde{M}_{n_k}(\omega)| < \frac{\varepsilon}{2}.$$

Now note that for $\ell = n_k, \dots, m_k$,

$$|X_\ell(\omega) - X_{n_k}(\omega)| \leq C\kappa \sum_{i=n_k+1}^{m_k} \gamma_i + |\tilde{M}_\ell(\omega) - \tilde{M}_{n_k}(\omega)| \leq \varepsilon.$$

Consequently, for these ℓ ,

$$|\nabla f(X_\ell(\omega))| \geq |\nabla f(X_{n_k}(\omega))| - \frac{\delta}{2} \geq \frac{\delta}{2},$$

and hence,

$$\sum_{\ell=n_k+1}^{m_k} \gamma_\ell |\nabla f(X_{\ell-1}(\omega))|^2 \geq \left(\frac{\delta}{2}\right)^2 \sum_{\ell=n_k+1}^{m_k} \gamma_\ell \rightarrow \left(\frac{\delta}{2}\right)^2 \frac{\varepsilon}{2\kappa}.$$

Since the intervals $([n_k + 1, m_k] : k \in \mathbb{N})$ are pairwise disjoint and since $\omega \in \mathbb{A}_\infty$ we get that

$$\sum_{k=N+1}^{\infty} \gamma_k \mathbb{1}_{\mathbb{A}_{k-1}}(\omega) |\nabla f(X_{k-1}(\omega))|^2 = \infty,$$

which contradicts $\omega \in \Omega_0$. Thus we proved that $(\nabla f(X_n))_{n \in \mathbb{N}_0}$ converges to zero on $\Omega_0 \cap \mathbb{A}_\infty$. \square

Proof of Theorem 1.1. Let $N \in \mathbb{N}$, $C > 0$ and $K \subset \mathbb{R}^d$ compact and consider the sets $(\mathbb{A}_n^{N,C,K} : n \geq N)$ given by

$$\mathbb{A}_n^{N,C,K} = \bigcap_{\ell=N}^n \{X_\ell \in K, |\nabla f(X_\ell)|^2 \leq C \langle \nabla f(X_\ell), -\Gamma_{\ell+1} \rangle, |\Gamma_{\ell+1}| \leq C |\nabla f(X_\ell)|,$$

$$\mathbb{E}[|D_{\ell+1}|^{1+\alpha_1} | \mathcal{F}_\ell] \leq C \sigma_{\ell+1}^{1+\alpha_1}, \mathbb{E}[D_{\ell+1} | \mathcal{F}_n] = 0\}.$$

The sets satisfy the assumptions of Lemma 3.1 and we thus have on $\mathbb{A}_\infty^{N,C,K} = \cap_{n \geq N} \mathbb{A}_n^{N,C,K}$ almost sure convergence of $(f(X_n))_{n \in \mathbb{N}}$ and, in the case

$$\sum \gamma_n = \infty, \quad \lim_{n \rightarrow \infty} \nabla f(X_n) = 0.$$

Now the statement follows, since for an increasing sequence of compact sets $(K_N : N \in \mathbb{N})$ with $\bigcup K_N = \mathbb{R}^d$ we have

$$\mathbb{C} = \bigcup_{N \in \mathbb{N}} \mathbb{A}_\infty^{N,N,K_N}. \quad (3.3)$$

It remains to prove almost sure convergence of $(X_n)_{n \in \mathbb{N}_0}$ on \mathbb{C} in the case where the set $\mathbb{C} = \{x : \nabla f(x) = 0\}$, does not contain a continuum of points. By (3.3), it suffices to show almost sure convergence on $\mathbb{A}_\infty^{N,N,K_N}$ for arbitrary $N \in \mathbb{N}$. First, we verify that, almost surely, $\lim_{n \rightarrow \infty} (X_n - X_{n-1}) = 0$. Indeed, as shown in the proof of Lemma 3.1 one has on $\mathbb{A}_\infty^{N,N,K_N}$ that $(\sum_{\ell=N+1}^n \gamma_\ell D_\ell)_{n \geq N}$ converges, almost surely, so that $\gamma_n D_n \rightarrow 0$ on $\mathbb{A}_\infty^{N,N,K_N}$. Moreover, $\gamma_n |\Gamma_{n+1}| \leq \gamma_n C |\nabla f(X_n)| \rightarrow 0$ on $\mathbb{A}_\infty^{N,N,K_N}$ so that $(X_n - X_{n-1})_{n \in \mathbb{N}}$ almost surely converges to zero on $\mathbb{A}_\infty^{N,N,K_N}$. Now note that

$$\begin{aligned} & \mathbb{P}(\{(X_n)_{n \in \mathbb{N}_0} \text{ diverges}\} \cap \mathbb{A}_\infty^{N,N,K_N}) \\ & \leq \sum_{i=1}^d \mathbb{P}((X_n^{(i)})_{n \in \mathbb{N}_0} \text{ diverges}, \nabla f(X_n) \rightarrow 0, X_n - X_{n-1} \rightarrow 0, X_m \in K_N, \forall m \geq N), \end{aligned}$$

where $X_n^{(i)}$ denotes the i -th coordinate of the random vector $X_n \in \mathbb{R}^d$.

If $\mathbb{P}(\{(X_n)_{n \in \mathbb{N}_0} \text{ diverges}\} \cap \mathbb{A}_\infty^{N,N,K_N})$ were strictly positive, one of the summands on the right-hand side would be strictly positive. Suppose that this were true for the i -th summand. In particular, for this i the respective event does not equal the empty set. We choose

$$\omega \in \{(X_n^{(i)})_{n \in \mathbb{N}_0} \text{ diverges}, \nabla f(X_n) \rightarrow 0, X_n - X_{n-1} \rightarrow 0, X_m \in K_N, \forall m \geq N\}.$$

Thus,

$$a := \liminf_{n \rightarrow \infty} X_n^{(i)}(\omega) \neq \limsup_{n \rightarrow \infty} X_n^{(i)}(\omega) =: b.$$

By $X_n(\omega) - X_{n-1}(\omega) \rightarrow 0$ we also get $\text{acc}((X_n^{(i)}(\omega))_{n \in \mathbb{N}_0}) = [a, b]$ and the principle of nested intervals produces for every $u \in [a, b]$ an accumulation point of $(X_n(\omega))_{n \in \mathbb{N}_0}$ with i -th coordinate equal to u . Hence, we defined an injective mapping that maps each $u \in [a, b]$ to a different accumulation point $\eta(u)$ of $(X_n(\omega))_{n \in \mathbb{N}_0}$. By choice of ω , $\eta(u)$ is a critical point of f and we thus constructed a continuum of critical points. \square

4 Proof of Theorem 1.2

The proof of Theorem 1.2 is arranged as follows. In Section 4.1, we provide moment estimates for the objective function value seen by the SGD where we fix a particular critical

level and take out realisations that undershoot the critical level by a certain amount (lower dropdown). In Section 4.2, we provide estimates for the drift term $(\sum_{\ell=1}^n \gamma_\ell \Gamma_\ell)_{n \in \mathbb{N}}$ in the setting of Section 4.1 on the event that no dropdown occurs. In Section 4.3, we provide estimates that later allow us to show that when a dropdown occurs for a particular critical level it is very likely that the limit of the target value of the SGD lies strictly below that critical level. In Section 4.4, we provide a technical lemma which shows applicability of the previous results in the context of Theorem 1.2. In Section 4.5, we provide properties of Łojasiewicz-functions and in Section 4.6 we combine the results to achieve the proof of Theorem 1.2.

4.1 An estimate for the target value in the case without lower dropdown

Technically, we will analyse the evolution of $(f(X_n))_{n \in \mathbb{N}_0}$ step by step and will thereby restrict attention to certain nice events. For this we use the concept of compatible events.

Definition 4.1. Let $N \in \mathbb{N}, \delta, C > 0, U \subset \mathbb{R}^d, (\sigma_n)_{n \geq N}$ a sequence of positive reals and $q \geq 2$. We call events $(\mathbb{A}_n)_{n \geq N}$ $(U, \delta, C, (\sigma_n), q)$ -compatible, if $(\mathbb{A}_n)_{n \geq N}$ is a sequence of decreasing events such that for all $n \geq N$:

- (i) $\mathbb{A}_n \in \mathcal{F}_n$ (adaptivity).
- (ii) $\mathbb{A}_n \subset \{X_n \in U, |\nabla f(X_n)|^2 \leq C \langle \nabla f(X_n), -\Gamma_{n+1} \rangle, |\Gamma_{n+1}| \leq C |\nabla f(X_n)|, \delta \geq \gamma_{n+1} |\Gamma_{n+1}| \}$ (boundedness and drift condition).
- (iii) $\mathbb{E}[\mathbb{1}_{\mathbb{A}_n} D_{n+1} | \mathcal{F}_n] = 0$ (martingale condition).
- (iv) $\mathbb{E}[\mathbb{1}_{\mathbb{A}_n} |D_{n+1}|^q | \mathcal{F}_n] \leq \sigma_{n+1}^q$ (moment condition).
- (v) $\mathbb{A}_{n+1} \subset \{\gamma_{n+1} |D_{n+1}| \leq \delta\}$ (excess condition).

The proof of the main result is based on several preliminary results. At first we provide an upper bound for the f -value of $(X_n)_{n \in \mathbb{N}_0}$ when approaching a critical level with a lower dropdown, meaning that we only consider realisations of the SGD that do not undershoot the critical level significantly. We will use the natural time scale associated with a stochastic approximation scheme which associates the n -th iterate X_n with the time $t_n = \sum_{\ell=1}^n \gamma_\ell$ and see that the f -value is bounded from above by a term arising from the stochastic noise and the function $(\Phi_t^R)_{t \geq 0}$, the worst-case bound for the f -value of the solution to the ODE (1.2) in a region satisfying a Łojasiewicz-inequality with exponent $\beta \in (1/2, 1)$ and constant $\sqrt{\eta} > 0$ starting in R , i.e. the solution to the ODE (see e.g. [33])

$$\dot{\Phi}_t^{(R)} = -\eta (\Phi_t^{(R)})^{2\beta}, \quad \Phi_0^{(R)} = R.$$

For a subset $U \subset \mathbb{R}^d$ we denote

$$\|\nabla f\|_{\text{Lip}(U)} = \sup_{x, y \in U, x \neq y} \frac{|\nabla f(x) - \nabla f(y)|}{|x - y|}.$$

Proposition 4.1. *Suppose that the following assumptions are satisfied:*

Łojasiewicz-assumption. *Let U be an open and bounded set containing a critical point x_0 and suppose that the Łojasiewicz-inequality holds on U with parameters $L > 0$ and $\beta \in (1/2, 1)$ meaning that for all $x \in U$,*

$$|\nabla f(x)| \geq L|f(x) - f(x_0)|^\beta.$$

Compatible events. *Let $N \in \mathbb{N}_0$, $\delta, C > 0$, $q \geq 2$ and let $(\sigma_n)_{n \geq N}$ and $(w_n)_{n \geq N}$ be sequences of positive reals. We denote by $(\mathbb{A}_n)_{n \geq N}$ a $(U, \delta, C, (\sigma_n), q)$ -compatible sequence of events. We let*

$$\mathbb{B}_N = \mathbb{A}_N \cap \{f(X_N) - f(x_0) \geq -w_N\},$$

and for $n > N$,

$$\mathbb{B}_n = \mathbb{A}_{n-1} \cap \{\gamma_n |D_n| < \delta, f(X_n) - f(x_0) \geq -w_n\},$$

and suppose that $\mathbb{B}_n \supset \mathbb{A}_n$ for $n \geq N$.

