J. Math. Study
doi: 10.4208/jms.v53n3.20.04

Vol. **53**, No. 3, pp. 297-315
September 2020

Boundedness of Rough Singular Integral Operators on Homogeneous Herz Spaces with Variable Exponents

Jinling Cai[1], Baohua Dong[2] and Canqin Tang[1,*]

[1] *College of Science, Dalian Maritime University, Dalian 116026, China;*
[2] *Department of Mathematics, Nanjing University of Information Science and Technology, Nanjing 210044, China.*

Received April 29, 2019; Accepted August 3, 2019;
Published online May 26, 2020.

Abstract. We establish the boundedness of rough singular integral operators on homogeneous Herz spaces with variable exponents. As an application, we obtain the boundedness of related commutators with BMO functions on homogeneous Herz spaces with variable exponents.

AMS subject classifications: 42B20, 42B35

Key words: Rough singular integral operator, commutator, Herz spaces with variable exponents, BMO spaces.

1 Introduction

Let $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n (n\geq 2)$ and Ω be a measurable function defined on $\mathbb{S}^{n-1}$. The rough singular integral operator T_Ω is defined by

$$T_\Omega f(x) = p.v.\int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\mathrm{d}y.$$

In 1952, Calderón and Zygmund first studied the operator T_Ω in [3] and proved that T_Ω is bounded in $L^p(\mathbb{R}^n)$ for $p\in(1,\infty)$ if $\Omega\in C^\infty(\mathbb{S}^{n-1})$ is a homogeneous function of degree zero and

$$\int_{\mathbb{S}^{n-1}} \Omega(x)\mathrm{d}\sigma(x) = 0. \tag{1.1}$$

*Corresponding author. *Email addresses:* `1875815902@qq.com` (J. Cai), `baohuadong@nuist.edu.cn` (B. Dong), `tangcq2000@dlmu.edu.cn` (C. Tang)

The condition on Ω can be weakened or revised; see [4, 8, 27]. Later, the results were further extended to the weight Lebesgue spaces by Duoandikoetxea [14]. Further development of the topic in other function spaces with constant exponents can be found in [5–7, 15, 16, 23] and the references therein.

Variable exponent function spaces received considerable attentions in recent decades [37] . They are important not only in theory as generalizations of classical function spaces, but also for their wide applications in the fields of fluid dynamics, elasticity dynamics, the differential equations with nonstandard growth. We refer to [1, 2, 12, 28] for the details. The rich development can be found in many research works of the theory of variable exponent function spaces. For example, Lebesgue spaces with variable exponent were studied in [10, 12, 19], Herz spaces with variable exponent were studied in [17, 26, 29], Morrey spaces with variable exponent were studied in [20, 32, 34], and some other type of function spaces with variable exponent can be found in [12, 13, 21, 25, 35, 36, 38–41].

Along with the development of the theory of variable exponent function spaces, the theories of the rough singular integral operators and their commutators on these function spaces with variable exponents have attracted many researchers' attentions. In the variable exponent Lebesgue spaces, Cruz-Uribe et al obtained the boundedness of the rough singular integral operator [10] . The related works were generalized to Herz spaces with variable exponents. For example, Wang proved the rough singular integral operator T_Ω and its commutators are bounded from the variable exponent Herz space $H\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)$ to the variable exponent Herz space $\dot{K}^{\alpha,p_2}_{q(\cdot)}(\mathbb{R}^n)$ [31]. Besides it, Wang et al considered the parameterized Littlewood-Paley operators and their commutators on Herz spaces with variable exponents $\dot{K}^{\alpha}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$ [33].

Motivated by the above works, in the paper, we devote to solve the boundedness of the rough singular integral operators and their commutators on the homogeneous Herz spaces with variable exponents $\dot{K}^{\alpha}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$. The rest of this paper is arranged as follows. In Section 2 we recall the definition of the homogeneous Herz spaces with variable exponents $\dot{K}^{\alpha}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$ and state our main results. The proofs of the main theorems will be proved in Sections 3 and 4, respectively.

Finally, some conventions should be explained. C is denoted by a positive constant whose value may be different from line to line. The symbol $A \lesssim B$ stands for the inequality $A \le CB$. Other notations will be explained when we meet it.

2 Preliminary and main results

Let $\lambda \in (0,\infty)$ and $p(\cdot): \mathbb{R}^n \to [1,\infty)$ be a measurable function. The Lebesgue space with variable exponent $p(\cdot)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) := \left\{ f \text{ is measurable}: \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} \mathrm{d}x < \infty \right\},$$