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Boundedness Characterization of Maximal Commutators on Orlicz Spaces in the Dunkl Setting

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Abstract. On the real line, the Dunkl operators

$$D_\nu(f)(x) := \frac{df(x)}{dx} + (2\nu+1)\frac{f(x)-f(-x)}{2x}, \qquad \forall x \in \mathbb{R},\ \forall \nu \geq -\tfrac{1}{2}$$

are differential-difference operators associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$, and on the $\mathbb{R}^d$ the Dunkl operators $\{D_{k,j}\}_{j=1}^d$ are the differential-difference operators associated with the reflection group $\mathbb{Z}_2^d$ on $\mathbb{R}^d$. In this paper, in the setting $\mathbb{R}$ we show that $b \in BMO(\mathbb{R}, dm_\nu)$ if and only if the maximal commutator $M_{b,\nu}$ is bounded on Orlicz spaces $L_\Phi(\mathbb{R}, dm_\nu)$. Also in the setting $\mathbb{R}^d$ we show that $b \in BMO(\mathbb{R}^d, h_k^2(x)dx)$ if and only if the maximal commutator $M_{b,k}$ is bounded on Orlicz spaces $L_\Phi(\mathbb{R}^d, h_k^2(x)dx)$.

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1 Introduction

Norm inequalities for several classical operators of harmonic analysis have been widely studied in the context of Orlicz spaces. It is well known that many of such operators fail to have continuity properties when they act between certain Lebesgue spaces and, in some situations, the Orlicz spaces appear as adequate substitutes. For example, the Hardy-Littlewood maximal operator is bounded on L^p for $1 < p < \infty$, but not on L^1, but

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using Orlicz spaces, we can investigate the boundedness of the maximal operator near $p=1$, see [13] and [4] for more precise statements.

Let T be the classical singular integral operator, the *commutator* $[b,T]$ generated by T and a suitable function b is given by

$$[b,T]f := bT(f) - T(bf). \tag{1.1}$$

A well-known result due to Coifman, *et al.* [3] (see also [11]) states that $b\in BMO(\mathbb{R}^n)$ if and only if the commutator $[b,T]$ is bounded on $L^p(\mathbb{R}^n)$ for $1<p<\infty$.

Maximal commutator of Hardy-Littlewood maximal operator M with a locally integrable function b is defined by

$$M_b f(x) = \sup_{B\ni x} \frac{1}{|B|}\int_B |b(x)-b(y)||f(y)|dy,$$

where the supremum is taken over all balls $B\subset\mathbb{R}^n$ containing x. We refer to [2] for a detailed investigation of the operators M_b and the commutator of the maximal operator $[b,M]$ and references therein. For the boundedness of these operators in Orlicz space $L^\Phi(\mathbb{R}^n)$ see for instance [5,7].

In [9], Dunkl introduced a family of first order differential-difference operators which play the role of the usual partial differentiation for the reflection group structure. For a real parameter $\nu\ge -1/2$, we consider the *Dunkl operator*, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$:

$$D_\nu(f)(x) := \frac{df(x)}{dx} + (2\nu+1)\frac{f(x)-f(-x)}{2x}, \quad \forall x\in\mathbb{R}.$$

Note that $D_{-1/2} = d/dx$.

In the setting $\mathbb{R}^d$ the Dunkl operators $\{D_{k,j}\}_{j=1}^d$, which are the differential-difference operators introduced by Dunkl in [9]. These operators are very important in pure mathematics and in physics. They provide useful tools in the study of special functions with root systems.

It is well known that maximal operators play an important role in harmonic analysis (see [21]). Harmonic analysis associated to the Dunkl transform and the Dunkl differential-difference operator gives rise to convolutions with a relevant generalized translation. In this paper, in the framework of this analysis in the setting $\mathbb{R}$, we study the boundedness of the maximal commutator $M_{b,\nu}$ and the commutator of the maximal operator, $[b,M_\nu]$, on Orlicz spaces $L_\Phi(\mathbb{R},dm_\nu)$, when b belongs to the space $BMO(\mathbb{R},dm_\nu)$, by which some new characterizations of the space $BMO(\mathbb{R},dm_\nu)$ are given. Also in the setting $\mathbb{R}^d$ we study the boundedness of the maximal commutator $M_{b,k}$ and the commutator of the maximal operator, $[b,M_k]$, on the Orlicz space $L_\Phi(\mathbb{R}^d,h_k^2(x)dx)$, when b belongs to the space $BMO(\mathbb{R}^d,h_k^2(x)dx)$, by which some new characterizations of the space $BMO(\mathbb{R}^d,h_k^2(x)dx)$ are given.