Main assumptions for asymptotic error term. *Let $(v_n)_{n \geq N}$ and $(\gamma_n)_{n \in \mathbb{N}}$ be decreasing sequences of positive reals that satisfy for a constant $\kappa > 0$ for all $n > N$,*

$$\frac{v_{n-1}}{v_n} - 1 \leq \kappa \gamma_n v_n^{2\beta-1}, \quad (4.1)$$

and

$$\begin{aligned} & (\delta^{-1} \|\nabla f\|_{L^\infty(U)} + 2 \|\nabla f\|_{\text{Lip}(U^{2\delta})}) (\gamma_n \sigma_n)^2 \\ & + \left(\|\nabla f\|_{L^\infty(U)}^q + (2 \|\nabla f\|_{\text{Lip}(U^{2\delta})})^{\frac{q}{2}} \right) w_n^{-(q-1)} (\gamma_n \sigma_n)^q \leq \gamma_n v_n^{2\beta}, \end{aligned} \quad (4.2)$$

where

$$U^{2\delta} = \{x \in \mathbb{R}^d : d(x, U) < 2\delta\}.$$

Additional technical assumptions. *We assume that for a constant $\delta' \in (0, 1)$ we have for all $n > N$ that $2C^3 \|\nabla f\|_{\text{Lip}(U^{2\delta})} \gamma_n \leq \delta'$ as well as*

$$\frac{1}{2} w_{n-1} \leq w_n \leq v_n \wedge 1, \quad (4.3a)$$

$$((C \vee C^{-1}) \gamma_n + 2 \|\nabla f\|_{\text{Lip}(U^{2\delta})} C^2 \gamma_n^2) \|\nabla f\|_{L^\infty(U)}^2 \leq w_n. \quad (4.3b)$$

Assumptions for the associated master equation. *Let $y_0 > 0, \eta > 0$ such that for*

$$g : \mathbb{R} \rightarrow \mathbb{R}, \quad y \mapsto \kappa y - (1 - \delta') C^{-1} L^2 |y|^{2\beta} + 1$$

one has for all $y \geq 0$,

$$\left(\sup_{n \geq N} \frac{v_n^{2\beta}}{v_{n-1}^{2\beta}} \right) g(y + y_0) \leq -\eta |y|^{2\beta}.$$

Associated differential equation. For $R \geq 0$ and $t \geq 0$ we let

$$\Phi_t^{(R)} = R((2\beta - 1)\eta R^{2\beta-1}t + 1)^{-\frac{1}{2\beta-1}}.$$

Result. Suppose that the above assumptions hold and that for a $R > 0$ with

$$R + (y_0 + 8)v_N \leq ((1 - \delta')2\beta C^{-1}L^2\gamma_{N+1})^{-\frac{1}{2\beta-1}}$$

one has

$$\mathbb{E}[\mathbb{1}_{\mathbb{B}_N}(f(X_N) - f(x_0))] \leq R.$$

Then one has, for all $n \geq N$,

$$\mathbb{E}[\mathbb{1}_{\mathbb{B}_n}(f(X_n) - f(x_0))] \leq \Phi_{t_n - t_N}^{(R)} + (y_0 + 8)v_n. \quad (4.4)$$

Proof. We assume without loss of generality that $f(x_0) = 0$ and we write, for $n > N$,

$$f(X_n) = f(X_{n-1}) + \gamma_n \langle \nabla f(X_{n-1}), \Gamma_n + D_n \rangle + R_n. \quad (4.5)$$

Taylor's formula gives that in the case where the segment joining X_{n-1} and X_n lies in $\tilde{U} := U^{2\delta}$, the remainder satisfies

$$|R_n| \leq 2\|\nabla f\|_{\text{Lip}(\tilde{U})}\gamma_n^2(|\Gamma_n|^2 + |D_n|^2). \quad (4.6)$$

We note that this is indeed the case when \mathbb{B}_n enters. With $\xi_n := \mathbb{E}[\mathbb{1}_{\mathbb{B}_n}f(X_n)]$ one has $\xi_n = \text{I} + \text{II}$, where

$$\begin{aligned} \text{I} &= -\mathbb{E}[\mathbb{1}_{\mathbb{A}_{n-1} \cap \{\gamma_n|D_n| < \delta\} \cap \{f(X_n) < -w_n\}}f(X_n)], \\ \text{II} &= \mathbb{E}[\mathbb{1}_{\mathbb{A}_{n-1} \cap \{\gamma_n|D_n| < \delta\}}f(X_n)]. \end{aligned}$$

We analyse the terms separately.

Analysis of I. On $\mathbb{A}_{n-1} \cap \{\gamma_n|D_n| < \delta\}$, one has that

$$\begin{aligned} |f(X_n) - f(X_{n-1})| &\leq (C\gamma_n + 2\|\nabla f\|_{\text{Lip}(\tilde{U})}C^2\gamma_n^2)\|\nabla f\|_{L^\infty(U)}^2 \\ &\quad + \|\nabla f\|_{L^\infty(U)}\gamma_n|D_n| + 2\|\nabla f\|_{\text{Lip}(\tilde{U})}\gamma_n^2|D_n|^2. \end{aligned}$$

Using that by assumption

$$(C\gamma_n + 2\|\nabla f\|_{\text{Lip}(\tilde{U})}C^2\gamma_n^2)\|\nabla f\|_{L^\infty(U)}^2 \leq w_n,$$

we get that on $\mathbb{A}_{n-1} \cap \{\gamma_n|D_n| < \delta\}$,

$$\begin{aligned} &\mathbb{1}_{\{|f(X_n) - f(X_{n-1})| \geq 3w_n\}}(|f(X_n) - f(X_{n-1})| - 3w_n) \\ &\leq \mathbb{1}_{\{\|\nabla f\|_{L^\infty(U)}\gamma_n|D_n| \geq w_n\}}\|\nabla f\|_{L^\infty(U)}\gamma_n|D_n| \\ &\quad + \mathbb{1}_{\{2\|\nabla f\|_{\text{Lip}(\tilde{U})}\gamma_n^2|D_n|^2 \geq w_n\}}2\|\nabla f\|_{\text{Lip}(\tilde{U})}\gamma_n^2|D_n|^2. \end{aligned}$$

Now,

$$\begin{aligned} & \mathbb{E} [\mathbb{1}_{\mathbb{A}_{n-1}} \mathbb{1}_{\{\|\nabla f\|_{L^\infty(U)} \gamma_n |D_n| \geq w_n\}} \|\nabla f\|_{L^\infty(U)} \gamma_n |D_n|] \leq \|\nabla f\|_{L^\infty(U)}^q w_n^{-(q-1)} (\gamma_n \sigma_n)^q, \\ & \mathbb{E} [\mathbb{1}_{\mathbb{A}_{n-1}} \mathbb{1}_{\{2\|\nabla f\|_{\text{Lip}(\tilde{U})} \gamma_n^2 |D_n|^2 \geq w_n\}} 2\|\nabla f\|_{\text{Lip}(\tilde{U})} \gamma_n^2 |D_n|^2] \\ & \leq (2\|\nabla f\|_{\text{Lip}(\tilde{U})})^{\frac{q}{2}} w_n^{-(\frac{q}{2}-1)} (\gamma_n \sigma_n)^q. \end{aligned}$$

Setting

$$C_1 = \|\nabla f\|_{L^\infty(U)}^q + (2\|\nabla f\|_{\text{Lip}(\tilde{U})})^{\frac{q}{2}},$$

and using $w_n \leq 1$ we get

$$\mathbb{E} [\mathbb{1}_{\mathbb{A}_{n-1} \cap \{\gamma_n |D_n| < \delta, |f(X_n) - f(X_{n-1})| \geq 3w_n\}} (|f(X_n) - f(X_{n-1})| - 3w_n)] \leq C_1 w_n^{-(q-1)} (\gamma_n \sigma_n)^q.$$

Recall that $\mathbb{A}_{n-1} \subset \mathbb{B}_{n-1}$ so that, on \mathbb{A}_{n-1} , $f(X_{n-1}) \geq -w_{n-1}$. Thus, using $w_{n-1} \leq 2w_n$ we get

$$\begin{aligned} \text{I} & \leq \mathbb{E} [\mathbb{1}_{\mathbb{A}_{n-1} \cap \{\gamma_n |D_n| < \delta\} \cap \{f(X_n) < -w_n\}} (|f(X_n) - f(X_{n-1})| + w_{n-1})] \\ & \leq 5w_n \mathbb{P}(\mathbb{A}_{n-1} \cap \{\gamma_n |D_n| < \delta\} \cap \{f(X_n) < -w_n\}) + C_1 w_n^{-(p-1)} (\gamma_n \sigma_n)^p. \end{aligned} \quad (4.7)$$

Analysis of II. Again, one has

$$\text{II} = \mathbb{E} [\mathbb{1}_{\mathbb{A}_{n-1} \cap \{\gamma_n |D_n| < \delta\}} (f(X_{n-1}) + \gamma_n \langle \nabla f(X_{n-1}), \Gamma_n + D_n \rangle + R_n)],$$

where the remainder satisfies inequality (4.6) since on $\mathbb{A}_{n-1} \cap \{\gamma_n |D_n| < \delta\}$ the segment connecting X_{n-1} and X_n lies in \tilde{U} . By assumption, $2C^3 \|\nabla f\|_{\text{Lip}(\tilde{U})} \gamma_n \leq \delta'$ so that

$$\begin{aligned} \text{II} & \leq \mathbb{E} [\mathbb{1}_{\mathbb{A}_{n-1} \cap \{\gamma_n |D_n| < \delta\}} (f(X_{n-1}) - (1 - \delta') C^{-1} \gamma_n |\nabla f(X_{n-1})|^2 \\ & \quad + \gamma_n \langle \nabla f(X_{n-1}), D_n \rangle + 2\|\nabla f\|_{\text{Lip}(\tilde{U})} \gamma_n^2 |D_n|^2)]. \end{aligned}$$

We note that

$$\begin{aligned} & \mathbb{E} [\mathbb{1}_{\mathbb{A}_{n-1} \cap \{\gamma_n |D_n| < \delta\}} \langle \nabla f(X_{n-1}), D_n \rangle] \\ & = -\mathbb{E} [\mathbb{1}_{\mathbb{A}_{n-1} \cap \{\gamma_n |D_n| \geq \delta\}} \langle \nabla f(X_{n-1}), D_n \rangle] \\ & \leq \delta^{-1} \|\nabla f\|_{L^\infty(U)} \gamma_n \sigma_n^2. \end{aligned}$$

Moreover, using that $C^{-1} \gamma_n \|\nabla f\|_{L^\infty(U)}^2 \leq w_n$ and $w_{n-1} \leq 2w_n$ we conclude

$$\begin{aligned} & -\mathbb{E} [\mathbb{1}_{\mathbb{B}_{n-1} \cap (\mathbb{A}_{n-1}^c \cup \{\gamma_n |D_n| \geq \delta\})} (f(X_{n-1}) - (1 - \delta') C^{-1} \gamma_n |\nabla f(X_{n-1})|^2)] \\ & \leq \mathbb{P}(\mathbb{B}_{n-1} \cap (\mathbb{A}_{n-1}^c \cup \{\gamma_n |D_n| \geq \delta\})) (2w_n + C^{-1} \gamma_n \|\nabla f\|_{L^\infty(U)}^2) \\ & \leq 3\mathbb{P}(\mathbb{B}_{n-1} \setminus \mathbb{B}_n) w_n. \end{aligned}$$

Consequently,

$$\begin{aligned}
& \mathbb{E}[\mathbb{1}_{\mathbb{A}_{n-1} \cap \{\gamma_n |D_n| < \delta\}} (f(X_{n-1}) - (1 - \delta')C^{-1}\gamma_n |\nabla f(X_{n-1})|^2)] \\
&= \mathbb{E}[\mathbb{1}_{\mathbb{B}_{n-1}} (f(X_{n-1}) - (1 - \delta')C^{-1}\gamma_n |\nabla f(X_{n-1})|^2)] \\
&\quad - \mathbb{E}[\mathbb{1}_{\mathbb{B}_{n-1} \cap (\mathbb{A}_{n-1}^c \cup \{\gamma_n |D_n| \geq \delta\})} (f(X_{n-1}) - (1 - \delta')C^{-1}\gamma_n |\nabla f(X_{n-1})|^2)] \\
&\leq \mathbb{E}[\mathbb{1}_{\mathbb{B}_{n-1}} (f(X_{n-1}) - (1 - \delta')C^{-1}\gamma_n |\nabla f(X_{n-1})|^2)] + 3\mathbb{P}(\mathbb{B}_{n-1} \setminus \mathbb{B}_n)w_n.
\end{aligned}$$

We conclude that altogether

$$\begin{aligned}
\mathbb{I} &\leq \mathbb{E}[\mathbb{1}_{\mathbb{B}_{n-1}} (f(X_{n-1}) - (1 - \delta')C^{-1}\gamma_n |\nabla f(X_{n-1})|^2)] \\
&\quad + 3\mathbb{P}(\mathbb{B}_{n-1} \setminus \mathbb{B}_n)w_n + (\delta^{-1}\|\nabla f\|_{L^\infty(U)} + 2\|\nabla f\|_{\text{Lip}(\tilde{U})})(\gamma_n \sigma_n)^2.
\end{aligned}$$

Estimating ξ_n in terms of a difference equation. Combining the estimates for I and II, see (4.7) above, we get

$$\begin{aligned}
\xi_n &\leq \mathbb{E}[\mathbb{1}_{\mathbb{B}_{n-1}} (f(X_{n-1}) - (1 - \delta')C^{-1}\gamma_n |\nabla f(X_{n-1})|^2)] + 8w_n \mathbb{P}(\mathbb{B}_{n-1} \setminus \mathbb{B}_n) \\
&\quad + \underbrace{(\delta^{-1}\|\nabla f\|_{L^\infty(U)} + 2\|\nabla f\|_{\text{Lip}(\tilde{U})})(\gamma_n \sigma_n)^2 + C_1 w_n^{-(q-1)}(\gamma_n \sigma_n)^q}_{\leq \gamma_n v_n^{2\beta}}. \tag{4.8}
\end{aligned}$$

