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O'Neil Inequality for Convolutions Associated with Gegenbauer Differential Operator and some Applications

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Abstract. In this paper we prove an O'Neil inequality for the convolution operator (G-convolution) associated with the Gegenbauer differential operator G_λ. By using an O'Neil inequality for rearrangements we obtain a pointwise rearrangement estimate of the G-convolution. As an application, we obtain necessary and sufficient conditions on the parameters for the boundedness of the G-fractional maximal and G-fractional integral operators from the spaces $L_{p,\lambda}$ to $L_{q,\lambda}$ and from the spaces $L_{1,\lambda}$ to the weak spaces $WL_{p,\lambda}$.

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1 Introduction

For $1 \le p \le \infty$, let $L_{p,\lambda}\left(\mathbb{R}_+, sh^{2\lambda}xdx\right)$ be the spaces of measurable functions on $\mathbb{R}_+ = (0,\infty)$ with the finite norm

$$\|f\|_{L_{p,\lambda}(\mathbb{R}_+)} = \left(\int_{\mathbb{R}_+} |f(chx)|^p sh^{2\lambda} xdx\right)^{\frac{1}{p}}, \qquad 1 \le p < \infty,$$
$$\|f\|_{L_{\infty,\lambda}} \equiv \|f\|_{L_\infty(\mathbb{R}_+)} = \underset{x\in\mathbb{R}_+}{ess\sup}|f(chx)|,$$

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where $0<\lambda<\frac{1}{2}$ is a fixed parameter.

Denote by A^{λ}_{cht} the shift operator (G-shift) (see [9])

$$A^{\lambda}_{cht}f(chx)=C_{\lambda}\int_0^{\pi} f(chxcht-shxsht\cos\varphi)\ (\sin\varphi)^{2\lambda-1}d\varphi\ ,$$

generated by Gegenbauer differential operator G_{λ}

$$G\equiv G_{\lambda}=\left(x^2-1\right)^{\frac{1}{2}-\lambda}\frac{d}{dx}\left(x^2-1\right)^{\lambda+\frac{1}{2}}\frac{d}{dx},\quad x\in(1,\infty),\ \lambda\in\left(0,\frac{1}{2}\right),$$

where

$$C_{\lambda}=\frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)\ \Gamma\left(\frac{1}{2}\right)}=\left(\int_0^{\pi}(\sin\varphi)^{2\lambda-1}d\varphi\right)^{-1}.$$

The Gegenbauer differential operator was introduced in [5]. For the properties of the Gegenbauer differential operator, we refer to [3,4,10–12].

The shift operator A^{λ}_{cht} generates the corresponding convolution (G-convolution)

$$(f\oplus g)(chx)=\int_{\mathbb{R}_+} f(cht)A^{\lambda}_{cht}\,g(chx)sh^{2\lambda}tdt.$$

The paper is organized as follows. In Section 2, we give some results needed to facilitate the proofs of our theorems. In Section 3, we show that an O'Neil inequality for rearrangements of the G-convolution holds. In Section 4, we prove an O'Neil inequality for G-convolution. In Section 5, we prove the boundedness of G-fractional maximal and G-fractional integral operators from the spaces $L_{p,\lambda}$ to $L_{q,\lambda}$ and from the spaces $L_{1,\lambda}$ to the weak spaces $WL_{q,\lambda}$. We show that the conditions on the boundedness cannot be weakened.

Further $A\lesssim B$ denotes that exists the constant $C>0$ such that $0<A\le CB$, moreover C can depend on some parameters. Symbol $A\approx B$ denote that $A\lesssim B$ and $B\lesssim A$.

2 Some auxiliary results

In this section we formulate some lemmas that will be needed later.

Lemma 2.1. *1) Let $1\le p\le\infty$, $f\in L_{p,\lambda}(\mathbb{R}_+)$, then for all $t\in\mathbb{R}_+$*

$$\left\|A^{\lambda}_{cht}\,f\right\|_{L_{p,\lambda}}\le\|f\|_{L_{p,\lambda}}.$$

2) Let $1\le p,\ r\le q\le\infty$, $\frac{1}{p'}+\frac{1}{q'}=\frac{1}{r}$, $pp'=p+p'$, $f\in L_{p,\lambda}(\mathbb{R}_+)$, $g\in L_{r,\lambda}(\mathbb{R}_+)$. Then $f\oplus g\in L_{q,\lambda}(\mathbb{R}_+)$ and

$$\|f\oplus g\|_{L_{q,\lambda}}\le\|f\|_{L_{p,\lambda}}\|g\|_{L_{r,\lambda}},$$

(see [9], *Lemmas* 2 *and* 4*).*