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Partial Fraction Decomposition of Matrices and Parallel Computing

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Abstract. We are interested in the design of parallel numerical schemes for linear systems. We give an effective solution to this problem in the following case: the matrix A of the linear system is the product of p nonsingular matrices A_i^m with specific shape: $A_i = I - h_i X$ for a fixed matrix X and real numbers h_i. Although having a special form, these matrices A_i arise frequently in the discretization of evolutionary Partial Differential Equations. For example, one step of the implicit Euler scheme for the evolution equation $u' = Xu$ reads $(I-hX)u^{n+1} = u^n$. Iterating m times such a scheme leads to a linear system $Au^{n+m} = u^n$. The idea is to express A^{-1} as a linear combination of elementary matrices A_i^{-1} (or more generally in term of matrices A_i^{-k}). Hence the solution of the linear system with matrix A is a linear combination of the solutions of linear systems with matrices A_i (or A_i^k). These systems are then solved simultaneously on different processors.

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1 Introduction

Let X be a real $n \times n$ square real (or complex) matrix and $(h_i)_{i=1}^p$ a collection of pairwise distinct real numbers. Matrices of the shape

$$A_i = I - h_i X \tag{1.1}$$

with I the $n \times n$ identity matrix, appear in many numerical schemes for differential equations. For small h_i (discretization parameters), such matrices are nonsingular. So is matrix

$$A = \prod_{i=1}^p A_i^m \tag{1.2}$$

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for any $m \in \mathbb{N}$. Note that matrices A_i commute and A is a polynomial of the single variable X. The problem of interest here is the following: given $y \in \mathbb{R}^n$, compute the unique solution $x \in \mathbb{R}^n$ of the linear system

$$Ax = b \tag{1.3}$$

using several processors at our disposal. The aim is to use parallel computing to save computational time. The key idea consists in expressing matrix A^{-1} as a linear combination of matrices $((A_i^{-k})_{i=1}^p)_{k=1}^m$. Since matrices A_i we consider come from the discretization of differential equations by time implicit scheme, our method falls into the family of "parallelization in time" algorithms. More precisely, it is a Direct Parallel Time Integration like the one developped by Y. Maday and E. Rønquist in [5], based on tensor-product space-time solvers. We refrer to the survey article by M. Gander [1] for a comprehensive overview of parallel time integration methods.

Our method is also related to exponential integrators. Consider the linear constant-coefficients system $u_t = Lu$ whose solution is $u(t) = e^{tL}u_0$. Applying p steps of an implicit scheme, we obtain a numerical evolution u_p defined by $Au_p = u_0$ with A the product of p matices A_i (see Section 2). Writing A^{-1} as a linear combination of A_i^{-1}′ is equivalent to seek for a rational approximation of the solution operator e^{tL}. Rational approximation of the exponential is an important problem in numerical analysis. Consult the survey paper by M. Hochbruck and A. Ostermann ([3]) on exponential integrators.

The paper is organised as follows.

- In Section 2, we present the algebraic problem to be solved.
- In Section 3, we explain how the method can be used to solve homgeneous evolution equations $u_t = \mathcal{L}u$.
- In Section 4, we consider nonhomgeneous evolution equations $u_t = \mathcal{L}u + f$ and derive a new parallel algorithm to solve such problems.
- The last section contains remarks on the reliability of the method and its limitations.

All computations have been done using FreeFem++, a Finite Elements software for the discretization of Partial Differential Equations [2].

2 The algebraic problem

We consider in this section the homogeneous linear time-evolution problem

$$\mathbf{u}_t = \mathcal{L}\mathbf{u}. \tag{2.1}$$

Section 4 is devoted the nonhomogenoeus case which requiers a different numerical treatment. Starting from an initial data $\mathbf{u}^0$, the time discretization of this equation by implicit