Using the Łojasiewicz-inequality together with the convexity of $x \mapsto |x|^{2\beta}$ we get

$$\xi_n \leq \xi_{n-1} - (1 - \delta')C^{-1}\mathbb{L}^2\gamma_n |\xi_{n-1}|^{2\beta} + 8w_n \mathbb{P}(\mathbb{B}_{n-1} \setminus \mathbb{B}_n) + \gamma_n v_n^{2\beta}.$$

By assumption, one has for $n > N$ that

$$\frac{v_{n-1}}{v_n} - 1 \leq \kappa \gamma_n v_n^{2\beta-1},$$

and using that $(v_n)_{n \geq N}$ is monotonically decreasing and $w_n \leq v_n$ for $\zeta_n := \xi_n/v_n$ and $g(y) = \kappa y - (1 - \delta')C^{-1}\mathbb{L}^2|y|^{2\beta} + 1$ we get

$$\zeta_n \leq \zeta_{n-1} + \gamma_n v_n^{2\beta-1} g(\zeta_{n-1}) + 8\mathbb{P}(\mathbb{B}_{n-1} \setminus \mathbb{B}_n). \tag{4.9}$$

A comparison argument for the difference equation (4.9). For $s > 0$, the mapping

$$T_s : \mathbb{R} \rightarrow \mathbb{R}, \quad y \mapsto y + s g(y)$$

is monotonically increasing on $(-\infty, ((1 - \delta')2\beta C^{-1}\mathbb{L}^2 s)^{-1/(2\beta-1)})$. We let

$$(\bar{\zeta}_n)_{n \geq N} = \left(\frac{\Phi_{t_n - t_N}^{(R)}}{v_n} + y_0 \right)_{n \geq N},$$

and note that, for $n > N$,

$$(\bar{\zeta}_{n-1} + 8)((1 - \delta')2\beta C^{-1}\mathbb{L}^2 \gamma_n v_n^{2\beta-1})^{\frac{1}{2\beta-1}}$$

$$\begin{aligned}
&\leq (v_{n-1}\bar{\zeta}_{n-1} + 8v_{n-1})((1 - \delta')2\beta C^{-1}\mathbb{L}^2\gamma_n)^{\frac{1}{2\beta-1}} \\
&\leq (v_N\bar{\zeta}_N + 8v_N)((1 - \delta')2\beta C^{-1}\mathbb{L}^2\gamma_{N+1})^{\frac{1}{2\beta-1}} \leq 1,
\end{aligned}$$

so that $T_{\gamma_n v_n^{2\beta-1}}$ is monotonically increasing on $(-\infty, \bar{\zeta}_{n-1} + 8]$. We prove by induction that for all $n \geq N$, $\zeta_n \leq \bar{\zeta}_n + 8\mathbb{P}(\mathbb{B}_N \setminus \mathbb{B}_n)$. By assumption, $\zeta_N = \bar{\zeta}_N/v_N \leq R/v_N \leq \bar{\zeta}_N$. Moreover, supposing that the statement is true for $n-1 \geq N$, we conclude that $\zeta_{n-1} \leq \bar{\zeta}_{n-1} + 8$ so that we get with (4.9) and the monotonicity of $T_{\gamma_n v_n^{2\beta-1}}$ on $(-\infty, \bar{\zeta}_{n-1} + 8]$ that

$$\begin{aligned}
\zeta_n &\leq T_{\gamma_n v_n^{2\beta-1}}(\bar{\zeta}_{n-1} + 8\mathbb{P}(\mathbb{B}_N \setminus \mathbb{B}_{n-1})) + 8\mathbb{P}(\mathbb{B}_{n-1} \setminus \mathbb{B}_n) \\
&= \bar{\zeta}_{n-1} + \gamma_n v_n^{2\beta-1} g(\bar{\zeta}_{n-1} + 8\mathbb{P}(\mathbb{B}_N \setminus \mathbb{B}_{n-1})) + 8\mathbb{P}(\mathbb{B}_N \setminus \mathbb{B}_n) \\
&\leq \frac{\Phi_{t_{n-1}-t_N}^{(R)}}{v_{n-1}} - \eta \gamma_n v_n^{2\beta-1} \left(\frac{\Phi_{t_{n-1}-t_N}^{(R)}}{v_n} \right)^{2\beta} + y_0 + 8\mathbb{P}(\mathbb{B}_N \setminus \mathbb{B}_n) \\
&\leq \frac{1}{v_n} \left(\Phi_{t_{n-1}-t_N}^{(R)} - \eta \gamma_n \left(\Phi_{t_{n-1}-t_N}^{(R)} \right)^{2\beta} \right) + y_0 + 8\mathbb{P}(\mathbb{B}_N \setminus \mathbb{B}_n). \tag{4.10}
\end{aligned}$$

Noting that

$$\dot{\Phi}_{t_{n-1}-t_N}^{(R)} = -\eta \left(\Phi_{t_{n-1}-t_N}^{(R)} \right)^{2\beta}$$

we get

$$-\eta \gamma_n \left(\Phi_{t_{n-1}-t_N}^{(R)} \right)^{2\beta} \leq \int_{t_{n-1}-t_N}^{t_n-t_N} \dot{\Phi}_u^{(R)} du = \Phi_{t_n-t_N}^{(R)} - \Phi_{t_{n-1}-t_N}^{(R)}.$$

With (4.10) it follows that $\zeta_n \leq \bar{\zeta}_n + 8\mathbb{P}(\mathbb{B}_N \setminus \mathbb{B}_n)$ which finishes the proof. \square

By slightly modifying the proof, one can derive a rate of convergence for the approximation of a local minimum in a neighbourhood that satisfies a Łojasiewicz-inequality. See also [71] for almost sure convergence rates.

Proposition 4.2. *Let $\beta \in (1/2, 1)$, $U \subset \mathbb{R}^d$ and $x_0 \in U$ such that ∇f is Lipschitz continuous on the convex hull of U , as well as $f(x) \geq f(x_0)$ and $|\nabla f(x)| \geq \mathbb{L}|f(x) - f(x_0)|^\beta$ for all $x \in U$. Let $(v_n)_{n \in \mathbb{N}_0}$ and $(\gamma_n)_{n \in \mathbb{N}}$ be decreasing sequences of positive reals such that $\gamma_n \rightarrow 0$ and there exists $\kappa > 0$ such that for all $n \in \mathbb{N}$,*

$$\frac{v_{n-1}}{v_n} - 1 \leq \kappa \gamma_n v_n^{2\beta-1}.$$

Assume that $f(X_0) \in L^1$ and there exists a $C \geq 0$ such that almost surely, for all $n \in \mathbb{N}_0$:

- (i) $X_n \in U$.
- (ii) $|\nabla f(X_n)|^2 \leq C \langle \nabla f(X_n), -\Gamma_{n+1} \rangle$ and $|\Gamma_{n+1}| \leq C |\nabla f(X_n)|$.
- (iii) $\mathbb{E}[D_{n+1} | \mathcal{F}_n] = 0$ and $\mathbb{E}[|D_{n+1}|^2 | \mathcal{F}_n] \leq \gamma_{n+1}^{-1} v_{n+1}^{2\beta}$.

Then there exists a constant $\tilde{C} \geq 0$ such that for all $n \in \mathbb{N}_0$,

$$\mathbb{E}[f(X_n) - f(x_0)] \leq \tilde{C}(\Phi_{t_n}^{(R)} + v_n). \quad (4.11)$$

The bounds in (4.4) and (4.11) depend on $(\Phi_{t_n}^{(R)})$ as well as the additional error term (v_n) that is defined by the size of the stochastic perturbation. In the next proposition we choose $(\Phi_{t_n}^{(R)})$ and (v_n) to be of the same order. This can be interpreted as computing a bound on the stochastic perturbation in order to recover the rate of convergence for the non-perturbed gradient descent algorithm in (4.4) and (4.11). If the perturbation is larger one can artificially increase β to satisfy the assumptions in Proposition 4.3.

Proposition 4.3. Let $\beta \in (1/2, 1)$ and $(\gamma_n)_{n \in \mathbb{N}}$ and $(\sigma_n)_{n \in \mathbb{N}}$ be sequences of strictly positive reals such that $(\gamma_n)_{n \in \mathbb{N}}$ is monotonically decreasing with $\lim_{n \rightarrow \infty} \gamma_n = 0$. If

$$\begin{aligned} (a) \quad \sigma_n^2 &= \mathcal{O}\left(\gamma_n^{-1} t_n^{-\frac{2\beta}{2\beta-1}}\right), \\ (b) \quad \sigma_n^q &= \mathcal{O}\left(\gamma_n^{1-q} t_n^{-\frac{2\beta+q-1}{2\beta-1}}\right), \\ (c) \quad \gamma_n &= \mathcal{O}\left(t_n^{-\frac{1}{2\beta-1}}\right), \end{aligned}$$

then for every Lipschitz function $\nabla f : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $U \subset \mathbb{R}^d$ bounded and $\delta > 0$ for sufficiently large constant C_w there exist constants κ and C_v such that for

$$v_n = C_v t_n^{-\frac{1}{2\beta-1}}, \quad w_n = C_w t_n^{-\frac{1}{2\beta-1}},$$

the main and additional assumptions of Proposition 4.1 are satisfied for large N . Moreover, for this choice of parameters, one can replace in Proposition 4.1, inequality (4.4) by

$$\mathbb{E}[\mathbb{1}_{B_n}(f(X_n) - f(x_0))] \leq \left(1 + (y_0 + 8) \frac{C_v}{R} \left((2\beta - 1)\eta R^{2\beta-1} + \frac{1}{t_N}\right)^{\frac{1}{2\beta-1}}\right) \Phi_{t_n - t_N}^{(R)}.$$

Proof. By choice of $(v_n)_{n \in \mathbb{N}}$, we get with the Taylor formula

$$-\frac{\Delta v_n}{v_n} \sim \frac{1}{2\beta-1} \gamma_n t_n^{-1} = \mathcal{O}(\gamma_n v_n^{2\beta-1}), \quad (4.12)$$

so that (4.1) is satisfied for large κ for all large n . Note that (4.2) is satisfied for $v_n = C_v t_n^{-1/(2\beta-1)}$ for a sufficiently large C_v for all large n if $(\gamma_n \sigma_n)^2$ and $(w_n^{-(q-1)} (\gamma_n \sigma_n)^q)$ are both of order $\mathcal{O}(\gamma_n t_n^{-2\beta/(2\beta-1)})$. Elementary computations show that the first, respectively second term is of order $\mathcal{O}(\gamma_n t_n^{-2\beta/(2\beta-1)})$, iff condition (a), respectively (b) holds. For the Eq. 4.3b, note that $w_n \leq v_n$ if $C_v \geq C_w$ and $w_n \leq 1$ for large enough n , since (c) implies that $t_n = \sum_{i=1}^n \gamma_i \rightarrow \infty$. The Eq. 4.3a holds for large n since $t_n = t_{n-1} + \gamma_n \sim t_{n-1}$. Moreover,

the third inequality holds for large n since $\gamma_n \rightarrow 0$. The rest follows since by definition of Φ and $(v_n)_{n \in \mathbb{N}}$ one has for $n \geq N$,

$$v_n \leq \frac{C_v}{R} \left((2\beta - 1)\eta R^{2\beta-1} + \frac{1}{t_N} \right)^{\frac{1}{2\beta-1}} \Phi_{t_n-t_N}^{(R)}.$$

The proof is complete. \square

Remark 4.1. (i) In the case $\gamma_n = C_\gamma n^{-\gamma}$ and $\sigma_n = C_\sigma n^\sigma$ for $\gamma \in (1/2, 1)$, $\sigma \in \mathbb{R}$ and $C_\gamma, C_\sigma > 0$ assumptions (a), (b) and (c) are satisfied, if

$$(a') \quad 2\sigma \leq \frac{(4\beta - 1)\gamma - 2\beta}{2\beta - 1},$$

$$(b') \quad \sigma \leq \frac{2\beta\gamma - 1}{2\beta - 1} - \frac{1}{q},$$

$$(c') \quad \gamma \geq \frac{1}{2\beta}.$$

In that case, we have

$$t_n \sim \frac{C_\gamma}{1 - \gamma} n^{1-\gamma}.$$

(ii) In the case $\gamma_n = C_\gamma n^{-1}$ and $\sigma_n = C_\sigma n^\sigma$ for $\sigma \in \mathbb{R}$ and $C_\gamma, C_\sigma > 0$ assumptions (a), (b) and (c) are satisfied if $\sigma < 1/2$ and $q \geq 2$. In that case, $t_n \sim C_\gamma \log(n)$.

