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The Monge-Ampère Equation for Strictly $(n-1)$-convex Functions with Neumann Condition

Bin Deng*

Department of Mathematics, University of Science and Technology of China, Hefei 230026, China.

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Abstract. A C^2 function on $\mathbb{R}^n$ is called strictly $(n-1)$-convex if the sum of any $n-1$ eigenvalues of its Hessian is positive. In this paper, we establish a global C^2 estimates to the Monge-Ampère equation for strictly $(n-1)$-convex functions with Neumann condition. By the method of continuity, we prove an existence theorem for strictly $(n-1)$-convex solutions of the Neumann problems.

AMS subject classifications: 35J60, 35A09

Key words: Neumann problem, $(n-1)$-convex, elliptic equation.

1 Introduction

Let $\Omega\subset\mathbb{R}^n$ be a bounded convex domain and $\nu(x)$ be the outer unit normal at $x\in\partial\Omega$. Suppose $f\in C^2(\Omega)$ is positive and $\phi\in C^3(\overline{\Omega})$. In this paper, we mainly consider the following equations of Monge-Ampère type with Neumann condition,

$$\begin{cases} \det(W)=f(x), & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu}=-u+\phi(x), & \text{in} \quad \partial\Omega. \end{cases} \tag{1.1}$$

where the matrix $W=(w_{\alpha_1\cdots\alpha_m,\beta_1\cdots\beta_m})_{n\times n}$, for $m=n-1$, with the elements as follows,

$$w_{\alpha_1\cdots\alpha_m,\beta_1\cdots\beta_m}=\sum_{i=1}^{m}\sum_{j=1}^{n}u_{\alpha_i j}\delta^{\alpha_1\cdots\alpha_{i-1}j\alpha_{i+1}\cdots\alpha_m}_{\beta_1\cdots\beta_{i-1}\beta_i\beta_{i+1}\cdots\beta_m}, \tag{1.2}$$

a linear combination of u_{ij}, where $u_{ij}=\frac{\partial^2 u}{\partial x_i\partial x_j}$ and $\delta^{\alpha_1\cdots\alpha_{i-1}\gamma\alpha_{i+1}\cdots\alpha_m}_{\beta_1\cdots\beta_{i-1}\beta_i\beta_{i+1}\cdots\beta_m}$ is the generalized Kronecker symbol. All indexes $i,j,\alpha_i,\beta_i,\cdots$ come from 1 to n.

*Corresponding author. *Email address:* bingomat@mail.ustc.edu.cn (B. Deng)

For general $1 \le m \le n-1$, the matrix $W \in \mathbb{R}^{C_n^m \times C_n^m}$, $C_n^m = \frac{n!}{m!(n-m)!}$, comes from the following operator $U^{[m]}$ as in [2] and [10]. First, note that $(u_{ij})_{n\times n}$ induces an operator U on $\mathbb{R}^n$ by

$$U(e_i) = \sum_{j=1}^{n} u_{ij} e_j, \quad \forall 1 \le i \le n,$$

where $\{e_1, e_2, \cdots, e_n\}$ is the standard orthogonal basis of $\mathbb{R}^n$. We further extend U to act on the real vector space $\wedge^m \mathbb{R}^n$ by

$$U^{[m]}(e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_m}) = \sum_{i=1}^{m} e_{\alpha_1} \wedge \cdots \wedge U(e_{\alpha_i}) \wedge \cdots \wedge e_{\alpha_m},$$

where $\{e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_m} \mid 1 \le \alpha_1 < \cdots < \alpha_m \le n\}$ is the standard basis for $\wedge^m \mathbb{R}^n$. Then W is the matrix of $U^{[m]}$ under this standard basis. It is convenient to denote the multi-index by $\overline{\alpha} = (\alpha_1 \cdots \alpha_m)$. We only consider the increasing multi-index, that is, $1 \le \alpha_1 < \cdots < \alpha_m \le n$. By the dictionary arrangement, we can arrange all increasing multi-indexes from 1 to C_n^m, and use $N_{\overline{\alpha}}$ denote the order number of the multi-index $\overline{\alpha} = (\alpha_1 \cdots \alpha_m)$, i.e., $N_{\overline{\alpha}} = 1$ for $\overline{\alpha} = (12 \cdots m)$, $\cdots$. We also use $\overline{\alpha}$ denote the index set $\{\alpha_1, \cdots, \alpha_m\}$ without confusion. It is not hard to see that

$$W_{N_{\overline{\alpha}} N_{\overline{\alpha}}} = w_{\overline{\alpha},\overline{\alpha}} = \sum_{i=1}^{m} u_{\alpha_i \alpha_i}, \tag{1.3}$$

$$W_{N_{\overline{\alpha}} N_{\overline{\beta}}} = w_{\overline{\alpha}\overline{\beta}} = (-1)^{|i-j|} u_{\alpha_i \beta_j}, \tag{1.4}$$

if the index set $\{\alpha_1, \cdots, \alpha_m\} \setminus \{\alpha_i\}$ equals to the index set $\{\beta_1, \cdots, \beta_m\} \setminus \{\beta_j\}$ but $\alpha_i \ne \beta_j$; and also

$$W_{N_{\overline{\alpha}} N_{\overline{\beta}}} = w_{\overline{\alpha}\overline{\beta}} = 0, \tag{1.5}$$

if the index sets $\{\alpha_1, \cdots, \alpha_m\}$ and $\{\beta_1, \cdots, \beta_m\}$ have more than one different element. Specifically, for $n=3, m=2$, we have

$$W = \begin{pmatrix} u_{11}+u_{22} & u_{23} & -u_{13} \\ u_{32} & u_{11}+u_{33} & u_{12} \\ -u_{31} & u_{21} & u_{22}+u_{33} \end{pmatrix}.$$

It follows that W is symmetrical and diagonal with $(u_{ij})_{n\times n}$ diagonal. The eigenvalues of W are the sums of eigenvalues of $(u_{ij})_{n\times n}$. Denoted by $\mu(D^2u) = (\mu_1, \cdots, \mu_n)$ the eigenvalues of the Hessian and by $\lambda(W) = (\lambda_1, \lambda_2, \cdots, \lambda_{C_n^m})$ the eigenvalues of W. Generally, for any $k = 1, 2, \cdots, C_n^m$, we define the k^{th} elementary symmetry function by

$$S_k(W) = S_k\big(\lambda(W)\big) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le C_n^m} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k},$$