4.2 Bounding the drift term in the case where no dropdown occurs

Proposition 4.4. Assume all assumptions of Proposition 4.1. Additionally assume that the sequence $(-\Delta v_n / \gamma_n)_{n \geq N}$ is decreasing and $(\gamma_n / (-\Delta \bar{\xi}_n))^{1/2} \bar{\xi}_n \rightarrow 0$ with

$$\bar{\xi}_n := \Phi_{t_n-t_N}^{(R)} + (y_0 + 8)v_n.$$

Then

$$\begin{aligned} & \sqrt{\frac{1-\delta'}{C^3}} \sum_{n=N+1}^{\infty} \gamma_n \mathbb{E}[\mathbb{1}_{\mathbb{B}_{n-1}} |\Gamma_n|] \\ & \leq \gamma_{N+1} \sqrt{\eta} R^\beta + \sqrt{\eta} \int_0^\infty (\Phi_u^{(R)})^\beta du \\ & \quad + \frac{1}{\sqrt{y_0 + 8}} \sum_{n=N+1}^{\infty} \sqrt{\frac{\gamma_n}{-\Delta v_n}} (-(y_0 + 8)\Delta v_n + \gamma_n v_n^{2\beta} + 8v_n \mathbb{P}(\mathbb{B}_{n-1} \setminus \mathbb{B}_n)). \end{aligned}$$

Proof. Using that $w_n \leq v_n$ with (4.8) we get

$$(1 - \delta') C^{-1} \gamma_n \mathbb{E}[\mathbb{1}_{\mathbb{B}_{n-1}} |\nabla f(X_{n-1})|^2] \leq \bar{\xi}_{n-1} - \bar{\xi}_n + \gamma_n v_n^{2\beta} + 8v_n \mathbb{P}(\mathbb{B}_{n-1} \setminus \mathbb{B}_n),$$

and hence,

$$\begin{aligned}
& \sqrt{(1-\delta')C^{-1}\gamma_n\mathbb{E}[\mathbf{1}_{\mathbb{B}_{n-1}}|\nabla f(X_{n-1})|]} \\
& \leq \sqrt{\gamma_n}\sqrt{(1-\delta')C^{-1}\gamma_n\mathbb{E}[\mathbf{1}_{\mathbb{B}_{n-1}}|\nabla f(X_{n-1})|^2]} \\
& \leq \sqrt{\gamma_n}\sqrt{-\Delta\tilde{\xi}_n + \gamma_nv_n^{2\beta} + 8v_n\mathbb{P}(\mathbb{B}_{n-1}\setminus\mathbb{B}_n)}.
\end{aligned} \tag{4.13}$$

With the Cauchy-Schwarz inequality we get that

$$\begin{aligned}
& \sum_{n=N+1}^{\infty} \sqrt{\gamma_n}\sqrt{-\Delta\tilde{\xi}_n + \gamma_nv_n^{2\beta} + 8v_n\mathbb{P}(\mathbb{B}_{n-1}\setminus\mathbb{B}_n)} \\
& \leq \left(\sum_{n=N+1}^{\infty} \sqrt{\gamma_n(-\Delta\tilde{\xi}_n)} \right)^{\frac{1}{2}} \left(\sum_{n=N+1}^{\infty} \sqrt{\frac{\gamma_n}{-\Delta\tilde{\xi}_n}} (-\Delta\tilde{\xi}_n + \gamma_nv_n^{2\beta} + 8v_n\mathbb{P}(\mathbb{B}_{n-1}\setminus\mathbb{B}_n)) \right)^{\frac{1}{2}}
\end{aligned} \tag{4.14}$$

with

$$\tilde{\xi}_n := \Phi_{t_n-t_N}^{(R)} + (y_0 + 8)v_n.$$

Recall that $\xi_n \leq \tilde{\xi}_n$ by Proposition 4.1.

We use partial summation to get an estimate for (4.14). By assumption, the sequence $(-\Delta v_n/\gamma_n)_{n>N}$ is decreasing. Moreover, for $\phi_n := \Phi_{t_n-t_N}^{(R)}$ we get with the mean value theorem that there exists $u_n \in (t_{n-1}-t_N, t_n-t_N)$ with $-\Delta\phi_n/\gamma_n = -\Phi_{u_n}^{(R)}$ so that $(-\Delta\phi_n/\gamma_n)_{n>N}$ is also decreasing. Altogether, we thus get that

$$(a_n)_{n>N} := \left(\left(\frac{\gamma_n}{-\Delta\tilde{\xi}_n} \right)^{\frac{1}{2}} \right)_{n>N}$$

is increasing so that its differences are non-negative. We conclude with partial summation that

$$\begin{aligned}
\sum_{n=N+1}^{\infty} a_n(-\Delta\tilde{\xi}_n) &= a_{N+1}\tilde{\xi}_N + \sum_{n=N+1}^{\infty} \Delta a_{n+1} \tilde{\xi}_n \\
&\leq a_{N+1}\tilde{\xi}_N + \sum_{n=N+1}^{\infty} \Delta a_{n+1} \tilde{\xi}_n \\
&= \sum_{n=N+1}^{\infty} a_n(-\Delta\tilde{\xi}_n),
\end{aligned}$$

where we used that $a_{n+1}\tilde{\xi}_n$ tends to zero. Consequently, we get with (4.13) and (4.14) that

$$\begin{aligned}
& \sqrt{(1-\delta')C^{-1}} \sum_{n=N+1}^{\infty} \gamma_n\mathbb{E}[\mathbf{1}_{\mathbb{B}_{n-1}}|\nabla f(X_{n-1})|] \\
& \leq \sum_{n=N+1}^{\infty} \sqrt{\frac{\gamma_n}{-\Delta\tilde{\xi}_n}} (-\Delta\tilde{\xi}_n + \gamma_nv_n^{2\beta} + 8v_n\mathbb{P}(\mathbb{B}_{n-1}\setminus\mathbb{B}_n)).
\end{aligned}$$

Note that

$$\sqrt{\frac{\Phi_0^{(R)} - \Phi_{t_{N+1}-t_N}^{(R)}}{\gamma_{N+1}}} \leq \sqrt{-\dot{\Phi}_0^{(R)}} \leq \sqrt{\eta} R^\beta,$$

and for $n > N + 1$,

$$\gamma_n \sqrt{\frac{\Phi_{t_{n-1}-t_N}^{(R)} - \Phi_{t_n-t_N}^{(R)}}{\gamma_n}} \leq \gamma_n \sqrt{-\dot{\Phi}_{t_{n-1}-t_N}^{(R)}} \leq \sqrt{\eta} \int_{t_{n-2}-t_N}^{t_{n-1}-t_N} (\Phi_u^{(R)})^\beta du,$$

where we used that $-\dot{\Phi}^{(R)}$ and $(\gamma_n)_{n>N}$ are monotonically decreasing. Hence, using that

$$-\Delta \bar{\xi}_n = \Phi_{t_{n-1}-t_N}^{(R)} - \Phi_{t_n-t_N}^{(R)} + (y_0 + 8)(-\Delta v_n)$$

we get

$$\begin{aligned} \sum_{n=N+1}^{\infty} \gamma_n \sqrt{\frac{-\Delta \bar{\xi}_n}{\gamma_n}} &\leq \gamma_{N+1} \sqrt{\eta} R^\beta + \sqrt{\eta} \int_0^\infty (\Phi_u^{(R)})^\beta du \\ &\quad + \sqrt{y_0 + 8} \sum_{n=N+1}^{\infty} \sqrt{\gamma_n (-\Delta v_n)}. \end{aligned}$$

Consequently,

$$\begin{aligned} &\sqrt{(1 - \delta')C^{-1}} \sum_{n=N+1}^{\infty} \gamma_n \mathbb{E}[\mathbb{1}_{\mathbb{B}_{n-1}} |\nabla f(X_{n-1})|] \\ &\leq \gamma_{N+1} \sqrt{\eta} R^\beta + \sqrt{\eta} \int_0^\infty (\Phi_u^{(R)})^\beta du \\ &\quad + \frac{1}{\sqrt{y_0 + 8}} \sum_{n=N+1}^{\infty} \sqrt{\frac{\gamma_n}{-\Delta v_n}} (-(y_0 + 8)\Delta v_n + \gamma_n v_n^{2\beta} + 8v_n \mathbb{P}(\mathbb{B}_{n-1} \setminus \mathbb{B}_n)). \end{aligned}$$

The proof is complete. \square

4.3 Technical analysis of lower dropdowns

Roughly speaking, the following two lemmas will later be used to show that for a certain critical level of the objective function, a lower dropdown (in the sense of the previous two propositions) entails that the SGD's target value converges to a value strictly below the respective critical level with high probability.

Lemma 4.1. *Let $(M_n)_{n \in \mathbb{N}}$ be a L^2 -martingale started in zero. Then for every $\kappa > 0$,*

$$\mathbb{P}\left(\sup_{\ell \in \mathbb{N}} (M_\ell - \langle M \rangle_\ell) \geq \kappa\right) \leq \frac{4}{\kappa^2} + \sum_{n \in \mathbb{N}_0} \frac{2^{n+3}}{(2^n + \kappa)^2} =: \phi(\kappa).$$

In particular, for every $\varepsilon > 0$ there exists a $\kappa > 0$ such that the above right-hand side is smaller than ε .

Proof. For $n \in \mathbb{N}_0$, let $T_n = \inf\{\ell \in \mathbb{N} : \langle M \rangle_{\ell+1} > 2^n\}$. Then,

$$\begin{aligned} \sup_{\ell=T_n+1, \dots, T_{n+1}} (M_\ell - \langle M \rangle_\ell) &\leq \sup_{\ell=1, \dots, T_n} M_\ell - 2^n, \\ \sup_{\ell=1, \dots, T_0} (M_\ell - \langle M \rangle_\ell) &\leq \sup_{\ell=1, \dots, T_0} M_\ell. \end{aligned}$$

We use Doob's L^2 -inequality to deduce that

$$\begin{aligned} \mathbb{P}\left(\sup_{\ell=1, \dots, T_{n+1}} M_\ell \geq 2^n + \kappa\right) &\leq 4(2^n + \kappa)^{-2} \mathbb{E}[M_{T_{n+1}}^2] \leq \frac{2^{n+3}}{(2^n + \kappa)^2}, \\ \mathbb{P}\left(\sup_{\ell=1, \dots, T_0} M_\ell \geq \kappa\right) &\leq \frac{4}{\kappa^2}. \end{aligned}$$

Therefore,

$$\mathbb{P}\left(\sup_{\ell \in \mathbb{N}} (M_\ell - \langle M \rangle_\ell) \geq \kappa\right) \leq \frac{4}{\kappa^2} + \sum_{n \in \mathbb{N}_0} \frac{2^{n+3}}{(2^n + \kappa)^2}.$$

The proof is complete. \square

Lemma 4.2. Let $N \in \mathbb{N}, \delta, C > 0$, $(\sigma_n)_{n \geq N}$ be a sequence of positive reals and let $(\mathbb{A}_n)_{n \geq N}$ be a sequence of $(U, \delta, C, (\sigma_n), 2)$ -compatible events. Moreover, suppose that $(\gamma_n \sigma_n^2)_{n \geq N}$ is decreasing and that for $\delta' \in (0, 1)$,

$$2C^3 \|\nabla f\|_{\text{Lip}(U^{2\delta})} \gamma_n \leq \delta'.$$

Then one has for $T > 0$ that

$$\begin{aligned} &\mathbb{P}\left(\sup_{n \geq N} \mathbf{1}_{\mathbb{A}_{n-1} \cap \{\gamma_n |D_n| < \delta\}} (f(X_n) - f(X_N)) \geq 2T\right) \\ &\leq \phi\left(\frac{1 - \delta'}{C \gamma_{N+1} \sigma_{N+1}^2} T\right) + 2\|\nabla f\|_{\text{Lip}(U^{2\delta})} \sum_{\ell=N+1}^{\infty} (\gamma_\ell \sigma_\ell)^2 \frac{1}{T}, \end{aligned}$$

where ϕ is as in Lemma 4.1.

Proof. By (4.5) and (4.6) we have on $\mathbb{A}_{n-1} \cap \{\gamma_n |D_n| \leq \delta\}$

$$\begin{aligned} f(X_n) &\leq f(X_{n-1}) - \gamma_n ((1 - \delta') C^{-1} |\nabla f(X_{n-1})|^2 + \langle \nabla f(X_{n-1}), D_n \rangle) \\ &\quad + 2\|\nabla f\|_{\text{Lip}(U^{2\delta})} \gamma_n^2 |D_n|^2. \end{aligned}$$

For $n \geq N$, we let

$$\begin{aligned} \Xi_n &= - \sum_{\ell=N+1}^n \mathbf{1}_{\mathbb{A}_{\ell-1}} \gamma_\ell ((1 - \delta') C^{-1} |f(X_{\ell-1})|^2 + \langle f(X_{\ell-1}), D_\ell \rangle), \\ \Xi'_n &= 2\|\nabla f\|_{\text{Lip}(U^{2\delta})} \sum_{\ell=N+1}^n \mathbf{1}_{\mathbb{A}_{\ell-1}} \gamma_\ell^2 |D_\ell|^2, \end{aligned}$$

and observe that on $\mathbb{A}_{n-1} \cap \{\gamma_n |D_n| \leq \delta\}$

$$f(X_n) - f(X_N) \leq \Xi_n + \Xi'_n.$$

Next, we deduce an estimate for the supremum of the process $(\Xi_n)_{n>N}$. In terms of the martingale

$$M_n = - \sum_{\ell=N+1}^n \mathbf{1}_{\mathbb{A}_{\ell-1}} \gamma_\ell \langle \nabla f(X_{\ell-1}), D_\ell \rangle,$$

we have

$$\begin{aligned} \langle M \rangle_n &= \sum_{\ell=N+1}^n \gamma_\ell^2 \mathbf{1}_{\mathbb{A}_{\ell-1}} \mathbb{E}[\langle \nabla f(X_{\ell-1}), D_\ell \rangle^2 | \mathcal{F}_{\ell-1}] \\ &\leq \sum_{\ell=N+1}^n \gamma_\ell^2 \mathbf{1}_{\mathbb{A}_{\ell-1}} |\nabla f(X_{\ell-1})|^2 \sigma_\ell^2. \end{aligned}$$

Using that $(\gamma_n \sigma_n^2)_{n>N}$ is monotonically decreasing we deduce that

$$\langle M \rangle_n \leq \gamma_{N+1} \sigma_{N+1}^2 \sum_{\ell=N+1}^n \mathbf{1}_{\mathbb{A}_{\ell-1}} \gamma_\ell |\nabla f(X_{\ell-1})|^2.$$

Consequently,

$$\Xi_n \leq M_n - \frac{1 - \delta'}{C \gamma_{N+1} \sigma_{N+1}^2} \langle M \rangle_n = a \left(\frac{1}{a} M_n - \left\langle \frac{1}{a} M \right\rangle_n \right)$$

for $a := (C \gamma_{N+1} \sigma_{N+1}^2) / (1 - \delta')$. With Lemma 4.1 we get

$$\mathbb{P} \left(\sup_{n>N} \Xi_n \geq T \right) \leq \mathbb{P} \left(\sup_{n>N} \frac{1}{a} M_n - \left\langle \frac{1}{a} M \right\rangle_n \geq \frac{T}{a} \right) \leq \phi \left(\frac{T}{a} \right). \quad (4.15)$$

Conversely,

$$\begin{aligned} \mathbb{E} \left[\sup_{n>N} \Xi'_n \right] &\leq 2 \|\nabla f\|_{\text{Lip}(U^{2\delta})} \mathbb{E} \left[\sum_{\ell=N+1}^{\infty} \mathbf{1}_{\mathbb{A}_{\ell-1}} \gamma_\ell^2 |D_\ell|^2 \right] \\ &\leq 2 \|\nabla f\|_{\text{Lip}(U^{2\delta})} \sum_{\ell=N+1}^{\infty} (\gamma_\ell \sigma_\ell)^2, \end{aligned}$$

and by the Markov inequality

$$\mathbb{P} \left(\sup_{n>N} \Xi'_n \geq T \right) \leq 2 \|\nabla f\|_{\text{Lip}(U^{2\delta})} \sum_{\ell=N+1}^{\infty} (\gamma_\ell \sigma_\ell)^2 \frac{1}{T}.$$

In combination with (4.15) we obtain the result. \square

4.4 Satisfiability of the assumptions

Proposition 4.5. *Let $\gamma \in (1/2, 1]$, $\sigma \in \mathbb{R}$ and $q \geq 2$ with*

$$3\gamma - 2\sigma > 2, \quad q > \frac{1}{2\gamma - \sigma - 1}, \quad (4.16)$$

and let, for positive constants C_γ and C_σ ,

$$(\gamma_n)_{n \in \mathbb{N}} = (C_\gamma n^{-\gamma})_{n \in \mathbb{N}}, \quad (\sigma_n)_{n \in \mathbb{N}} = (C_\sigma n^\sigma)_{n \in \mathbb{N}}.$$

Suppose that f satisfies a Łojasiewicz-inequality on a bounded set U . Then there exists $\beta \in (1/2, 1)$ such that for

$$(v_n)_{n \in \mathbb{N}} = (C_v t_n^{-\frac{1}{2\beta-1}})_{n \in \mathbb{N}}, \quad (w_n)_{n \in \mathbb{N}} = (C_w t_n^{-\frac{1}{2\beta-1}})_{n \in \mathbb{N}}$$

with $C_w, C_v > 0$ sufficiently large, one has for sufficiently large N and all compatible events $(\mathbb{B}_n)_{n \geq N}$ in the sense of Proposition 4.1 that

$$\sum_{n=N+1}^{\infty} \gamma_n \mathbb{E}[\mathbb{1}_{\mathbb{B}_{n-1}} | \Gamma_n |] < \infty.$$

Moreover, as $N \rightarrow \infty$,

$$\frac{w_N}{\gamma_{N+1} \sigma_{N+1}^2} \rightarrow \infty, \quad \sum_{\ell=N+1}^{\infty} (\gamma_\ell \sigma_\ell)^2 w_N^{-1} \rightarrow 0.$$

Proof. First note that since U is bounded we can conclude that the validity of a Łojasiewicz-inequality for a $\beta_0 \in (1/2, 1)$ entails a Łojasiewicz-inequality for every $\beta \in [\beta_0, 1)$. By Proposition 4.3 and Remark 4.1, one can choose for every large C_w , appropriate constants C_v and κ such that all assumptions of Proposition 4.1 are satisfied, if

$$(a') \quad 2\sigma \leq \frac{(4\beta - 1)\gamma - 2\beta}{2\beta - 1},$$

$$(b') \quad \sigma \leq \frac{2\beta\gamma - 1}{2\beta - 1} - \frac{1}{q},$$

$$(c') \quad \gamma \geq \frac{1}{2\beta}.$$

By sending β to one, one easily verifies by using (4.16) and $\gamma > 1/2$, that (a'), (b') and (c') are satisfied for a $\beta \in [\beta_0, 1)$. In view of Proposition 4.4, it suffices to verify the additional condition and finiteness of the series appearing in the latter proposition.

We still need to consider the additional conditions imposed in Proposition 4.4. First, we show that the sequence $(-\Delta v_n / \gamma_n)_{n \geq 2}$ is eventually decreasing. One has

$$\frac{-\Delta v_n}{\gamma_n} - \frac{-\Delta v_{n+1}}{\gamma_{n+1}} = \frac{-\Delta v_n \gamma_{n+1} + \Delta v_{n+1} \gamma_n}{\gamma_n \gamma_{n+1}},$$

$$\begin{aligned} -\Delta v_n &= \frac{C_v}{2\beta-1} t_n^{-\frac{2\beta}{2\beta-1}} \gamma_n + \frac{C_v \beta}{(2\beta-1)^2} t_n^{-\frac{4\beta-1}{2\beta-1}} \gamma_n^2 + o\left(t_n^{-\frac{4\beta-1}{2\beta-1}} \gamma_n^2\right), \\ -\Delta v_{n+1} &= \frac{C_v}{2\beta-1} t_n^{-\frac{2\beta}{2\beta-1}} \gamma_{n+1} - \frac{C_v \beta}{(2\beta-1)^2} t_n^{-\frac{4\beta-1}{2\beta-1}} \gamma_{n+1}^2 + o\left(t_n^{-\frac{4\beta-1}{2\beta-1}} \gamma_n^2\right), \end{aligned}$$

so that

$$-\Delta v_n \gamma_{n+1} + \Delta v_{n+1} \gamma_n = \frac{C_v \beta}{(2\beta-1)^2} t_n^{-\frac{4\beta-1}{2\beta-1}} \gamma_n \gamma_{n+1} (\gamma_n + \gamma_{n+1}) + o\left(t_n^{-\frac{4\beta-1}{2\beta-1}} \gamma_n^3\right),$$

which is positive for large n .

Further, using (4.12) we get

$$\left(\frac{\gamma_n}{-\Delta v_n}\right)^{\frac{1}{2}} v_n = \mathcal{O}((v_n^{-1} t_n)^{\frac{1}{2}} v_n) = \mathcal{O}(v_n^{1-\beta}),$$

and with $v_n \sim C_v((2\beta-1)\eta)^{1/(2\beta-1)} \Phi_{t_n-t_N}^{(R)}$ we have $(\gamma_n/(-\Delta \bar{\xi}_n))^{1/2} \bar{\xi}_n \rightarrow 0$.

Next, we show that the series appearing in Proposition 4.4 is finite. Indeed, we have

$$\begin{aligned} \sqrt{\gamma_n(-\Delta v_n)} &\sim \sqrt{\frac{C_v}{2\beta-1}} \gamma_n t_n^{-\frac{\beta}{2\beta-1}}, \\ \sqrt{\frac{\gamma_n}{-\Delta v_n}} \gamma_n v_n^{2\beta} &\sim C_v^{2\beta-\frac{1}{2}} \sqrt{2\beta-1} \gamma_n t_n^{-\frac{\beta}{2\beta-1}}, \end{aligned}$$

so that for $\gamma < 1$ summability follows with $\beta/(2\beta-1) > 1$ for all $\beta \in (1/2, 1)$ and in the case $\gamma = 1$ summability follows from Cauchy's condensation test. Moreover, as stated before $(\gamma_n/(-\Delta v_n))^{1/2} v_n = \mathcal{O}(v_n^{1-\beta})$ tends to zero.

It remains to discuss the asymptotic statements in the proposition. The term

$$\frac{w_N}{\gamma_{N+1} \sigma_{N+1}^2} \sim \frac{C_w}{C_\gamma C_\sigma^2} (t_n)^{-\frac{1}{2\beta-1}} n^{\gamma-2\sigma}$$

tends to infinity, iff $\beta > (1-2\sigma)/(2\gamma-4\sigma)$. As consequence of (4.16) the right-hand side is smaller than 3/4 and for sufficiently large β , $w_N/(\gamma_{N+1} \sigma_{N+1}^2)$ tends to infinity. Moreover,

$$\sum_{\ell=N+1}^{\infty} (\gamma_\ell \sigma_\ell)^2 = \mathcal{O}(N^{-2\gamma+2\sigma+1})$$

is of order $o(w_n)$, iff $\beta > (\gamma-2\sigma)/(4\gamma-4\sigma-2)$ with the right-hand side being strictly smaller than one. \square

4.5 Properties of Łojasiewicz-functions

We summarise some properties of Łojasiewicz-functions. Note that the proofs only use that f is C^1 but not that the differentials are locally Lipschitz continuous. A similar statement for a uniform Łojasiewicz-inequality around a compact level set can be found in [3, Lemma 1].

Lemma 4.3. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a Łojasiewicz-function and $K \subset \mathbb{R}^d$ be an arbitrary compact set.

(i) The set of critical levels

$$\mathcal{L}_K = \{f(x) : x \in \nabla f^{-1}(\{0\}) \cap K\}$$

is finite so that f has at most a countable number of critical levels.

(ii) For every critical level $\ell \in \mathcal{L}_K$ there exists an open neighbourhood $U \supset f^{-1}(\{\ell\}) \cap K$, $\mathbb{L} > 0$, $\beta \in [1/2, 1)$ such that for every $y \in U$,

$$|\nabla f(y)| \geq \mathbb{L}|f(y) - \ell|^\beta.$$

(iii) For a neighbourhood as in (ii), there exists $\varepsilon > 0$ such that

$$f^{-1}((\ell - \varepsilon, \ell + \varepsilon)) \cap K \subset U.$$

Proof. (i) Let $K \subset U$ be a compact set. For every $x \in \nabla f^{-1}(\{0\})$ we can choose an open neighbourhood U_x of x on which the Łojasiewicz-inequality holds for parameters $\mathbb{L}^{(x)} > 0$ and $\beta^{(x)} \in [1/2, 1)$. By compactness of $\nabla f^{-1}(\{0\}) \cap K$ we conclude that $\nabla f^{-1}(\{0\}) \cap K \subset \bigcup_{x \in \nabla f^{-1}(\{0\}) \cap K} U_x$ has a finite cover, say $\bigcup_{x \in \mathcal{X}} U_x$. We show that $\mathcal{L}_K = f(\mathcal{X})$. Indeed, every $y \in \nabla f^{-1}(\{0\}) \cap K$ lies in a neighbourhood U_x with $x \in \mathcal{X}$ and one has

$$0 = |\nabla f(y)| \geq \mathbb{L}^{(x)}|f(y) - f(x)|^{\beta^{(x)}},$$

so that $f(y) = f(x) \in f(\mathcal{X})$.

(ii) Let $x \in K \cap f^{-1}(\{\ell\})$. If x lies in $\nabla f^{-1}(\{0\})$, then x admits an open and bounded neighbourhood U_x on which the Łojasiewicz-inequality holds with appropriate parameters $\mathbb{L}^{(x)} > 0$ and $\beta^{(x)} \in [1/2, 1)$. If x does not lie in $\nabla f^{-1}(\{0\})$, then $|\nabla f(x)| > 0$ and by continuity we can choose a neighbourhood U_x so that for every $y \in U_x$,

$$\begin{aligned} |\nabla f(y)| &> \frac{|\nabla f(x)|}{2}, \\ |f(y) - f(x)|^{\frac{1}{2}} &< \frac{|\nabla f(x)|}{2}. \end{aligned}$$

Thus the Łojasiewicz-inequality holds on U_x with parameters 1 and 1/2. By compactness of $K \cap f^{-1}(\{\ell\})$ there exists a finite set $\mathcal{X} \subset K \cap f^{-1}(\{\ell\})$ with $f^{-1}(\{\ell\}) \cap K \subset \bigcup_{x \in \mathcal{X}} U_x$. We let $\beta := \max_{x \in \mathcal{X}} \beta^{(x)}$ and assuming that on $U := \bigcup_{x \in \mathcal{X}} U_x$, $|f - \ell|$ is bounded by R we conclude that for every $y \in U$ there exists $x \in \mathcal{X}$ with $y \in U_x$ so that

$$|\nabla f(y)| \geq \mathbb{L}^{(x)}|f(y) - \ell|^{\beta^{(x)}} \geq \frac{\mathbb{L}^{(x)}}{R^{\beta - \beta^{(x)}}}|f(y) - \ell|^\beta,$$

and the Łojasiewicz-inequality holds on U with parameters $\min_{x \in \mathcal{X}} \mathbb{L}^{(x)} / R^{\beta - \beta^{(x)}}$ and β .

(iii) Suppose that for every $\varepsilon > 0$, $K \cap f^{-1}((\ell - \varepsilon, \ell + \varepsilon)) \not\subset U$. Then we can pick a $K \cap U^c$ -valued sequence $(x_n)_{n \in \mathbb{N}}$ with $|f(x_n) - \ell| \leq 1/n$. Since K is compact we can assume without loss of generality that $(x_n)_{n \in \mathbb{N}}$ converges (otherwise we choose an appropriate subsequence). Then by continuity of f , $f(x_n) \rightarrow f(x) = \ell$ and $x \in K \cap f^{-1}(\{\ell\})$. But this entails that all but finitely many of the entries of $(x_n)_{n \in \mathbb{N}}$ have to lie in U since U is an open neighbourhood of $f^{-1}(\{\ell\}) \cap K$ causing a contradiction. \square

4.6 Proof of Theorem 1.2

Let K be a compact set and $Y \in \mathcal{L}_K$ a critical level of f on K . By Lemma 4.3 we can choose an open and bounded set $U \supset f^{-1}(\{Y\}) \cap K$ and parameters $\mathbb{L}_0 > 0$ and $\beta_0 \in (1/2, 1)$ such that f admits the Łojasiewicz-inequality on U , i.e. that for all $y \in U$,

$$|\nabla f(y)| \geq \mathbb{L}_0 |f(y) - Y|^{\beta_0}.$$

Again by Lemma 4.3 we can pick $\varepsilon > 0$ with $U \supset f^{-1}((Y - \varepsilon, Y + \varepsilon))$. Based on parameters $N \geq N_0$ and $C > 0$ we consider events $(\mathbb{A}_n)_{n \geq N}$ given by

$$\begin{aligned} \mathbb{A}_N &= \{X_N \in U, |\nabla f(X_N)|^2 \leq C \langle \nabla f(X_N), -\Gamma_{N+1} \rangle, |\Gamma_{N+1}| \leq C |\nabla f(X_N)|, \\ &\quad \mathbb{E}[|D_{N+1}|^q | \mathcal{F}_N] \leq C \sigma_n^q \text{ and } \mathbb{E}[D_{N+1} | \mathcal{F}_N] = 0\}, \end{aligned}$$

and for $n > N$,

$$\begin{aligned} \mathbb{A}_n &= \mathbb{A}_{n-1} \cap \{X_n \in U, |\nabla f(X_n)|^2 \leq C \langle \nabla f(X_n), -\Gamma_{n+1} \rangle, |\Gamma_{n+1}| \leq C |\nabla f(X_n)| \\ &\quad \gamma_n |D_n| < \delta, \mathbb{E}[|D_{n+1}|^q | \mathcal{F}_n] \leq C \sigma_n^q \text{ and } \mathbb{E}[D_{n+1} | \mathcal{F}_n] = 0\}. \end{aligned}$$

Provided that N is sufficiently large such that, on \mathbb{A}_n ,

$$\gamma_{n+1} |\Gamma_{n+1}| \leq \gamma_{n+1} C \|\nabla f\|_{L^\infty(U)} \leq \delta$$

for all $n \geq N$, the family $(\mathbb{A}_n)_{n \geq N}$ is $(U, \delta, C, (C\sigma_n), q)$ -compatible.

Now for every $\beta \in [\beta_0, 1)$ there exists a $\mathbb{L} > 0$ such that f also admits the Łojasiewicz-inequality with parameters β and \mathbb{L} on U and we can choose a quadruple (N_0, β, C_v, C_w) which satisfies the statement of Proposition 4.5 for $\sigma_n = Cn^\sigma$. We assume without loss of generality that $Y = 0$ mentioning that we could consider $\bar{f} = f - Y$ instead of f and fix a sufficiently large $N \geq N_0$, so that $(\mathbb{A}_n)_{n \geq N}$ is $(U, \delta, C, (C\sigma_n), q)$ -compatible.

Additionally, we let $\mathbb{B}_N = \mathbb{A}_N \cap \{-w_N \leq f(X_N)\}$ and for $n > N$,

$$\mathbb{B}_n = \mathbb{B}_{n-1} \cap \mathbb{A}_{n-1} \cap \{\gamma_n |D_n| < \delta, -w_n \leq f(X_n)\}.$$

We note that $(\mathbb{B}_n)_{n \geq N}$ satisfies a representation as in Lemma 4.1 for $(\mathbb{A}'_n)_{n \geq N}$ given by $\mathbb{A}'_N = \mathbb{A}_N$ and for $n > N$,

$$\begin{aligned} \mathbb{A}'_n &= \mathbb{B}_n \cap \{X_n \in U, |\nabla f(X_n)|^2 \leq C \langle \nabla f(X_n), -\Gamma_{n+1} \rangle, |\Gamma_{n+1}| \leq C |\nabla f(X_n)|, \\ &\quad \mathbb{E}[|D_{n+1}|^q | \mathcal{F}_n] \leq C \sigma_n^q \text{ and } \mathbb{E}[D_{n+1} | \mathcal{F}_n] = 0\}. \end{aligned}$$

In particular, one has $\mathbb{A}'_n \subset \mathbb{A}_n$ and $(\mathbb{A}'_n)_{n \geq N}$ is $(U, \delta, C, (C\sigma_n), q)$ -compatible if this is the case for $(\mathbb{A}_n)_{n \geq N}$. Note that for $\mathbb{B}_\infty = \bigcap_{n \geq N} \mathbb{B}_n$ (and respectively for \mathbb{A}_∞ and \mathbb{A}'_∞) we have that

$$\mathbb{B}_\infty = \mathbb{A}'_\infty = \mathbb{A}_\infty \cap \{F(X_n) \geq -w_n, \forall n \geq N\}.$$

By Propositions 4.4 and 4.5, one has that on \mathbb{A}'_∞ , almost surely, $\sum_{n > N} \gamma_n |\Gamma_n| < \infty$. Moreover, the L^2 -martingale $(\bar{M}_n)_{n \geq N} = (\sum_{\ell=N+1}^n \gamma_\ell \mathbb{1}_{\mathbb{A}_{\ell-1}} D_\ell)_{n \geq N}$ converges, almost surely since

$$\mathbb{E}[\langle \bar{M} \rangle_\infty] \leq \sum_{\ell=N+1}^{\infty} C^2 (\gamma_\ell \sigma_\ell)^2.$$

Hence, we have almost sure convergence of $(X_n)_{n \in \mathbb{N}_0}$ on \mathbb{A}'_∞ . Now consider the stopping time T given by

$$T(\omega) = \inf \{n \geq N : f(X_n(\omega)) < -w_n, \omega \in \mathbb{A}_n\}.$$

For fixed $N' \geq N$ we apply Lemma 4.2 to estimate the probability

$$\mathbb{P}\left(\sup_{n \geq N'} f(X_n) - f(X_{N'}) \geq w_{N'} \mid T = N'\right).$$

Note that $(\mathbb{A}''_n)_{n \geq N'}$ with $\mathbb{A}''_n = \{T = N'\} \cap \mathbb{A}_n$ is $(U, \delta, C, (C\sigma_n), 2)$ -compatible under the conditional distribution $\mathbb{P}(\cdot \mid T = N')$. Recall that $2C^3 \|\nabla f\|_{\text{Lip}(U^{2\delta})} \gamma_n \leq \delta'$ so that

$$\begin{aligned} & \mathbb{P}\left(\sup_{n > N'} \mathbb{1}_{\mathbb{A}_{n-1} \cap \{\gamma_n |D_n| < \delta\}} (f(X_n) - f(X_{N'})) \geq w_{N'} \mid T = N'\right) \\ & \leq \phi\left(\frac{1 - \delta'}{2C^3 \gamma_{N'+1} \sigma_{N'+1}^2} w_{N'}\right) + 4C^2 \|\nabla f\|_{\text{Lip}(U^{2\delta})} \sum_{\ell=N'+1}^{\infty} (\gamma_\ell \sigma_\ell)^2 \frac{1}{w_{N'}} =: \rho_{N'}. \end{aligned}$$

By Proposition 4.5, $\rho_{N'} \rightarrow 0$ as $N' \rightarrow \infty$ so that also $\bar{\rho}_N := \sup_{n \geq N} \rho_n$ converges to zero. We conclude that

$$\begin{aligned} & \mathbb{P}((\mathbb{A}_\infty \setminus \mathbb{B}_\infty) \cap \{\lim_{n \rightarrow \infty} f(X_n) = 0\}) \\ & = \mathbb{P}(\mathbb{A}_\infty \cap \{T < \infty, \lim_{n \rightarrow \infty} f(X_n) = 0\}) \\ & \leq \sum_{N'=N}^{\infty} \mathbb{P}(T = N') \mathbb{P}\left(\sup_{n > N'} \mathbb{1}_{\mathbb{A}_{n-1} \cap \{\gamma_n |D_n| < \delta\}} (f(X_n) - f(X_{N'})) \geq w_{N'} \mid T = N'\right) \leq \bar{\rho}_N. \end{aligned}$$

We recall that \mathbb{A}_∞ and \mathbb{B}_∞ depend on the choice of N and in the following we write $\bar{\mathbb{A}}_N$ and $\bar{\mathbb{B}}_N$ for the respective events. Moreover, we denote

$$\begin{aligned} \bar{\mathbb{M}}_N^{C, \sigma, q} &:= \bigcap_{n=N}^{\infty} \{|\nabla f(X_n)|^2 \leq C \langle \nabla f(X_n), -\Gamma_{n+1} \rangle, |\Gamma_{n+1}| \leq C |\nabla f(X_n)|, \\ & \quad \mathbb{E}[|D_{n+1}|^q | \mathcal{F}_n] \leq (C\sigma_{n+1})^q \text{ and } \mathbb{E}[D_{n+1} | \mathcal{F}_n] = 0\}, \\ \bar{\mathbb{M}}^{C, \sigma, q} &= \bigcup_{N \in \mathbb{N}} \bar{\mathbb{M}}_N^{C, \sigma, q}, \quad \mathbb{U}_N := \bigcap_{n=N}^{\infty} \{X_n \in U\}, \quad \mathbb{U} := \bigcup_{N \in \mathbb{N}} \mathbb{U}_N. \end{aligned}$$

We note that

$$(\tilde{\mathbb{M}}_N^{C,\sigma,q} \cap \mathbb{U}_N) \setminus \bar{\mathbb{A}}_N = \tilde{\mathbb{M}}_N^{C,\sigma,q} \cap \mathbb{U}_N \cap \{\exists n > N : \gamma_n |D_n| \geq \delta\},$$

and estimate

$$\begin{aligned} & \mathbb{P}(\tilde{\mathbb{M}}_N^{C,\sigma,q} \cap \mathbb{U}_N \cap \{\exists n > N : \gamma_n |D_n| \geq \delta\}) \\ & \leq \sum_{n=N+1}^{\infty} \mathbb{P}(\gamma_n |D_n| \geq \delta \mid \mathbb{E}[|D_n|^2 \mid \mathcal{F}_{n-1}] \leq (C\sigma_n)^2) \leq \delta^{-1} \sum_{n=N+1}^{\infty} C^2(\gamma_n \sigma_n)^2. \end{aligned}$$

By Theorem 1.1, we have almost surely convergence of $(f(X_n))_{n \in \mathbb{N}_0}$ on $\tilde{\mathbb{M}}_N^{C,\sigma,q} \cap \mathbb{U}_N$ and since f has a unique critical level on U we thus have $f(X_n) \rightarrow 0$, almost surely on $\tilde{\mathbb{M}}_N^{C,\sigma,q} \cap \mathbb{U}_N$. We conclude that

$$\begin{aligned} & \mathbb{P}(\tilde{\mathbb{M}}_N^{C,\sigma,q} \cap \mathbb{U}_N \cap \{(X_n) \text{ does not converge}\}) \\ & \leq \mathbb{P}((\tilde{\mathbb{M}}_N^{C,\sigma,q} \cap \mathbb{U}_N) \setminus \bar{\mathbb{A}}_N) + \mathbb{P}(\bar{\mathbb{A}}_N \setminus \bar{\mathbb{B}}_N) \\ & \leq \delta^{-1} \sum_{n=N+1}^{\infty} C^2(\gamma_n \sigma_n)^2 + \bar{\rho}_N \rightarrow 0, \quad \text{as } N \rightarrow \infty. \end{aligned}$$

Thus we obtain almost sure convergence of $(X_n)_{n \in \mathbb{N}_0}$ on the monotone limit

$$\mathbb{M}^{C,\sigma,q} \cap \mathbb{U} = \bigcup_{N \in \mathbb{N}} \tilde{\mathbb{M}}_N^{C,\sigma,q} \cap \mathbb{U}_N.$$

Obviously, this is also true on $\mathbb{G} \cap \mathbb{M}^{\sigma,q} \cap \mathbb{U}$ since

$$\mathbb{G} \cap \mathbb{M}^{\sigma,q} = \bigcup_{C \in \mathbb{N}} \tilde{\mathbb{M}}^{C,\sigma,q}.$$

Now note that there is only a finite number of distinct critical levels on K , say ℓ_1, \dots, ℓ_J , and for each critical level ℓ_j , $j = 1, \dots, J$, we can choose an open Łojasiewicz-neighbourhood U_j as above. By elementary analysis it follows that for all sequences $(x_n)_{n \in \mathbb{N}_0}$ with (a) all but finitely many x_n lie in K , (b) $\lim_{n \rightarrow \infty} f(x_n)$ exists and (c) $\lim_{n \rightarrow \infty} \nabla f(x_n) = 0$, one has that $(f(x_n))_{n \in \mathbb{N}_0}$ converges to one of the critical levels ℓ_1, \dots, ℓ_J and all but finitely many of its entries are in the respective open set U_j . By Theorem 1.1, we have that in the case where $\mathbb{G} \cap \mathbb{M}^{\sigma,q}$ enters and all but finitely many of the entries $(X_n)_{n \in \mathbb{N}_0}$ lie in K that all but finitely many entries of $(X_n)_{n \in \mathbb{N}_0}$ lie in a single set U_j and we thus obtain with the above that $(X_n)_{n \in \mathbb{N}_0}$ converges almost surely. Since this is true for every compact set K we obtain the result.

5 Analytic neural networks

In this section, we discuss neural networks that satisfy the assumptions of Theorem 1.2. We use similar notation as in [56]. Let $\rho : \mathbb{R} \rightarrow \mathbb{R}$ denote a function, the activation function.

We fix a depth $L \in \mathbb{N}$ and the number of neurons $d_{\text{in}} := N_0, \dots, N_L :=: d_{\text{out}}$ in each of the layers and we denote by

$$\mathcal{P}_N := \prod_{\ell=1}^L (\mathbb{R}^{N_\ell \times N_{\ell-1}} \times \mathbb{R}^{N_\ell}),$$

the set of all parametrizations of networks with architecture $N = (N_0, \dots, N_L)$. The architecture N is related to the realization map $\mathcal{R}_N : \mathcal{P}_N \rightarrow C(\mathbb{R}^{d_{\text{in}}}, \mathbb{R}^{d_{\text{out}}})$ given by

$$\Theta = (A_\ell, b_\ell)_{\ell=1}^L \mapsto \mathcal{R}_N(\Theta) = \text{Aff}_{A_L, b_L} \circ \rho^{\otimes N_{L-1}} \circ \text{Aff}_{A_{L-1}, b_{L-1}} \circ \dots \circ \rho^{\otimes N_1} \circ \text{Aff}_{A_1, b_1},$$

where for $p, q \in \mathbb{N}$ for each $A \in \mathbb{R}^{p,q}$ and $b \in \mathbb{R}^p$, $\text{Aff}_{A,b} : \mathbb{R}^q \rightarrow \mathbb{R}^p, x \mapsto Ax + b$ and $\rho^{\otimes p} : \mathbb{R}^p \rightarrow \mathbb{R}^p$ is the mapping that applies ρ component-wise.

Proposition 5.1. *If $\rho : \mathbb{R} \rightarrow \mathbb{R}$ is analytic, then the mapping*

$$\mathcal{P}_N \times \mathbb{R}^{d_{\text{in}}} \rightarrow \mathbb{R}^{d_{\text{out}}}, \quad (\Theta, x) \mapsto \mathcal{R}_N(\Theta, x) = \mathcal{R}_N(\Theta)(x)$$

is analytic.

Proof. Note that for each $\ell = 1, \dots, L$ the mapping

$$\mathcal{P}_N \times \mathbb{R}^{N_{\ell-1}} \rightarrow \mathcal{P}_N \times \mathbb{R}^{N_\ell}, \quad (\Theta, x) \mapsto (\Theta, A_\ell x + b_\ell)$$

with (A_ℓ, b_ℓ) being the ℓ -th entry of Θ is analytic. Moreover, for each $\ell = 1, \dots, L-1$ the mapping

$$\mathcal{P}_N \times \mathbb{R}^{N_\ell} \rightarrow \mathcal{P}_N \times \mathbb{R}^{N_\ell}, \quad (\Theta, x) \mapsto (\Theta, \rho^{\otimes N_\ell}(x))$$

is analytic. The mapping in the proposition may be written as composition of the above analytic functions. Hence, it is also analytic. \square

We give sufficient conditions that imply analyticity and, thus, the existence of local Łojasiewicz-inequalities for the objective function $f(\Theta) = \mathbb{E}[\mathcal{L}(\mathcal{R}(\Theta, X), Y)]$, where X and Y are compactly supported random variables. In practice, the expectation is often taken as an empirical average over a finite data set, so that X and Y are clearly compactly supported.

Theorem 5.1. *We assume the above setting and let $\rho : \mathbb{R} \rightarrow \mathbb{R}$ and $\mathcal{L} : \mathbb{R}^{d_{\text{out}}} \times \mathbb{R}^{d_{\text{out}}} \rightarrow \mathbb{R}$ be analytic functions. Then for compactly supported $\mathbb{R}^{d_{\text{in}}}$ - and $\mathbb{R}^{d_{\text{out}}}$ -valued random variables X and Y , the function*

$$f : \mathcal{P}_N \rightarrow \mathbb{R}, \quad \Theta \mapsto \mathbb{E}[\mathcal{L}(\mathcal{R}(\Theta, X), Y)]$$

is analytic.

Proof. As consequence of Proposition 5.1 the mapping

$$\mathcal{P}_N \times \mathbb{R}^{d_{\text{in}}} \times \mathbb{R}^{d_{\text{out}}} \rightarrow \mathbb{R}, \quad (\Theta, x, y) \mapsto \mathcal{L}(\mathcal{R}(\Theta, x), y)$$

is analytic. We prove that generally for $p, q \in \mathbb{N}$, an analytic function $G : \mathbb{R}^{p+q} \rightarrow \mathbb{R}$ and a compactly supported \mathbb{R}^q -valued random variable Z , the function $\mathbb{R}^p \ni x \mapsto \mathbb{E}[G(x, Z)]$ is analytic. This then implies the statement of the theorem.

We will use that a function is analytic if and only if it is C^∞ and satisfies locally an estimate as (5.1) below, see for instance [40, Proposition 2.2.10]. Pick $x_0 \in \mathbb{R}^p$ and a compact set $K \subset \mathbb{R}^q$ on which Z is supported. Then there exists for each $z_0 \in \mathbb{R}^q$, an open set $U_{z_0} \supset \{(x_0, z_0)\}$, $C_{z_0} < \infty$ and $R_{z_0} \in (0, 1]$ such that for all $(x, z) \in U_{z_0}$ one has for every multiindex μ ,

$$\left| \frac{\partial^{|\mu|} G}{\partial x^\mu}(x, z) \right| \leq C_{z_0} \cdot \frac{\mu!}{R_{z_0}^{|\mu|}}. \quad (5.1)$$

Now

$$\bar{U}_{z_0} := \{z \in \mathbb{R}^q : (x_0, z) \in U_{z_0}\} \supset \{z_0\}$$

is open and there is a finite cover (a finite subset $\mathcal{Z} \subset \mathbb{R}^q$) with $\bigcup_{z' \in \mathcal{Z}} \bar{U}_{z'} \supset K$. We let $R := \min_{z' \in \mathcal{Z}} R_{z'} > 0$ and $C = \max_{z' \in \mathcal{Z}} C_{z'}$ and note that for every $(x, z) \in \bigcup_{z' \in \mathcal{Z}} U_{z'}$ one has

$$\left| \frac{\partial^{|\mu|} G}{\partial x^\mu}(x, z) \right| \leq C \cdot \frac{\mu!}{R^{|\mu|}}.$$

Now $\bigcup_{z' \in \mathcal{Z}} U_{z'}$ is open and covers $\{x_0\} \times K$ so that there exists an open neighbourhood V of x_0 with $V \times K \subset \bigcup_{z' \in \mathcal{Z}} U_{z'}$ and we observe that for every $x \in V$ and every multiindex μ ,

$$\mathbb{E} \left[\frac{\partial^{|\mu|} G}{\partial x^\mu}(x, Z) \right] \leq C \cdot \frac{\mu!}{R^{|\mu|}}.$$

Using that $G(\cdot, z)$ is C^∞ for every $z \in \mathbb{R}^q$ we obtain the result. \square

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References

- [1] P.-A. Absil, R. Mahony, and B. Andrews, Convergence of the iterates of descent methods for analytic cost functions, *SIAM J. Optim.*, 16(2):531–547, 2005.
- [2] S. Arora, N. Cohen, W. Hu, and Y. Luo, Implicit regularization in deep matrix factorization, *Adv. Neural Inf. Process. Syst.*, Vol. 32, 2019.
- [3] H. Attouch and J. Bolte, On the convergence of the proximal algorithm for nonsmooth functions involving analytic features, *Math. Program.*, 116:5–16, 2009.
- [4] J.-F. Aujol, C. Dossal, and A. Rondepierre, Convergence rates of the heavy-ball method under the Łojasiewicz property, *Math. Program.*, 198(1):195–254, 2023.
- [5] F. Bach and E. Moulines, Non-asymptotic analysis of stochastic approximation algorithms for machine learning, *Adv. Neural Inf. Process. Syst.*, 24:451–459, 2011.

- [6] F. Bach and E. Moulines, Non-strongly-convex smooth stochastic approximation with convergence rate $O(1/n)$, *Adv. Neural Inf. Process. Syst.*, 26:773–781, 2013.
- [7] R. Bassily, M. Belkin, and S. Ma, On exponential convergence of SGD in non-convex over-parametrized learning, *arXiv:1811.02564*, 2018.
- [8] M. Benaïm, Dynamics of stochastic approximation algorithms, in: *Séminaire de Probabilités XXXIII. Lecture Notes in Mathematics*, Springer, Vol. 1709, 1999.
- [9] A. Benveniste, M. Métivier, and P. Priouret, *Adaptive Algorithms and Stochastic Approximations*, in: *Applications of Mathematics*, Springer-Verlag, Vol. 22, 1990.
- [10] D. P. Bertsekas and J. N. Tsitsiklis, Gradient convergence in gradient methods with errors, *SIAM J. Optim.*, 10(3):627–642, 2000.
- [11] P. Cheridito, A. Jentzen, and F. Rossmannek, Non-convergence of stochastic gradient descent in the training of deep neural networks, *J. Complexity*, 64:101540, 2021.
- [12] R. Chill, A. Haraux, and M. A. Jendoubi, Applications of the Łojasiewicz-Simon gradient inequality to gradient-like evolution equations, *Anal. Appl. (Singap.)*, 7(4):351–372, 2009.
- [13] L. Chizat, E. Oyallon, and F. Bach, On lazy training in differentiable programming, *Adv. Neural Inf. Process. Syst.*, 32:2937–2947, 2019.
- [14] E. Chouzenoux, J.-B. Fest, and A. Repetti, A Kurdyka-Łojasiewicz property for stochastic optimization algorithms in a non-convex setting, *arXiv:2302.06447*, 2023.
- [15] Y. Cooper, Global minima of overparameterized neural networks, *SIAM J. Math. Data Sci.*, 3(2):676–691, 2021.
- [16] H. B. Curry, The method of steepest descent for non-linear minimization problems, *Quart. Appl. Math.*, 2(3):258–261, 1944.
- [17] S. Dereich, A. Jentzen, and S. Kassing, On the existence of minimizers in shallow residual ReLU neural network optimization landscapes, *SIAM J. Numer. Anal.*, 2023. To appear.
- [18] S. Dereich and S. Kassing, Cooling down stochastic differential equations: Almost sure convergence, *Stochastic Process. Appl.*, 152:289–311, 2022.
- [19] S. Dereich and S. Kassing, Central limit theorems for stochastic gradient descent with averaging for stable manifolds, *Electron. J. Probab.*, 28:1–48, 2023.
- [20] S. Dereich and S. Kassing, On the existence of optimal shallow feedforward networks with ReLU activation, *J. Mach. Learn.*, 3(1):1–22, 2024.
- [21] S. Dereich and T. Müller-Gronbach, General multilevel adaptations for stochastic approximation algorithms of Robbins-Monro and Polyak-Ruppert type, *Numer. Math.*, 142(2):279–328, 2019.
- [22] S. Du, J. Lee, H. Li, L. Wang, and X. Zhai, Gradient descent finds global minima of deep neural networks, in: *Proceedings of the 36th International Conference on Machine Learning*, PMLR, 97:1675–1685, 2019.
- [23] M. Duflo, *Algorithmes Stochastiques*, in: *Mathématiques et Applications*, Springer-Verlag, Vol. 23, 1996.
- [24] I. Fatkhullin, J. Etesami, N. He, and N. Kiyavash, Sharp analysis of stochastic optimization under global Kurdyka-Łojasiewicz inequality, *Adv. Neural Inf. Process. Syst.*, 35:15836–15848, 2022.
- [25] B. Fehrman, B. Gess, and A. Jentzen, Convergence rates for the stochastic gradient descent method for non-convex objective functions, *J. Mach. Learn. Res.*, 21:136, 2020.
- [26] A. A. Gaivoronski, Convergence properties of backpropagation for neural nets via theory of stochastic gradient methods. Part 1, *Optim. Methods Softw.*, 4(2):117–134, 1994.
- [27] B. Gess and S. Kassing, Convergence rates for momentum stochastic gradient descent with noise of machine learning type, *arXiv:2302.03550*, 2023.
- [28] B. Gess, S. Kassing, and V. Konarovskiy, Stochastic modified flows, mean-field limits and dynamics of stochastic gradient descent, *J. Mach. Learn. Res.*, 25(30):1–27, 2024.
- [29] R. Gower, O. Sebbouh, and N. Loizou, SGD for structured nonconvex functions: Learning rates, mini-batching and interpolation, in: *Proceedings of The 24th International Conference on Artificial Intelligence and Statistics*, PMLR, 1315–1323, 2021.
- [30] L. Grippo, A class of unconstrained minimization methods for neural network training, *Optim. Methods Softw.*, 4(2):135–150, 1994.
- [31] S. Gunasekar, B. E. Woodworth, S. Bhojanapalli, B. Neyshabur, and N. Srebro, Implicit regularization in matrix factorization, *Adv. Neural Inf. Process. Syst.*, Vol. 30, 2017.

- [32] M. Gurbuzbalaban, U. Simsekli, and L. Zhu, The heavy-tail phenomenon in SGD, in: *International Conference on Machine Learning*, PMLR, 3964–3975, 2021.
- [33] A. Haraux, Some applications of the Łojasiewicz gradient inequality, *Commun. Pure Appl. Anal.*, 11:2417–2427, 2012.
- [34] A. Jentzen and A. Riekert, On the existence of global minima and convergence analyses for gradient descent methods in the training of deep neural networks, *J. Mach. Learn.*, 1(2):141–246, 2022.
- [35] A. Jentzen and T. Welti, Overall error analysis for the training of deep neural networks via stochastic gradient descent with random initialisation, *Appl. Math. Comput.*, 455:127907, 2023.
- [36] B. Karimi, B. Miasojedow, E. Moulines, and H.-T. Wai, Non-asymptotic analysis of biased stochastic approximation scheme, in: *Conference on Learning Theory*, PMLR, 1944–1974, 2019.
- [37] H. Karimi, J. Nutini, and M. Schmidt, Linear convergence of gradient and proximal-gradient methods under the Polyak-Łojasiewicz condition, in: *European Conference on Machine Learning and Knowledge Discovery in Databases*, Springer, 9851:795–811, 2016.
- [38] A. Khaled and P. Richtárik, Better theory for SGD in the nonconvex world, *Transactions on Machine Learning Research*, 2023.
- [39] P. D. Khanh, B. S. Mordukhovich, and D. B. Tran, A new inexact gradient descent method with applications to nonsmooth convex optimization, *Optim. Methods Softw.*, 1–29, 2024.
- [40] S. G. Krantz and H. R. Parks, *A Primer of Real Analytic Functions*, Springer Science & Business Media, 2002.
- [41] H. J. Kushner and G. G. Yin, *Stochastic approximation and recursive algorithms and applications*, in: *Stochastic Modelling and Applied Probability*, Springer-Verlag, Vol. 35, 2003.
- [42] Y. Lei, T. Hu, G. Li, and K. Tang, Stochastic gradient descent for nonconvex learning without bounded gradient assumptions, *IEEE Trans. Neural Netw. Learn. Syst.*, 31(10):4394–4400, 2019.
- [43] X. Li and A. Milzarek, A unified convergence theorem for stochastic optimization methods, *Adv. Neural Inf. Process. Syst.*, 35:33107–33119, 2022.
- [44] X. Li, A. Milzarek, and J. Qiu, Convergence of random reshuffling under the Kurdyka-Łojasiewicz inequality, *SIAM J. Optim.*, 33(2):1092–1120, 2023.
- [45] S. Łojasiewicz, Une propriété topologique des sous-ensembles analytiques réels, *Les équations aux dérivées partielles*, 117:87–89, 1963.
- [46] S. Łojasiewicz, *Ensembles Semi-Analytiques*, IHES (Bures-sur-Yvette), 1965.
- [47] Z.-Q. Luo and P. Tseng, Analysis of an approximate gradient projection method with applications to the backpropagation algorithm, *Optim. Methods Softw.*, 4(2):85–101, 1994.
- [48] H. Luschgy, *Martingale in Diskreter Zeit: Theorie und Anwendungen*. Springer-Verlag, 2012.
- [49] O. L. Mangasarian and M. V. Solodov, Serial and parallel backpropagation convergence via nonmonotone perturbed minimization, *Optim. Methods Softw.*, 4(2):103–116, 1994.
- [50] P. Mertikopoulos, N. Hallak, A. Kavis, and V. Cevher, On the almost sure convergence of stochastic gradient descent in non-convex problems, *Adv. Neural Inf. Process. Syst.*, 33:1117–1128, 2020.
- [51] A. Milzarek and J. Qiu, Convergence of a normal map-based Prox-SGD method under the KL inequality, *arXiv:2305.05828*, 2023.
- [52] Q. Nguyen, On connected sublevel sets in deep learning, in: *Proceedings of the 36th International Conference on Machine Learning*, PMLR, 97:4790–4799, 2019.
- [53] J. Nocedal and S. J. Wright, *Numerical Optimization*, Springer, 1999.
- [54] J. J. Palis and W. De Melo, *Geometric theory of dynamical systems: An Introduction*, Springer Science & Business Media, 2012.
- [55] P. Petersen, M. Raslan, and F. Voigtlaender, Topological properties of the set of functions generated by neural networks of fixed size, *Found. Comput. Math.*, 21:375–444, 2021.
- [56] P. Petersen and F. Voigtlaender, Optimal approximation of piecewise smooth functions using deep ReLU neural networks, *Neural Netw.*, 108:296–330, 2018.
- [57] B. T. Poljak, Gradient methods for minimizing functionals, *J. Comput. Math. Math. Phys.*, 3:643–653, 1963.
- [58] B. T. Polyak, A new method of stochastic approximation type, *Avtomat. i Telemekh.*, 7:98–107, 1990.
- [59] B. T. Polyak and A. B. Juditsky, Acceleration of stochastic approximation by averaging, *SIAM J. Control Optim.*, 30(4):838–855, 1992.

- [60] B. T. Polyak and Y. Z. Tsypkin, Pseudogradient adaptation and training algorithms, *Avtomat. i Telemekh.*, 3:45–68, 1973.
- [61] T. Qin, L. Zhou, and D. Xiu, Reducing parameter space for neural network training, *Theor. Appl. Mech. Lett.*, 10(3):170–181, 2020.
- [62] Q. Rebjock and N. Boumal, Fast convergence to non-isolated minima: Four equivalent conditions for C^2 functions, *arXiv:2303.00096*, 2023.
- [63] H. Robbins and S. Monro, A stochastic approximation method, *Ann. Math. Statistics*, 22(3):400–407, 1951.
- [64] D. Ruppert, Almost sure approximations to the Robbins-Monro and Kiefer-Wolfowitz processes with dependent noise, *Ann. Probab.*, 10:178–187, 1982.
- [65] U. Simsekli, L. Sagun, and M. Gurbuzbalaban, A tail-index analysis of stochastic gradient noise in deep neural networks, in: *International Conference on Machine Learning*, PMLR, 5827–5837, 2019.
- [66] D. Soudry, E. Hoffer, M. S. Nacson, S. Gunasekar, and N. Srebro, The implicit bias of gradient descent on separable data, *J. Mach. Learn. Res.*, 19(1):2822–2878, 2018.
- [67] V. B. Tadic, Convergence and convergence rate of stochastic gradient search in the case of multiple and non-isolated extrema, *arXiv:0907.1020*, 2009.
- [68] V. B. Tadic, Convergence and convergence rate of stochastic gradient search in the case of multiple and non-isolated extrema, *Stochastic Process. Appl.*, 125(5):1715–1755, 2015.
- [69] S. Vaswani, F. Bach, and M. Schmidt, Fast and faster convergence of SGD for over-parameterized models and an accelerated perceptron, in: *The 22nd International Conference on Artificial Intelligence and Statistics*, PMLR, 1195–1204, 2019.
- [70] H. Walk, Foundations of stochastic approximation, in: *Stochastic approximation and optimization of random systems. DMV Seminar*, Birkhäuser, Vol. 17, 1992.
- [71] S. Weissmann, S. Klein, W. Azizian, and L. Döring, Almost sure convergence rates of stochastic gradient methods under gradient domination, *arXiv:2405.13592*, 2024.
- [72] S. Wojtowytsch, Stochastic gradient descent with noise of machine learning type Part I: Discrete time analysis, *J. Nonlinear Sci.*, 33(3):45, 2023.
- [73] S. Wojtowytsch, Stochastic gradient descent with noise of machine learning type Part II: Continuous time analysis, *J. Nonlinear Sci.*, 34(1):16, 2024.
- [74] B. Woodworth, S. Gunasekar, J. D. Lee, E. Moroshko, P. Savarese, I. Golan, D. Soudry, and N. Srebro, Kernel and rich regimes in overparametrized models, in: *Conference on Learning Theory*, PMLR, 3635–3673, 2020.
- [75] L. Wu, C. Ma, and W. E, How SGD selects the global minima in over-parameterized learning: A dynamical stability perspective, *Adv. Neural Inf. Process. Syst.*, Vol. 31, 2018.
- [76] Y. Xie, X. Wu, and R. Ward, Linear convergence of adaptive stochastic gradient descent, in: *International Conference on Artificial Intelligence and Statistics*, PMLR, 1475–1485, 2020.
- [77] Z. Zhu, J. Wu, B. Yu, L. Wu, and J. Ma, The anisotropic noise in stochastic gradient descent: Its behavior of escaping from sharp minima and regularization effects, in: *International Conference on Machine Learning*, PMLR, 7654–7663, 2019.